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Controllability of Lie-Poisson Reduced Dynamics [1]

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Abstract

In this paper we present sufficient conditions for the controllability of the reduced dynamics of a class of mechanical systems with symmetry. We prove conditions (boundedness of coadjoint orbits and existence of a radially unbounded Lyapunov function) under which the drift vector field (of the reduced system) is weakly positively Poisson stable (WPPS). The WPPS nature of the drift vector field along with the Lie algebra rank condition is used to show controllability of the reduced system. We discuss the dynamics, Lie-Poisson reduction, and controllability of hovercraft and underwater vehicles, all treated as rigid bodies.

1. Introduction

A geometric approach to the study of mechanical systems has had a profound influence in recent years on our understanding of dynamics and control aspects. Playing an essential role in this are recent developments in reduction theory (cf. [1]), i.e. the exploitation of invariance of the controlled dynamics to a group of transformations (the symmetry group). The existence of a symmetry group permits dropping of the dynamics to a lower dimensional (reduced) space. Lagrangian reduction involves dropping the Euler-Lagrange equations to the quotient of the velocity phase space given by the symmetry group while Hamiltonian reduction involves projecting the Poisson bracket to the reduced (quotient) space which also inherits a Poisson structure (attributed to Lie and Berezin-Kirillov-Kostant-Souriau, see the work of Weinstein for historical remarks [2]). In particular, if the configuration space of the system can be identified with a Lie group G, a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on T^*G/G which is isomorphic to $\mathcal{G}^*$ the dual of the Lie algebra of G. The reduced bracket is now the Lie-Poisson bracket. The complete dynamics can then be reconstructed from the reduced system.

The state space of a large class of mechanical systems such as hovercraft, spacecraft underwater vehicle etc. can be identified with a Lie group G. The dynamics (defined on T^*G) of these systems subject to external forces can often be written in the form of a control system as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \quad x \in T^*G, u = (u_1, \cdots u_m) \qquad (1)$$

where $f(x) = \sum_{i=1}^{n}\{x_i, H\}\frac{\partial}{\partial x_i}$, H is the Hamiltonian and $\{\cdot,\cdot\}$ is the canonical Poisson bracket defined on T^*G. If the vector fields f and g_i's are G-invariant[1] this allows us to drop the the vector fields f and g_i's from T^*G to T^*G/G and the reduced dynamics take the form

$$\dot{\mu} = \widetilde{f}(\mu) + \sum_{i=1}^{m} \widetilde{g}_i(\mu)\widetilde{u}_i \qquad (2)$$

where $\mu \in T^*G/G$, $\widetilde{f}$ and $\widetilde{g}_i$ are the projections of f and g_i on T^*G/G. As we shall see in section 2 in this particular situation $\widetilde{f} = \sum_{i=1}^{m}\{\mu_i, \widetilde{H}\}_-\frac{\partial}{\partial \mu_i}$ where $\{\cdot,\cdot\}_-$ is the minus Lie-Poisson bracket and $\widetilde{H}$ is the reduced Hamiltonian defined on T^*G/G.

In this paper we present sufficient conditions for controllability of affine nonlinear control systems of the form (2) where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field. We show that depending on the existence of a radially unbounded Lyapunov type function, the drift vector field (of the reduced system (2)) is weakly positively Poisson stable (WPPS). The WPPS nature of the drift vector field along with the Lie algebra rank condition is used to show controllability of the reduced system. Our result gives a manageable tool to check for controllability of a wide class of mechanical systems with symmetry.

The paper is organized as follows. In section 2 we present a brief overview of Lie-Poisson reduction. In section 3 we present our main result on controllability of reduced dynamics. In section 4 we discuss in some detail, the dynamics, and reduced space controllability of the hovercraft and the underwater vehicle. Conclusions and future work are discussed in section 5.

2. Lie-Poisson Reduction

Recall (cf. [1, 3]) that if G is a symmetry group acting on a Poisson manifold M, then the quotient manifold M/G inherits a Poisson structure so that when-

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[1]We do not require that g_i's be Hamiltonian

ever $\widetilde{P}, \widetilde{Q} : M/G \to \mathbb{R}$ correspond to G invariant functions, $P, Q : M \to \mathbb{R}$, their Poisson bracket $\{\widetilde{P}, \widetilde{Q}\}_{M/G}$ corresponds to the G-invariant function $\{P, Q\}_M$. If $H : M \to \mathbb{R}$ is a G-invariant Hamiltonian, then H descends to $\widetilde{H} : M/G \to \mathbb{R}$ and determines the reduced dynamics on M/G. The solutions of the reduced Hamiltonian system on M/G are projections of the solutions of the complete system defined on M. In particular if $M \stackrel{\triangle}{=} T^*G$ and G acts on itself by left translations then $M/G \equiv \mathcal{G}^*$, the dual of the Lie algebra of G, a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on $\mathcal{G}^*_-$ (the space $\mathcal{G}^*$ associated with the minus Poisson structure). The reduced bracket is now the minus Lie-Poisson bracket, $\{\cdot,\cdot\}_-$ defined in its coordinate free form by $\{\widetilde{P}, \widetilde{Q}\}_-(x) = - < x, [\nabla \widetilde{P}(x), \nabla \widetilde{Q}(x)] >$. Let $\{X_1, \cdots, X_r\}$ and $\{X_1^\flat, \cdots, X_r^\flat\}$ be a basis for the Lie algebra $\mathcal{G}$ and a basis for the dual space $\mathcal{G}^*$ respectively, i.e. $< X_i^\flat, X_j >= \delta_j^i$. Any $\mu \in \mathcal{G}^*_-$ can be expressed as $\mu = \sum_{i=1}^r \mu_i X_i^\flat$ and the Lie-Poisson bracket of two differentiable functions $F, H \in C^\infty(\mathcal{G}^*)$ is given by

$$\{F, H\}_-(\mu) = - \sum_{i,j,k=1}^{r} c_{ij}^k \mu_k \frac{\partial F}{\partial \mu_i} \frac{\partial H}{\partial \mu_j} \tag{3}$$

where $c_{ij}^k, i, j, k = 1, \cdots, r$ are the structure constants of $\mathcal{G}$ relative to the basis $\{X_1, \cdots, X_r\}$. Equivalently (3) can be written as

$$\{F, H\}_-(\mu) = \nabla F^T \Lambda(\mu) \nabla H \tag{4}$$

where $[\Lambda(\mu)]_{ij} = -\sum_{k=1}^r c_{ij}^k \mu_k$. The rank of Poisson tensor Λ determines the nontrivial Casimirs of μ. The Lie-Poisson reduced dynamics on $\mathcal{G}^*$ can now be expressed as

$$\dot{\mu} = \{\mu, \widetilde{H}\}_- \tag{5}$$

The ith component on the right hand side being $\{\mu_i, \widetilde{H}\}_-$, $i = 1, \cdots, r$ and $\widetilde{H}$ is the reduced Hamiltonian. The induced symplectic foliation by the Lie-Poisson bracket on $\mathcal{G}^*_-$ has a particularly nice interpretation in terms of the dual to the adjoint representation of the underlying Lie group G on the Lie algebra $\mathcal{G}$. This is given by the following theorem, which appears to be due to Kirillov, Arnold, Kostant and Souriau, though similar ideas first appear in the work of Lie, Berezin and Weil. (See [1] for historical comments and references.)

Theorem 1 *Let G be a connected Lie group with coadjoint representation Ad^*G on $\mathcal{G}^*$. Then the orbits of Ad^*G are immersed submanifolds of $\mathcal{G}^*$ and are precisely the leaves of the symplectic foliation induced by the Lie-Poisson bracket on $\mathcal{G}^*$. Moreover, for each $g \in G$, the coadjoint map Ad^*G is a Poisson mapping on $\mathcal{G}^*$ preserving the leaves of the foliation.*

As flows of (5) remain on coadjoint orbits on which they started, the geometry of coadjoint orbits plays an important role in understanding the dynamics of the Lie-Poisson reduced equations.

3. Poisson Stability and Controllability

Consider an affine nonlinear control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, u = (u_1, \cdots u_m) \in U \subset \mathbb{R}^m. \tag{6}$$

Assume $x \in M \subset \mathbb{R}^n$, where M is a smooth manifold. Let $L(x) = \{\text{span} X(x) \mid X \text{ vector field in } \mathcal{L}, x \in M\}$ where $\mathcal{L}$, the accessibility Lie algebra, is the smallest subalgebra of the Lie algebra of vector fields on M that contains $f, g_1, \cdots, g_m$. The system is said to satisfy the accessibility Lie algebra rank condition (LARC) if $L(x) = T_x M$, $\forall x \in M$. We know that if (6) satisfies the Lie algebra rank condition (LARC) then it is locally accessible, and in addition if $f = 0$ then LARC implies that the system is controllable. While the kinematic equations of motion can often be written as a drift free system, once dynamics are included LARC does not imply controllability. Proving controllability is usually much harder than proving accessibility. In [4] sufficient conditions are given, in terms of a "group action", that a locally accessible system is also locally reachable. In [5] sufficient conditions for the controllability of a conservative dynamical polysystem on a compact Riemannian manifold are presented. More recently this result was extended by [6] to dynamical polysystems where the drift vector field was required to be weakly positively Poisson stable. We prove conditions under which the Lie-Poisson reduced dynamics (5) are WPPS. Before we present our results we introduce some definitions and related theorems regarding Poisson stable systems. We follow the development in [6, 7, 8].

Let X be a smooth complete vector field on M and let $\phi_t^X(\cdot)$ denote its flow. A point $p \in M$ is called *positively Poisson stable* for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$, such that $\phi_t^X(p) \in V_p$. The vector field X is called positively Poisson stable if the set of Poisson stable points for X is dense in M. A point $p \in M$ is a nonwandering point of X if for all $T > 0$, any neighborhood V_p of p, there exists a time $t > T$ such that $\phi_t^X(V_p) \bigcap V_p \neq \phi$, where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$. One should observe here that though positive Poisson stability of a vector field is a sufficient condition that the nonwandering set is the entire manifold M, there could exist weaker conditions under which the nonwandering set is M. This gives rise to the definition: a vector field X is called *weakly positively Poisson stable* (WPPS) if its nonwandering set is M. The following theorem on controllability is due to Kuang-Yow Lian et. all. [6]. Earlier versions of this theorem where the hypothesis required f to be only Poisson Stable, are due to Lobry [5], Bonnard and Crouch [9].

Theorem 2 *If the system (6), where U contains $\{u \mid |u_i| \leq K_i \neq 0, i, \cdots, m\}$, is such that f is a weakly positively Poisson stable vector field, then the system is controllable if the accessibility LARC is satisfied.*

In the setting of Lie-Poisson reduced dynamics we can

make the following observation.

Theorem 3 *Let G be a Lie group that acts on itself by left (right) translations. Let $H : T^*G \to \mathbb{R}$ be a left (right) invariant Hamiltonian. Then,*

(i) If G is a compact group, the coadjoint orbits of $\mathcal{G}^* = T^*G/G$ are bounded and the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ defined by $X_{\widetilde{H}}(\mu) = \{\mu, \widetilde{H}\}_{-(+)}$ is WPPS.

(ii) If G is a noncompact group then the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS if there exists a function $V : \mathcal{G}^* \to \mathbb{R}$ such that $V(\mu)$ is bounded below, $V(\mu) \to \infty$ as $\|\mu\| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.

Here $\widetilde{H} = H|_{\mathcal{G}^*}$ is the restriction of H to the quotient manifold $\mathcal{G}^* = T^*G/G$ and $\{\cdot,\cdot\}_{-(+)}$ is the induced minus (plus) Lie-Poisson bracket on the quotient manifold $\mathcal{G}^* = T^*G/G$.

Proof: (i) The projection $\lambda : T^*G \to \mathcal{G}^*_-$ is a Poisson map, and the Poisson manifold $\mathcal{G}^*_-$ is symplectically foliated by coadjoint orbits, i.e. it is a disjoint union of symplectic leaves that are just the coadjoint orbits. Any Hamiltonian system on $\mathcal{G}^*_-$ leaves invariant the symplectic leaves and hence restricts to a canonical Hamiltonian system on a leaf. To study the dynamics of a particular system with initial condition $\mu(0) \in \mathcal{G}^*_-$, we therefore restrict attention to the coadjoint orbit through $\mu(0)$. By hypothesis, each coadjoint orbit is compact. The flow starting at $\mu(0)$ preserves the symplectic volume measure on the orbit. Hence by the Poincaré Recurrence Theorem, we know that for almost every point $p \in \mathcal{G}^*_-$ and any neighborhood V_p of p there exists a time $t > T$ such that $\phi_t^X(p)$ returns to V_p i.e. $X_{\widetilde{H}}$ is WPPS.
(ii) Let $D = \{\mu \mid V(\mu) \leq E\}$ and Orb$(\cdot)$ denote the coadjoint orbit through $\mu(0)$ in $\mathcal{G}^*_-$. Then the integral curve of $X_{\widetilde{H}}$ starting at $\mu(0) \in D$ lies entirely in the set $S = D \cap \text{Orb}(\cdot)$. Since S closed and bounded in $\mathcal{G}^*_-$, it is compact in Orb$(\cdot)$, and hence as before $X_{\widetilde{H}}$ is WPPS. ■

In many situations the function $H_\phi = \widetilde{H} + \phi(C_i)$ where $\widetilde{H}$ is the reduced Hamiltonian and C_i are the Casimirs are a good choice for $V(\cdot)$.

Remark: In our present setting of Lie-Poisson reduced dynamics, WPPS conditions in theorem 2 can be verified whenever the hypotheses of theorem 3 hold. Once WPPS property of the drift vector field has been established, theorem 2 can be used to conclude controllability.

4. Examples

In this section we discuss the controllability of the reduced dynamics of hovercraft, spacecraft and underwater vehicles. These systems satisfy hypotheses of theorem 3.

4.1. Hovercraft - Planar Rigid Body with a Thruster

We identify the position and orientation of the planar rigid body, with a thruster that exerts a force that does not pass through the center of mass (see Fig. 1), with an element of $SE(2)$.[2] Observe that the Lagrangian

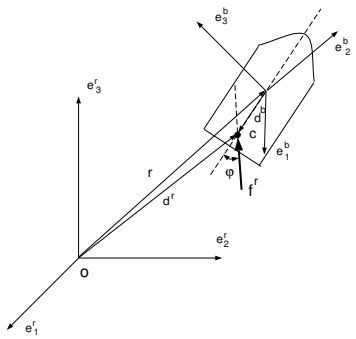

Figure 1: Planar rigid body with thruster.$\{e_i^r, e_2^r, e_3^r\}$ is the inertial frame and $\{e_1^b, e_2^b, e_3^b\}$ is the body fixed frame

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}(I\Omega^2 + m\|\dot{r}\|^2) \tag{7}$$

is invariant under the lifted action of G on $TSE(2)$ and the reduced Hamiltonian defined on $se(2)^*$ is given by

$$\widetilde{H} = \frac{1}{2}(\frac{\Pi^2}{I} + \frac{\|P\|^2}{m}). \tag{8}$$

Here Ω denotes the body angular momentum, I the moment of inertia m the mass, P_1, P_2 are the convected linear momentum and Π is the body angular momentum. Hence Hamiltonian dynamics on $T^*SE(2)$ (which is also left invariant under the action of $SE(2)$) can be reduced to a Hamiltonian system on $se(2)^*$. Assume that the line of action of the force is fixed (i.e. ϕ is fixed) but its direction can be reversed. Let $(P_1, P_2, \Pi) \in se(2)^*$. Projecting the external force appropriately on $se(2)^*$ (see [10] for a complete description of the reduction procedure), written as a control system the reduced dynamics on $se(2)^*$ are given by

$$\dot{P}_1 = P_2\Pi/I + \alpha u \tag{9-a}$$
$$\dot{P}_2 = -P_1\Pi/I + \beta u \tag{9-b}$$
$$\dot{\Pi} = d\beta u \tag{9-c}$$

where $\alpha = \cos\phi, \beta = \sin\phi$ and $u \in [1, -1]$. We refer to (9) as the **Jet-Puck** dynamics.

Proposition 1 *The jet-puck dynamics (9) are controllable if $\beta \neq 0$.*

Proof: We first show that LARC is satisfied. Observe that

$$\begin{aligned}&\det(g, [[f,g],g], [[f,g],[[f,g],g]])\\ &= -4\tfrac{(d\beta)^4}{I^3}(\beta^2 + \alpha^2)\\ &= -4\tfrac{(d\beta)^4}{I^3} \text{ (since } \alpha^2 + \beta^2 = 1)\end{aligned}$$

where $f = (P_2\Pi/I, -P_1\Pi/I, 0)^T$ and $g = (\alpha, \beta, d\beta)^T$. Hence $\dim(\text{span}\mathcal{L}_{\{f,g\}})(p) = 3 \ \forall p \in se(2)^*$ as long as $\beta = \sin\phi \neq 0$, i.e. as long as the line of action of the force

[2] In the rest of the discussion an element of $SE(n)$, $n = 2, 3$ is given by the pair (R, r) where $R \in SO(n)$ and $r \in \mathbb{R}^n$ is a vector from the origin of the inertial frame to the origin of the body frame.

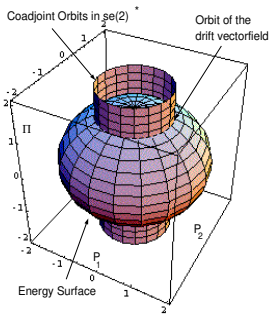

Figure 2: Energy surface and coadjoint orbits in $se(2)^*$

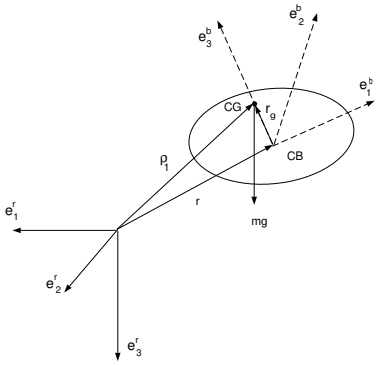

Figure 3: Autonomous Underwater Vehicle. $\{e_1^r, e_2^r, e_3^r\}$ is an inertial frame of reference and $\{e_1^b, e_2^b, e_3^b\}$ is a body frame fixed on the vehicle at its center of buoyancy (CB).

does not pass through the center of mass. The reduced Hamiltonian $\widetilde{H} = \frac{\Pi^2}{2I} + \frac{\|P\|^2}{2m}$ is bounded below, radially unbounded and is such that $\dot{\widetilde{H}} = 0$, along trajectories of (9). Hence it follows from theorem 3 that f is WPPS and hence from theorem 2 we conclude that the jet-puck dynamics are controllable. ■

In fact one observes that every orbit of f is periodic and hence trivially Poisson stable.

Remark: Observe that the coadjoint orbits in $se(2)^*$ are cylinders $\{(P_1, P_2, \Pi) \in \Re^3 \mid P_1^2 + P_2^2 = constant \neq 0\}$. The surfaces defined by $D = \{(P_1, P_2, \Pi) \mid \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{\Pi^2}{2I} = const\}$ are ellipsoids. From theorem 3 the integral curves of the the vector field $\frac{P_2\Pi}{I}\frac{\partial}{\partial P_1} - \frac{P_1\Pi}{I}\frac{\partial}{\partial P_2}$ are restricted to the set $S = D \cap \mathrm{Orb}(\cdot)$, which in this case is simply S^1 (see Fig. (2)).

In [10] we show that the complete (unreduced) system, defined on T^*G is locally strongly accessible. To investigate small time local controllability of the dynamics recall that a control system (1) is said to be *small time locally controllable* (STLC) from $x_0 \in M$ if it is locally accessible from x_0 and x_0 is in the interior of $R^V(x_0, \leq T)$ [3] for all $T > 0$ and each neighborhood V of x_0. If this holds for any $x_0 \in M$ then the system is called STLC.

Proposition 2 *The jet puck dynamics defined (9) are not STLC from the origin.*

Proof: Define $\mathcal{S}^1(X_1, X_2)$ is the linear span of vector fields X_1, X_2, and the brackets $(adX_1)^j X_2$ for $j \geq 1$ and $\widetilde{u}$ as $\{\widetilde{u} \mid f(x_0) + \widetilde{u}g(x_0) = 0\}$. With $\widetilde{u} = 0$ observe that $\mathcal{S}^1(f,g)(0)$ is a one-dimensional space spanned by $\alpha\frac{\partial}{\partial P_1} + \beta\frac{\partial}{\partial P_2} + d\beta\frac{\partial}{\partial \Pi}$ while $[g,[f,g]](0) = -2\frac{d}{I}\beta^2\frac{\partial}{\partial P_1} + 2\frac{d}{I}\beta\alpha\frac{\partial}{\partial P_2}$. Hence $[g,[f,g]](0) \notin \mathcal{S}^1(f,g)(0)$. The result follows from [11], proposition 6.3. ■

Observe that the jet-puck dynamics are similar to those of the reduced dynamics of a rigid body, symmetric about any one principal axis, with one control torque aligned with an axis having a nonzero axis component along all principal axes, allowing us to make similar controllability conclusions (c.f. [12, 13, 14])

4.2. Autonomous Underwater Vehicle

In this section we discuss the controllability of the reduced dynamics for a neutrally buoyant underwater vehicle (UV), subject to control torques and forces, with non-coincident centers of buoyancy and gravity.[4] We assume that the underwater vehicle is submerged in an infinitely large volume of incompressible, inviscid, irrotational fluid which is at rest at infinity (c.f. [15, 16, 17]). We identify the position and orientation of the underwater vehicle with an element of $SE(3)$. Observe that the Lagrangian $L : TSE(3) \to \mathbb{R}$ given by

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}(\Omega^T J\Omega + 2\Omega^T Dv + v^T Mv + 2mgl(i_g . Ri_3))$$

is invariant under the action of the group

$$G = \{(R, r) \in SE(3) \mid R^T i_g = i_g\} = SE(2) \times \mathbb{R}.$$

Ω, and v denote the body angular velocity, and the linear velocity components along the body frame. J is the sum of the body inertia matrix plus the added inertia matrix due to the flow of the fluid. Similarly M is sum of the mass matrix and the added mass matrix associated with the fluid, and D accounts for cross terms. i_g is a unit vector in the direction of gravity in inertial coordinates. The reduced Hamiltonian on $\mathcal{S}^*$ (see below) is

$$\widetilde{H}(\Pi, P, \Gamma) = \frac{1}{2}(\Pi^T A\Pi + 2\Pi^T B^T P + P^T CP - 2mgl(\Gamma . i_3)),$$

where $\Pi = J\Omega + Dv$ and $P = Mv + D^T\Omega$ are angular and linear components of the impulse of the system and Γ is a unit vector (expressed in body coordinates) pointing in the direction of gravity. For details on the constants A, B and C we refer the reader to [17]. Hence the Hamiltonian system on $T^*SE(3)$ (which is also left invariant under the

[3] $R^V(x_0, \leq T) = \bigcup_{0 \leq t \leq T} R^V(x_0, t)$ where $R^V(x_0, T)$ is the reachable set from x_0 at time $T > 0$, following trajectories which remain for $t \leq T$ in the neighborhood V of x_0

[4] Let $r_g = li_3, l > 0$ denote the position of the center of gravity w.r.t the center of buoyancy, where i_3 denotes a unit vector in body coordinates along e_3^b axis. See Fig. 3.

action of $SE(2) \times \mathbb{R})$ can be reduced to a Hamiltonian system on $\mathcal{S}^*$, the dual of the Lie algebra of the semi-direct product $S = SE(3) \times_\rho \mathbb{R}^3$ (see [17]).

The Lie-Poisson reduced dynamics are then given by

$$\dot{\Pi} = \Pi \times \Omega + P \times v - mgl\Gamma \times i_3 \quad (10\text{-a})$$
$$\dot{P} = P \times \Omega \quad (10\text{-b})$$
$$\dot{\Gamma} = \Gamma \times \Omega \quad (10\text{-c})$$

Proposition 3 *The Lie-Poisson reduced Hamiltonian vector field, given by r.h.s of (10), defined on $\mathcal{S}^*$ is WPPS.*

Proof: Choose $V(\Pi, P, \Gamma) = \widetilde{H}(\Pi, P, \Gamma) + \Gamma^T\Gamma$. Observing that V is radially unbounded and that $\dot{V} = 0$ along trajectories of (10) the result follows from theorem 3. ■

Remark : In the case of coincident center of gravity and center of buoyancy the Hamiltonian system on $T^*SE(3)$ is left invariant under the $SE(3)$ action of rotations and translations, and we can derive a set of reduced Lie-Poisson dynamics on $se(3)^*$. The Lie-Poisson reduced equations on $se(3)^*$ are given by

$$\dot{\Pi} = \Pi \times (A\Pi) + P \times CP \quad (11\text{-a})$$
$$\dot{P} = P \times A\Pi \quad (11\text{-b})$$

$A = diag(\frac{1}{I_1}, \frac{1}{I_2}, \frac{1}{I_3})$ and $C = diag(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3})$ Choosing $V(\Pi, P) = \widetilde{H} = \frac{1}{2}(\Pi^T A\Pi + P^T CP)$, the reduced Hamiltonian, we observe that the Lie-Poisson reduced Hamiltonian vector field defined on $se(3)^*$ is WPPS.

Assume that we have three controls u_1, u_2, u_3 such that u_1, u_2 provides pure torques and u_3 provides a pure force. The reduced dynamics with controls for the underwater vehicle with coincident center of buoyancy and center of mass are

$$\dot{\Pi} = \Pi \times (A\Pi) + P \times CP + U_1 \quad (12\text{-a})$$
$$\dot{P} = P \times A\Pi + U_2 \quad (12\text{-b})$$

with $U_1 = (u_1, u_2, 0)^T$ and $U_2 = (u_3, 0, 0)^T$

Proposition 4 *The reduced dynamics (12) of the underwater vehicle with coincident center of buoyancy and center of gravity are controllable if $I_1 \neq I_2$.*

Proof: Observing that

$det(g_1, g_2, g_3, [[f, g_1], g_2], [[f, g_2], g_3], [[[f, g_2], [f, g_3]], g_1])$
$= \frac{I_1 - I_2}{I_1 I_2^2 I_3} \neq 0$, if $I_1 \neq I_2$

where

$$f = (\Pi \times (A\Pi) + P \times CP, P \times A\Pi)^T$$
$$g_1 = (1, 0, 0, 0, 0, 0)^T, g_2 = (0, 1, 0, 0, 0, 0)^T$$
$$g_3 = (0, 0, 0, 1, 0, 0)^T$$

i.e. $\dim(\text{span}\mathcal{L}_{\{f, g_1, g_2, g_3\}})(p) = 6$, $\forall p \in se(3)^*$, and that f is WPPS the result follows from theorem 2. ■

5. Conclusions and Future Work

We present a sufficient condition for controllability of a class of mechanical systems with symmetry. The Lie-Poisson reduced dynamics and controllability of hovercraft, spacecraft and underwater vehicle are discussed. Current research includes the design of feedback laws to stabilize the origin of the reduced system, the use of periodic controls to generate loops in the base space, and thereby steer in the fiber.

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