

**ENCE 201 Final Exam, Open Notes and Open Book**

Name : \_\_\_\_\_

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**Exam Format and Grading.** This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are five questions. Partial credit will be given for partially correct answers, so please show all your working.

**IMPORTANT:** Please see the **class web page for instructions on how to submit your exam paper**. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
Total	70	

**Question 1: 20 points.**

Consider the family of matrix equations  $Ax = b$ , where:

$$\begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ a + 2 \end{bmatrix}. \quad (1)$$

**[1a]** (4 pts). Derive an expression for the  $\det(A)$  as a function of 'a'. Determine the values of 'a' for which matrix A will be singular.

**[1b]** (6 pts). Based on your solutions to part 1a, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

**[1c]** (10 pts). Use the method of Gauss Elimination (i.e., row operations followed by back substitution) to show that solutions to matrix equation 1 can be written:

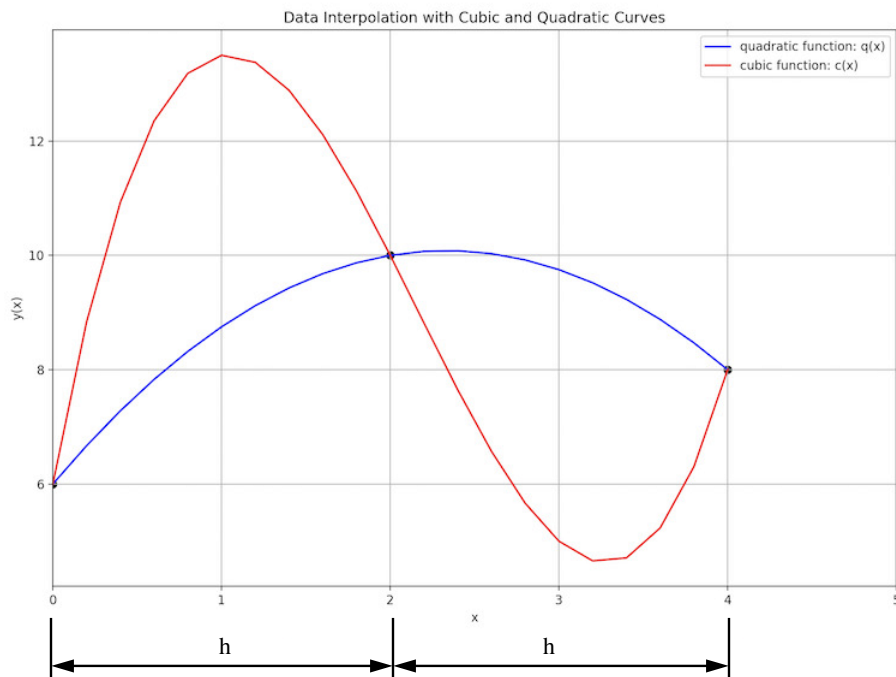
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a + 25)/(7a + 28) \\ (10a + 54)/(7a + 28) \\ 1/(a + 4) \end{bmatrix}. \quad (2)$$

This is a hand calculation, so show all of your working.

Question **[1c]** continued:

**Question 2: 20 points.**

This question covers numerical integration of quadratic and cubic functions with the Trapezoid Rule and Simpson's Rule.



**Figure 1.** Interpolation of three data points with quadratic and cubic functions.

To motivate the problem setup, Figure 1 shows a scenario where quadratic (i.e.,  $q(x) = q_0 + q_1x + q_2x^2$ ) and cubic (i.e.,  $c(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ ) curves interpolate three equally spaced data points.

**[2a]** (4 pts). Derive an expression for the integral:

$$I_q = \int_0^{2h} q(x) dx \quad (3)$$

in terms of  $q_0$ ,  $q_1$ ,  $q_2$  and  $h$ .

**[2b]** (4 pts). Derive an expression for the integral:

$$I_c = \int_0^{2h} c(x) dx \quad (4)$$

in terms of  $c_0$ ,  $c_1$ ,  $c_2$ ,  $c_3$  and  $h$ .

**[2c]** (6 pts). Show that one step of Simpson's Rule integrates the cubic equation exactly !!!

**[2d]** (6 pts). Now suppose that equation 3 is approximated by one step of the Trapezoid Rule. Develop a symbolic expression for the actual error. Compute the error estimate (which we have covered in class). Is the latter less than in actual error?

**Question 3: 10 points**

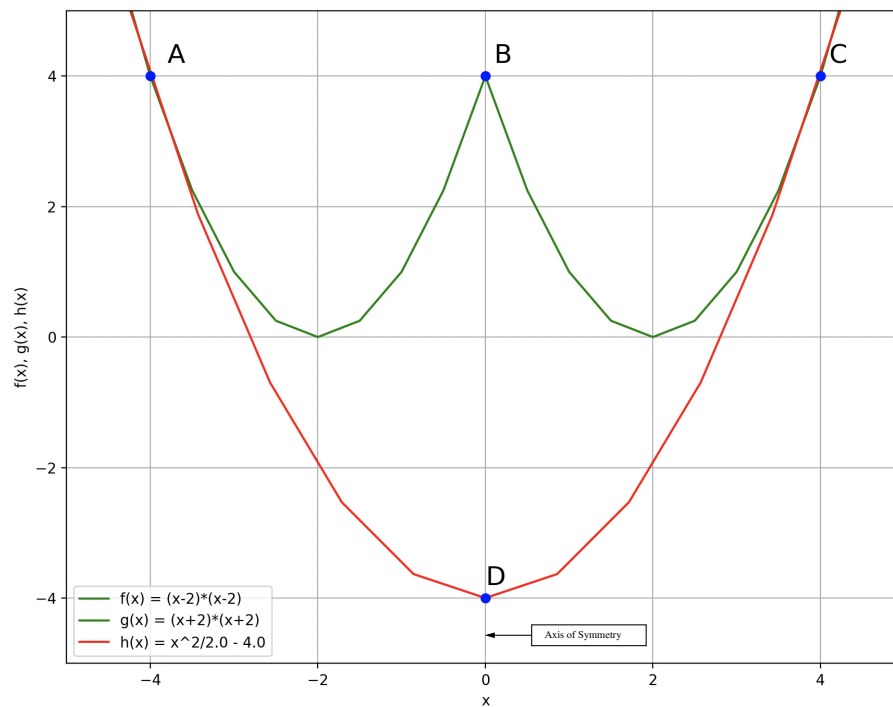
The family of equations:

$$f(x) = [x - 2]^2. \tag{5}$$

$$g(x) = [x + 2]^2. \tag{6}$$

$$h(x) = \left[ \frac{x^2 - 8}{2} \right]. \tag{7}$$

defines the region A-B-C-D shown in Figure 2.



**Figure 2.** Region A-B-C-D.

Theoretical considerations indicate that equation  $h(x)$  is tangent to  $f(x)$  and  $g(x)$  at their intersection (i.e., at points A and C). Also notice an axis of symmetry in region A-B-C-D along the line B-D – this observation can be used to simplify the analysis.



**[3a]** (3 pts). Using calculus, or otherwise, show that the area of region A-B-C-D is  $64/3$ .

**[3b]** (4 pts). Demonstrate how you can use **one step** of Simpson's Rule, to obtain a high-accuracy estimate the area of region A-B-C-D.

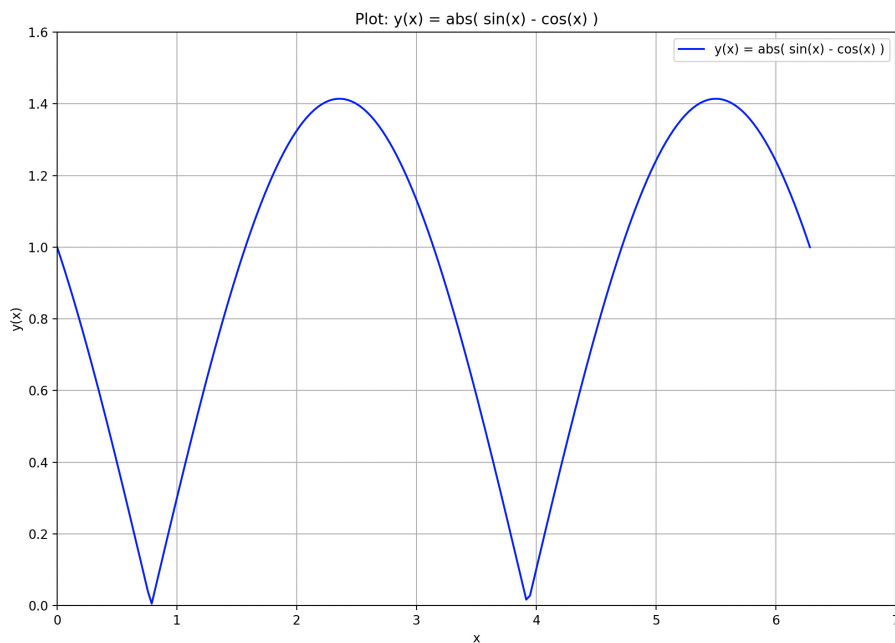
**[3c]** (3 pts). What is the expected error with Simpson's Rule? Briefly compare and discuss the analytical and numerical results.

**Question 4.** (10 pts.)

Consider the integral:

$$I = \int_0^{2\pi} |\sin(x) - \cos(x)| dx. \quad (8)$$

and the plot shown in Figure 3:



**Figure 3.** Graph of  $y(x) = |\sin(x) - \cos(x)|$  from 0 to  $2\pi$ .

Notice that  $y(x) = |\sin(x) - \cos(x)|$  is periodic. The values of  $y(x)$  can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2	5pi/8
y(x)	1.0000	0.5412	0.0000	0.5412	1.0000	1.3066

x	3pi/4	7pi/8	pi	9pi/8	5pi/4	11pi/8
y(x)	1.4142	1.3066	1.0000	0.5412	0.0000	0.5412

x	6pi/4	13pi/8	7pi/4	15pi/8	2pi
y(x)	1.0000	1.3066	1.4142	1.3066	1.0000

**[4a]** (2 pts). Briefly explain how you can use the periodic nature of  $y(x)$  to simplify the evaluation of equation 8.

**[4b]** (3 pts). Compute the analytical solution to equation 8 This is a hand calculation, so show all of your working.

**[4c]** (5 pts). Now suppose that equation 8 is evaluated using only four steps of the Trapezoid Rule. With your answer to part [4a] in mind, what is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

**Question 5: 10 points**

Consider the family of integration problems:

$$I_n = \int_0^{\pi/2} \left[ \frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \right] dx \quad (9)$$

where  $n$  is an integer. When  $n = 2$ , for example, we have:

$$I_2 = \int_0^{\pi/2} \left[ \frac{\sin^2(x)}{\sin^2(x) + \cos^2(x)} \right] dx = \int_0^{\pi/2} \sin^2(x) dx = \frac{\pi}{4}. \quad (10)$$

You can check this result via integration by parts (calculus), or Wolfram Alpha, or ChatGPT 3.5/4.

That was a little too easy – how about we set  $n = 5$ . The values for  $f(x) = \sin^5(x)/(\sin^5(x) + \cos^5(x))$  can be summarized as follows:

x		0.0		pi/8		pi/4		3pi/8		pi/2
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f(x)		0.0		0.012046		0.5		0.987954		1.0

**[5a]** (3 pts). Use the method of Romberg integration to obtain an  $O(h^4)$  accurate estimate of equation 9.

**[5b]** (4 pts). Evaluate equation 9 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

Sometimes in calculus a simple change of variables (or adjustment in your point of view) can transform a problem from one that seems impossible into something quite simple. With that in mind:

**[5c]** (3 pts). Show that:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \quad (11)$$

Hence, or otherwise, prove:

$$I_n = \int_0^{\pi/2} \left[ \frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \right] dx = \frac{\pi}{4}. \quad (12)$$

for **any integer**  $n$ . Show all of your working.