Controllability of Lie-Poisson Reduced Dynamics *

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Abstract

In this paper we present sufficient conditions for controllability of Lie-Poisson reduced dynamics of a class of mechanical systems with symmetry. We prove conditions (boundedness of coadjoint orbits and existence of a radially unbounded Lyapunov function) under which the drift vector field (of the reduced system) is weakly positively Poisson stable (WPPS). The WPPS nature of the drift vector field along with the Lie algebra rank condition is used to show controllability of the reduced system. We discuss the dynamics, Lie-Poisson reduction, and controllability of hovercraft, spacecraft and underwater vehicles, all treated as rigid bodies.

1 Introduction

A geometric approach to the study of mechanical systems has had a profound influence in recent years on our understanding of dynamics and control aspects. Playing an essential role in this are recent developments in reduction theory (cf. [1]), i.e. the exploitation of invariance of the controlled dynamics to a group of transformations (the symmetry group). The existence of a symmetry group permits dropping of the dynamics to a lower dimensional (reduced) space. Lagrangian reduction [2, 3] involves dropping the Euler-Lagrange equations to the quotient of the velocity phase space given by the symmetry group while Hamiltonian reduction involves projecting the Poisson bracket to the reduced (quotient) space which also inherits a Poisson structure (attributed to Lie

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and Berezin-Kirillov-Kostant-Souriau, see the work of Weinstein for historical remarks [4]). In particular, if the configuration space of the system can be identified with a Lie group G, a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on T^*G/G which is isomorphic to $\mathcal{G}^*$ the dual of the Lie algebra of G. The reduced bracket is now the Lie-Poisson bracket. The complete dynamics can then be reconstructed from the reduced system. The geometric phase [5] associated with a trajectory of the reduced system describes the motion of the complete (lifted) system. There has also been recent progress in the area of control in the presence of nonholonomic constraints [6, 7, 3, 8].

For a large class of mechanical systems, the configuration space can be identified with a Lie group G. Often the dynamics of such systems are G-invariant and hence they can be reduced to obtain a set of Lie-Poisson reduced dynamics on T^*G/G. Examples of such systems include hovercraft, spacecraft and underwater vehicles modeled as rigid bodies. The design and control of autonomous versions of these vehicles has been of much recent interest. For example the amphibious versatility of hovercraft has given them a role in specialized applications including search and rescue, emergency medical services, ice breaking, Arctic off-shore exploration, and recreational activities [9]. Certain environmental aspects (such as ice-roughness, Arctic rubble fields etc.) also provide a niche for operations by hovercraft. Similarly a growing industry in underwater vehicles for deep sea explorations has led to the demand for more versatile, robust and high performance autonomous vehicles that can cope with actuator failures, disturbances, exploit sensor based local navigation etc.

In this paper we discuss the controllability of the Lie-Poisson reduced dynamics of a class of mechanical systems which include as examples hovercraft, spacecraft and bottom heavy underwater vehicles. In each case we identify the configuration space with a Lie group G. The G-invariance of the Hamiltonian and the forcing term (control) is used to obtain a set of Lie-Poisson equations on T^*G/G which is isomorphic to $\mathcal{G}^*$ the dual of the Lie algebra of G. We show that depending on the existence of a radially unbounded Lyapunov type function, the drift vector field (of the reduced system) is weakly positively Poisson stable (WPPS). The WPPS nature of the drift vector field along with the Lie algebra rank condition is used to show controllability of the reduced system.

The paper is organized as follows. In section 2 we present a brief overview of Lie-Poisson reduction. In section 3 we present our main result on controllability of Lie-Poisson reduced dynamics. In section 4 we discuss in some detail, the dynamics, reduction and reduced space controllability of the hovercraft, the spacecraft and the underwater vehicle. Conclusions and future work is discussed in section 5.

2 Lie-Poisson Reduction

Recall (cf. [1, 10]) that if G is a symmetry group acting on a Poisson manifold M, then the quotient manifold M/G inherits a Poisson structure so that whenever $\widetilde{P}, \widetilde{Q} : M/G \to \mathbb{R}$

correspond to G invariant functions, $P, Q : M \to \mathbb{R}$, their Poisson bracket $\{\widetilde{P}, \widetilde{Q}\}_{M/G}$ corresponds to the G-invariant function $\{P, Q\}_M$. If $H : M \to \mathbb{R}$ is a G-invariant Hamiltonian, then H descends to $\widetilde{H} : M/G \to \mathbb{R}$ and determines the reduced dynamics on M/G. The solutions of the reduced Hamiltonian system on M/G are projections of the solutions of the complete system defined on M. In particular if $M \stackrel{\triangle}{=} T^*G$ and G acts on itself by left translations then $M/G \equiv \mathcal{G}^*$, the dual of the Lie algebra of G, a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on $\mathcal{G}^*_-$ (the space $\mathcal{G}^*$ associated with the minus Poisson structure). The reduced bracket is now the minus Lie-Poisson bracket, $\{\cdot, \cdot\}_-$ defined in its coordinate free form

$$\{P, Q\}_-(x) = - < x, [\nabla P(x), \nabla Q(x)] > .$$

Let $\{X_1, \cdots, X_r\}$ and $\{X_1^b, \cdots, X_r^b\}$ be a basis for the Lie algebra $\mathcal{G}$ and the dual basis $\mathcal{G}^*$ respectively, i.e. $< X_i^b, X_j >= \delta_j^i$. Any $\mu \in \mathcal{G}^*_-$ can be expressed as $\mu = \sum_{i=1}^r \mu_i X_i^b$ and the Lie-Poisson bracket of two differentiable functions $F, H \in C^\infty(\mathcal{G}^*)$ is given by

$$\{F, H\}_-(\mu) = - \sum_{i,j,k=1}^{r} c_{ij}^k \mu_k \frac{\partial F}{\partial \mu_i} \frac{\partial H}{\partial \mu_j} \tag{1}$$

where $c_{ij}^k, i, j, k = 1, \cdots, r$ are the structure constants of $\mathcal{G}$ relative to the basis $\{X_1, \cdots, X_r\}$. Equivalently (1) can be written as

$$\{F, H\}_-(\mu) = \nabla F^T \Lambda(\mu) \nabla H \tag{2}$$

where

$$[\Lambda(\mu)]_{ij} = - \sum_{k=1}^{r} c_{ij}^k \mu_k$$

The rank of Poisson tensor Λ determines the nontrivial Casimirs of μ. The Lie-Poisson reduced dynamics can now be expressed

$$\dot{\mu_i} = \{\mu_i, \widetilde{H}\}_-, \quad i = 1, \cdots, r \tag{3}$$

where $\widetilde{H}$ is the reduced Hamiltonian.

In the rest of this paper we will denote the Lie-Poisson reduced dynamics with the following notation.

$$\dot{\mu} = \{\mu, \widetilde{H}\}_- \tag{4}$$

The ith component on the right hand side being $\{\mu_i, , \widetilde{H}\}_-$, $i = 1, \cdots, r$.

The induced symplectic foliation by Lie-Poisson bracket on $\mathcal{G}^*$ has a particularly nice interpretation in terms of the dual to the adjoint representation of the underlying Lie group G on the Lie algebra $\mathcal{G}$. This is given by the following theorem, which appears to be due to Kirillov[11, 12], Arnold [13], Kostant [14] and Souriau [15], though similar ideas first appear in the work of Lie, Borel and Weil.

Theorem 1 *Let G be a connected Lie group with coadjoint representation Ad^*G on $\mathcal{G}^*$. Then the orbits of Ad^*G are immersed submanifolds of $\mathcal{G}^*$ and are precisely the leaves of the symplectic foliation induced by the Lie-Poisson bracket on $\mathcal{G}^*$. Moreover, for each $g \in G$, the coadjoint map Ad^*G is a Poisson mapping on $\mathcal{G}^*$ preserving the leaves of the foliation.*

As flows of (4) remain on coadjoint orbits on which they started, the geometry of coadjoint orbits plays an important role in understanding the dynamics of the Lie-Poisson reduced equations.

3 Poisson Stability and Controllability

The state space of a large class of mechanical systems such as hovercraft, spacecraft underwater vehicle etc. can be identified with a Lie group G. The Hamiltonian dynamics (defined on T^*G) of these systems subject to external forces can be written in the form of a control system as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{5}$$

where $x \in T^*G$, $f(x) = \{x, H\}$ and $u = (u_1, \cdots u_m)$. (H is the Hamiltonian defined on T^*G). Here we do not assume that g_i are Hamiltonian vector fields. Often we observe that the vector fields f and g_i's are G-invariant. This allows us to drop the the vector fields f and g_i from T^*G to T^*G/G and the reduced dynamics take the form

$$\dot{\mu} = \widetilde{f}(\mu) + \sum_{i=1}^{m} \widetilde{g}_i(\mu)\widetilde{u_i} \tag{6}$$

where $\mu \in T^*G/G$, $\widetilde{f}$ and $\widetilde{g}_i$ are the projections of f and g on T^*G/G. From the discussion in section 2 we know that $\widetilde{f} = \{\mu, \widetilde{H}\}_-$ where $\widetilde{H}$ is the reduced Hamiltonian. Hence (6) can be written as

$$\dot{\mu} = \{\mu, \widetilde{H}\}_- + \sum_{i=1}^{m} \widetilde{g}_i(\mu)\widetilde{u_i} \tag{7}$$

Studying controllability of systems of the form (7) or of more general systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in \mathbb{R}^n, \quad u = (u_1, \cdots u_m) \in U \tag{8}$$

is usually a hard problem. We know that if a system of the form (8) satisfies the Lie algebra rank condition (LARC) then it is locally accessible, and in addition if $f = 0$ then LARC implies that the system is controllable. (See appendix for details.) While the kinematic equations of motion can often be written as a drift free system, once dynamics are included LARC does not imply controllability. Proving controllability is usually much harder than proving accessibility. In [16] sufficient conditions are given, in terms of a "group action", that a locally accessible system is also locally reachable. In [17] sufficient conditions for the controllability of a conservative dynamical polysystem on a compact Riemannian manifold are presented. More recently this result was extended by [18] to dynamical polysystems where the drift vector field was required to be weakly positively Poisson stable. We extend this result to Lie-Poisson reduced dynamics. We prove conditions under which the reduced dynamics are controllable. Before we present our results we introduce some definitions and related theorems regarding Poisson stable systems. We follow the development in [18, 19, 20, 21, 22, 23].

Let X be a smooth complete vector field on M and let $\phi_t^X(\cdot)$ denote its flow.

Definition: A point $p \in M$ is called *positively Poisson stable* for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$, such that $\phi_t^X(p) \in V_p$. The vector field X is called positively Poisson stable if the set of Poisson stable points for X is dense in M.

Definition: A point $p \in M$ is called nonwandering point of X if for all $T > 0$, any neighborhood V_p of p, there exists a time $t > T$ such that $\phi_t^X(V_p) \bigcap V_p \neq \phi$, where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$.

One should observe here that though the positive Poisson stability is a sufficient condition that the nonwandering set of a positively Poisson stable vector field is the entire manifold M, there could exist weaker conditions under which the nonwandering set is M. This gives rise to the definition of a weakly positively Poisson stable (WPPS) vector field.

Definition: The vector field X is called weakly positively Poisson stable if its nonwandering set is M.

The following theorem on controllability is due to Kuang-Yow Lian et. all. [18]. Earlier versions of this theorem and the corollary that follows, where the hypothesis required f to be only Poisson Stable, are due to Lobry [17], Bonnard and Crouch [24].

Theorem 2 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \;\; u = (u_1, \cdots u_m) \in U \subset \mathbb{R}^m$$

where U contains $\{u \mid |u_i| \leq M_i \neq 0, i, \cdots, m\}$ is such that f is a weakly positively Poisson stable vector field, then the system is controllable if the accessibility LARC is satisfied.

Corollary 1 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \;\; u = (u_1, \cdots u_m) \in U$$

is such that f is a weakly positively Poisson stable vector field, and accessibility LARC is satisfied, then the system with controls constrained by $u_i \in \{-M_i, M_i\}, M_i > 0, \; i = 1, \cdots, m$ is controllable.

In the setting of Lie Poisson reduced dynamics we can make the following observation.

Theorem 3 *Let G be a Lie group that acts on itself by left (right) translations. Let $H : T^*G \to \mathbb{R}$ be a left (right) invariant Hamiltonian. Then,*

(i) If G is a compact group, the coadjoint orbits of $\mathcal{G}^* = T^*G/G$ are bounded and the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ defined by $X_{\widetilde{H}}(\mu) = \{\mu, \widetilde{H}\}_{-(+)}$ is WPPS.

(ii) If G is a noncompact group then the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS if there exists a function $V : \mathcal{G}^* \to \mathbb{R}$ such that $V(\mu)$ is bounded below, $V(\mu) \to \infty$ as $\|\mu\| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.

Here $\widetilde{H} = H|_{\mathcal{G}^*}$ is the restriction of H to the quotient manifold $\mathcal{G}^* = T^*G/G$ and $\{\cdot, \cdot\}_{-(+)}$ is the induced minus (plus) Lie-Poisson bracket on the quotient manifold $\mathcal{G}^* = T^*G/G$.

Proof: (i) The projection $\lambda : T^*G \to \mathcal{G}^*_-$ is a Poisson map, and the Poisson manifold $\mathcal{G}^*_-$ is symplectically foliated by coadjoint orbits, i.e. it is a disjoint union of symplectic leaves that are just the coadjoint orbits. Any Hamiltonian system on $\mathcal{G}^*_-$ leaves invariant the symplectic leaves and hence restricts to a canonical Hamiltonian system on a leaf. To study the dynamics of a particular system with initial condition $\mu(0) \in \mathcal{G}^*_-$, we therefore restrict attention to the coadjoint orbit through $\mu(0)$. By hypothesis, each coadjoint orbit is compact. The flow starting at $\mu(0)$ preserves the symplectic volume measure on the orbit. Hence by the Poincaré Recurrence Theorem, we know that for almost every point $p \in \mathcal{G}^*_-$ and any neighborhood V_p of p there exists a time $t > T$ such that $\phi_t^X(p)$ returns to V_p i.e. $X_{\widetilde{H}}$ is WPPS.

(ii)Let $D = \{\mu \mid V(\mu) \leq E\}$ and Orb$(\cdot)$ denote the coadjoint orbit through $\mu(0)$ in $\mathcal{G}_-^*$. Then the integral curve of $X_{\widetilde{H}}$ starting at $\mu(0)i \in D$ lies entirely in the set $S = D \cap \text{Orb}(\cdot)$. Since S closed and bounded in $\mathcal{G}_-^*$, it is compact in Orb$(\cdot)$, and hence as before $X_{\widetilde{H}}$ is WPPS. $\square$

In many situations the function $H_\phi = \widetilde{H} + \phi(C_i)$ where $\widetilde{H}$ is the reduced Hamiltonian and C_i are the Casimirs are a good choice for $V(\cdot)$.

Remark: In our present setting of Lie-Poisson reduced dynamics, WPPS conditions in theorem 2 can be verified whenever the hypotheses of theorem 3 hold. Once WPPS of the drift vector field has been established theorem 2 can be used to conclude controllability.

4 Examples

In this section we discuss the controllability of the Lie-Poisson reduced dynamics of hovercraft, spacecraft and the underwater vehicles. These systems satisfy conditions of theorem 3. The kinematics and dynamics of these examples can also be found in [1, 25, 26] for completeness we present necessary details here.

4.1 Hovercraft - Planar Rigid Body with a Thruster

In this section we discuss the dynamics of a planar rigid body with a thruster. The configuration of the system is shown in Figure 1. Let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference fixed at **O** and $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the rigid body $\mathcal{B}$ at its center of mass. Since the rigid body is restricted to move in the $e_1^r e_2^r$ plane, a typical material point q^b in the rigid body is then represented in the inertial frame as $q^r = Rq^b + r$ where R is an element of $SO(2)$, the special orthogonal group of 2×2 matrices and $r = (x, y)$ is a vector from **O** to the center of mass of $\mathcal{B}$. Hence at any instant, the configuration $X(t)$ of $\mathcal{B}$ can be uniquely identified by the pair (R, r) or equivalently as an element of $SE(2)$, the Special Euclidean group of 2×2 matrices. Recall

$$SE(2) \stackrel{\triangle}{=} \{\begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \mid R \in SO(2), r \in \mathbb{R}^2\}$$

Let us assume that the thruster is mounted at the point C defined by the vector d^b in body coordinates and d^r in the inertial frame of reference. The thrusters exerts a force f^r in inertial coordinates such that the line of action of the force passes through C and makes an angle ϕ with the vector d^b. We now derive the equations of motion of a rigid body subject to a force f^r along a specified line of action.

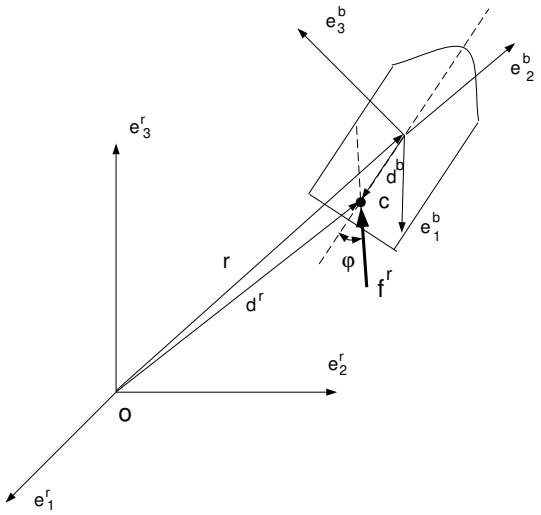

Figure 1: Planar rigid body with thruster

4.1.1 Symmetry and Reduction

We assume for now that the rigid body (which will later be approximated to a hovercraft) has sufficient lift and can glide on the surface with no friction. The Lagrangian $L : TSE(2) \to \mathbb{R}$ for this case is simply the kinetic energy, i.e.

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2} I \Omega^2 + \frac{m}{2} \|\dot{r}\|^2 \tag{9}$$

where m is the total mass, I is the moment of inertia of $\mathcal{B}$ in the body frame, Ω is the scalar body angular velocity about the center of mass. The corresponding Hamiltonian is given by

$$H = \frac{1}{2I} \Pi^2 + \frac{\|p\|^2}{2m} \tag{10}$$

where $\Pi = I\Omega$ is the body angular momentum and $p = m\dot{r}$ is the spatial linear momentum.

Collecting together the Newton-Euler balance laws one can write the dynamics in spatial (inertial) variables (R, r, π, p) as

$$\dot{R} = R\widehat{\Omega} \tag{11-a}$$
$$\dot{r} = p/m \tag{11-b}$$
$$\dot{\pi} = d^r \times f^r \tag{11-c}$$
$$\dot{p} = f^r \tag{11-d}$$

where

$$\widehat{\Omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Omega.$$

We observe that the lifted action of G on $TSE(2)$ defined by

$$\begin{aligned} &\phi_{g^*} : TSE(2) \to TSE(2) \\ &\quad (R, r, \dot{R}, \dot{r}) \mapsto (\bar{R}R, \bar{R}r + \bar{r}, \bar{R}\dot{R}, \bar{R}\dot{r}) \end{aligned}$$

leaves the Lagrangian (9) invariant. Hence the Hamiltonian is also G-invariant. We can now induce a Hamiltonian on the quotient space, $T^*SE(2)/SE(2)$, and express the dynamics in terms of the appropriate reduced variables. The quotient space $T^*SE(2)/SE(2)$ is isomorphic to $se(2)^*$, the dual of the Lie algebra of $SE(2)$ and the reduced variables are:

$$\begin{aligned} &\Pi, \text{ the body angular momentum,} \\ &P = R^T p, \text{the convected linear momentum.} \end{aligned}$$

The reduced Hamiltonian $\widetilde{H}$ is given by

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m} \tag{12}$$

Choosing

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$
$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as a basis for $se(2)$ we have the commutation relations: $[X_1, X_2] = 0$, $[X_1, X_3] = -X_2$ and $[X_2, X_3] = X_1$. The Lie-Poisson bracket of two differentiable functions G, H on $se(2)^*$ is then given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H \tag{13}$$

where $\mu = (P_1, P_2, \Pi) \in se(2)^*$ and

$$\Lambda = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

The reduced equations take the form

$$\dot{\mu} = \{\mu, \widetilde{H}\}_- + f_{ext} \tag{14}$$

where f_{ext} is the external force projected appropriately. In the present setting $f_{ext} = (|F|\cos\phi, |F|\sin\phi, |F||d|\sin\phi)^T$, where $F = R^T f^r$ and $|\cdot|$ denotes the norm (see [25] for more details).

Depending on the control authority we distinguish two versions of the problem.

Case 1: The Jet-Puck Problem: Here we assume that the line of action of the force is fixed (i.e. ϕ is fixed) but its direction can be reversed. Written as a control system, equations (13) take the form

$$\begin{aligned}
\dot{P}_1 &= P_2\Pi/I + \alpha u && \text{(15-a)}\\
\dot{P}_2 &= -P_1\Pi/I + \beta u && \text{(15-b)}\\
\dot{\Pi} &= d\beta u && \text{(15-c)}
\end{aligned}$$

where $\alpha = \cos\phi, \beta = \sin\phi$ and $u \in [1, -1]$.

Case 2: The Hovercraft Problem: Here we assume that we now have control over both the magnitude of the thrust and ϕ. The equations now take the form

$$\begin{aligned}
\dot{P}_1 &= P_2\Pi/I + u_1\cos(u_2) && \text{(16-a)}\\
\dot{P}_2 &= -P_1\Pi/I + u_1\sin(u_2) && \text{(16-b)}\\
\dot{\Pi} &= du_1\sin(u_2) && \text{(16-c)}
\end{aligned}$$

where $u_1 \in [-1, 1]$ and $u_2 \in [\phi_{min}, \phi_{max}]$

4.1.2 Controllability of the Reduced Dynamics

Proposition 1: The jet-puck dynamics defined by (9) are controllable if $\sin\phi \neq 0$.

Proof: We first show that LARC is satisfied. To show that

$$\dim(\text{span}\mathcal{L}_{\{f,g\}})(p) = 3, \quad \forall p \in se(2)^*$$

where $f = (P_2\Pi/I, -P_1\Pi/I, 0)^T$ and $g = (\alpha, \beta, d\beta)^T$, observe that

$$\begin{aligned}
&\det(g, [[f,g],g], [[f,g],[[f,g],g]]) \\
&= det \begin{bmatrix} \alpha & 2\frac{d}{I}\beta^2 & -2\frac{d^2}{I^2}\beta^2\alpha \\ \beta & -2\frac{d}{I}\beta\alpha & -2\frac{d^2}{I^2}\beta^3 \\ d\beta & 0 & 0 \end{bmatrix} \\
&= -4\frac{(d\beta)^4}{I^3}(\beta^2+\alpha^2) \\
&= -4\frac{(d\beta)^4}{I^3} \ \text{(since } \alpha^2+\beta^2 = 1)
\end{aligned}$$

Hence $\dim(\text{span}\mathcal{L}_{\{f,g\}})(p) = 3 \ \forall p \in se(2)^*$ as long as $\beta = \sin\phi \neq 0$, i.e. as long as the line of action of F does not pass through the center of mass.

We observe that the reduced Hamiltonian

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m} \tag{17}$$

is bounded below, radially unbounded and is such that $\dot{\widetilde{H}} = 0$. Hence it follows from theorem 3 that f is WPPS and hence from theorem 2 we conclude that the jet-puck dynamics are controllable. □

In fact one observes that every orbit of f is periodic and hence trivially Poisson stable.

Remark: Observe that the coadjoint orbits in $se(2)^*$ are cylinders

$$\{(P_1, P_2, \Pi) \in \Re^3 \mid P_1^2 + P_2^2 = constant \neq 0\}.$$

The surfaces defined by $D = \{(P_1, P_2, \Pi) \mid \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{\Pi^2}{2I} = const\}$ are ellipsoids. From theorem 3 the integral curves of the the vector field $\frac{P_2\Pi}{I}\frac{\partial}{\partial P_1} - \frac{P_1\Pi}{I}\frac{\partial}{\partial P_2}$ are restricted to the set $S = D \cap \text{Orb}(\cdot)$, which in this case is simply S^1 (see Fig. (2)).

Proposition 2: The hovercraft dynamics defined by (16) are controllable.

Proof: In (16) setting $u_2 = k$, where k is some constant not equal to zero, the equations reduce to those of the jet-puck and hence from Proposition 1 the dynamics are controllable. □

4.1.3 Cotangent Space Controllability

We now investigate the controllability in the cotangent space. We show that the complete (unreduced) system is locally strongly accessible. The complete set of equations for the

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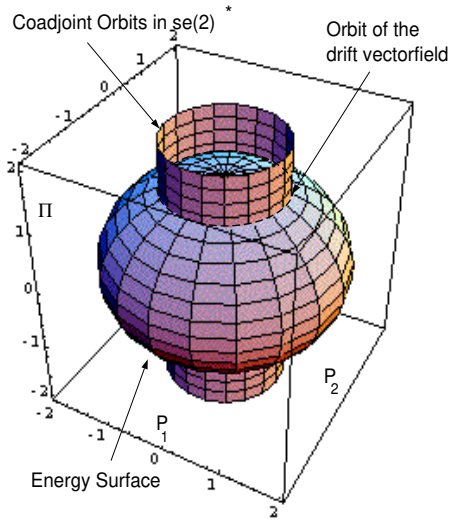

Figure 2: Energy surface and coadjoint orbits in $se(2)^*$

jet-puck take the form

$$\dot{x} = p_1/m \quad (18\text{-a})$$
$$\dot{y} = p_2/m \quad (18\text{-b})$$
$$\dot{\theta} = \Pi/I \quad (18\text{-c})$$
$$\dot{p_1} = (\cos(\theta+\phi))u \quad (18\text{-d})$$
$$\dot{p_2} = (\sin(\theta+\phi)u \quad (18\text{-e})$$
$$\dot{\Pi} = (d\sin\phi)u \quad (18\text{-f})$$

Proposition 3 : For the system defined by (18) the cotangent space $T^*SE(2)$ is locally strongly accessible.

Proof: To show that

$$\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 6,\ \forall p \in T^*SE(2)$$

where $f = (p_1/m, p_2/m, \Pi/I, 0, 0, 0)^T$,
and $g = (0, 0, 0, \cos(\theta+\phi), \sin(\theta+\phi), d\sin\phi)^T$, observe that

$$\begin{aligned} &det(g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \\ &= 16K(\cos^2(\theta+\phi) + \sin^2(\theta+\phi))^2 \\ &= 16K \end{aligned}$$

where

$$\xi_1 = [f,g],\ \xi_2 = [[f,g],g],\ \xi_3 = [[[f,g],f],g],$$
$$\xi_4 = [[[[f,g],f],g],[f,g]],$$
$$\xi_5 = [[[[f,g],f],g],[[f,g],g]] \text{ and}$$
$$K = (\tfrac{1}{m})^2(\tfrac{1}{I})^7(d\sin\phi)^8$$

Hence again if $\sin\phi \neq 0$, $\dim(\text{span}\mathcal{L}_{\{f,g\}})(p) = 6, \quad \forall p \in T^*SE(2)$. Also $[f,X] \in \text{span}(g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5), \forall X \in \{g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5\}$. Hence the complete system is locally strongly accessible. □

4.1.4 Small-Time local controllability

Recall that a control system (8) is said to be *small time locally controllable* (STLC) from $x_0 \in M$ if it is locally accessible from x_0 and x_0 is in the interior of $R^V(x_0, \leq T)$ [1] for all $T > 0$ and each neighborhood V of x_0. If this holds for any $x_0 \in M$ then the system is called STLC.

In [27] Sussmann proves the following sufficient condition for scalar input systems.

Theorem 4 *Consider an analytic system*

$$\dot{x} = f_0(x) + f_1(x)u \quad |u(t)| \leq A \tag{19}$$

and a point x_0 such that

$$[f_1,[f_0,f_1]](x_o) \notin \mathcal{S}^1(f_0 + \widetilde{u}f_1, f_1)(x_0)$$

where $\mathcal{S}^1(X_1,X_2)$ is the linear span of X_1, X_2, and the brackets $(adX_1)^jX_2$ for $j \geq 1$ and $\{\widetilde{u} \mid f_0(x_0) + \widetilde{u}f_1(x_0) = 0\}$. Then (19) is not STLC from x_0.

We use the above result to show that the jet-puck dynamics are not STLC

Proposition 4 The jet puck dynamics defined (18) are not STLC from the origin.

Proof: It is sufficient to consider STLC of the reduced dynamics. With $\widetilde{u} = 0$ observe that $\mathcal{S}^1(f,g)(0)$ is a one-dimensional space spanned by $\alpha\frac{\partial}{\partial P_1} + \beta\frac{\partial}{\partial P_2} + d\beta\frac{\partial}{\partial \Pi}$ while $[g,[f,g]](0) = -2\frac{d}{I}\beta^2\frac{\partial}{\partial P_1} + 2\frac{d}{I}\beta\alpha\frac{\partial}{\partial P_2}$. Hence $[g,[f,g]](0) \notin \mathcal{S}^1(f,g)(0)$ □

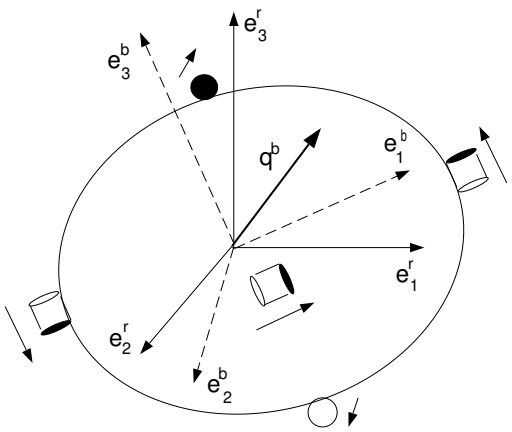

Figure 3: Spacecraft with gas jets

4.2 Attitude control of spacecraft

We now discuss the dynamics describing spacecraft attitude control with gas jet actuators. Let $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the rigid body (spacecraft) $\mathcal{B}$ at its center of mass and let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference with origin coincident with the origin of the body fixed frame (see Fig 3). A typical material point q^b in the rigid body is then represented in the inertial frame as $q^r = Rq^b$ where R is an element of $SO(3)$, the special orthogonal group of 3×3 matrices. Hence the configuration space of the rigid body may be identified with $SO(3)$, the velocity space is with the tangent bundle $TSO(3)$ and the momentum phase space with the cotangent bundle $T^*SO(3)$. Let $b_1, \cdots b_m$ be the axis about which the corresponding control torque of magnitude $\|b_i\|u_i$ is applied by means of opposing pairs of gas jets. The dynamic equations for the controlled space craft are then given by

$$\dot{R} = R\widehat{\Omega} \tag{20-a}$$

$$I\dot{\Omega} = I\Omega \times \Omega + \sum_{i=1}^{m} b_i u_i \tag{20-b}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity, $\widehat{\Omega}$ is a 3×3 skew symmetric matrices given by

$$\widehat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

[1] $R^V(x_0, \leq T) = \bigcup_{0 \leq t \leq T} R^V(x_0, t)$ where $R^V(x_0, T)$ is the reachable set from x_0 at time $T > 0$, following trajectories which remain for $t \leq T$ in the neighborhood V of x_0

and $I = \text{diag}(I_1, I_2, I_3)$ is the inertia matrix. In the rest of the discussion $\hat{}$ defines a map $\hat{}: \mathbb{R}^3 \to so(3)$, such that $\widehat{\alpha}\beta = \alpha \times \beta,\ \alpha, \beta \in \mathbb{R}^3$. Thus

$$\widehat{\alpha} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$$

4.2.1 Symmetry and Reduction

The Lagrangian $L : TSO(3) \to \mathbb{R}$ is again simply the kinetic energy and is given by

$$L(R, \dot{R}) = \frac{1}{2} < \Omega, I\Omega >$$

and the corresponding Hamiltonian $H : T^*SO(3) \to \mathbb{R}$ is given by

$$\frac{1}{2} < \Pi, I^{-1}\Pi >$$

where $\Pi = I\Omega$ is the body angular momentum. Observe that the lifted action of $g = \bar{R} \in SO(3)$ on $TSO(3)$ defined as

$$\begin{aligned} \phi_{g^*} &: TSO(3) \to TSO(3) \\ &(R, \dot{R}) \mapsto (\bar{R}R, \bar{R}R\widehat{\Omega}) \end{aligned}$$

the lifted action of $g = \bar{R} \in SO(3)$ on $TSO(3)$ defined as leaves the Lagrangian (and hence also the Hamiltonian) invariant. Hence one can induce a Hamiltonian on the quotient space, $T^*SO(3)/SO(3)$, and express the dynamics in terms of the appropriate reduced variables. The quotient space $T^*SO(3)/SO(3)$ is isomorphic to $so(3)^*$, the dual of the Lie algebra of $SO(3)$ and the reduced variables are $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ the body angular momentum. Choosing

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and}$$
$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as a basis of $so(3)$ and with the commutation relations $[X_1, X_2] = X_3$, $[X_3, X_1] = X_2$ and $[X_3, X_2] = -X_1$, the Lie-Poisson bracket of two differentiable functions G, H on $so(3)^*$ is given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H \tag{21}$$

where $\mu = (\Pi_1, \Pi_2, \Pi_3) \in so(3)^*$ and

$$\Lambda = \begin{bmatrix} 0 & -\Pi_3 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_1 \\ -\Pi_2 & \Pi_1 & 0 \end{bmatrix}.$$

The reduced equations take the form

$$\dot{\Pi} = f(\Pi) + + \sum_{i=1}^{m} \widetilde{b}_i u_i \tag{22}$$

where $f(\Pi) = (\frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3, \frac{I_3 - I_1}{I_1 I_3}\Pi_2\Pi_3, \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2)^T$, $\widetilde{b_i} = R^T b_i$. Assuming that we have only one control and $I_1 = I_2$ (22) can be written as

$$\dot{\Pi_1} = \frac{(I_1 - I_3)}{I_1 I_3}\Pi_2\Pi_3 + \alpha u \tag{23-a}$$

$$\dot{\Pi_2} = -\frac{(I_1 - I_3)}{I_1 I_3}\Pi_1\Pi_3 + \beta u \tag{23-b}$$

$$\dot{\Pi_3} = \gamma u \tag{23-c}$$

4.2.2 Controllability of reduced dynamics

Observe the similar structure of base space equations for the jet-puck and those of the controlled Euler equations. Hence we can make similar claims regarding controllability and STLC (see also [28, 29]). Proofs are omitted as they are similar to those of the jet-puck dynamics.

Proposition 5 The spacecraft dynamics defined by (23) are controllable if $\alpha^2 + \beta^2 \neq 0$ and $\gamma \neq 0$

Proposition 6 The spacecraft dynamics (23) are not STLC from the origin.

Remark: The coadjoint orbits in $so(3)^*$ are spheres (see Fig 4)

$$\{(\Pi_1, \Pi_2, \Pi_3) \in \mathbb{R}^3 \mid \Pi_1^2 + \Pi_2^2 + \Pi_3^2 = const\}.$$

In this case since the coadjoint orbits are compact manifolds one can conclude from theorem 3 that the drift vector field is WPPS. Fig (4) shows the intersection of the coadjoint orbits and the energy surface.

b

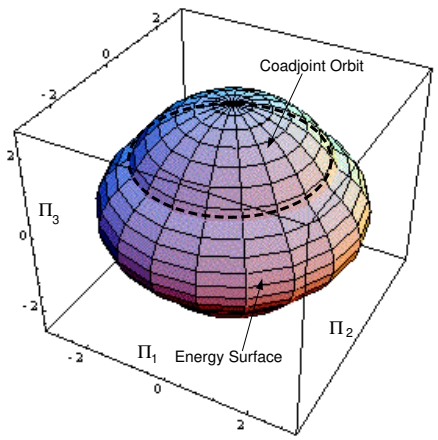

Figure 4: Energy surface and coadjoint orbits in $se(2)^*$

4.3 Autonomous Underwater Vehicle

In this section we discuss the reduced space controllability for a neutrally buoyant underwater vehicle (UV). We distinguish between the cases of coincident and noncoincident centers of buoyancy and gravity. The Lie-Poisson dynamics for these cases have been derived in [30, 31, 26]. We only present a brief overview of the Lie-Poisson dynamics and then use the results of theorem 2 and theorem 3 to make necessary conclusions about controllability.

Let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference (see fig 5) fixed at **O** and $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the vehicle at its center of buoyancy (CB). A material point q^b in the UV is then represented in the inertial frame as $q^r = Rq^b + r$ where R is an element of $SO(3)$, the special orthogonal group of 3×3 matrices and $r = (x, y, z)$ is a vector from **O** to the center of buoyancy (CB). Hence at any instant, the configuration $X(t)$ of the UV can be uniquely identified by the pair (R, r) or equivalently as an element of $SE(3)$, the Special Euclidean group of 3×3 matrices. Recall

$$SE(3) \triangleq \{\begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \mid R \in SO(3), r \in \mathbb{R}^3\}$$

While deriving the dynamics we assume that the UV is submerged in an infinitely large mass of incompressible, inviscid fluid. Further, we assume that the flow is irrotational (the motion of the fluid is entirely due to that of the UV). Under these assumptions the motion of the fluid can be characterized by the existence of a single valued potential ϕ which satisfies

$$\nabla^2 \phi = 0$$

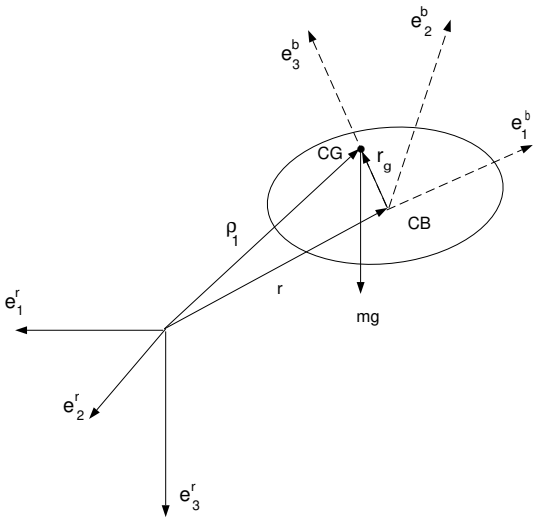

Figure 5: Autonomous Underwater Vehicle

$$\begin{aligned} \nabla\phi &= 0 \text{ at infinity} \\ -\frac{\partial\phi}{\partial n} &= n.(v + \Omega \times r^b) \quad \text{at body surface,} \end{aligned}$$

where r^b is a vector from the CB to the vehicle's surface, n is the unit outward normal vector of the vehicle, $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ are the body angular velocities, and $v = (v_1, v_2, v_3)^T$ are the linear velocity components along the body-fixed frame. Under these assumptions Kirchhoff showed that

$$\phi = v_1\phi_1 + v_2\phi_2 + v_3\phi_3 + \Omega_1\chi_1 + \Omega_2\chi_2 + \Omega_3\chi_3 \tag{24}$$

where $\phi_1, \phi_2, \phi_3, \chi_1, \chi_2, \chi_3$ are functions of x, y, z determined by the configuration of the surface of the solid. Using the form of ϕ as expressed in (23) the kinetic energy of the fluid

$$T_f = \frac{1}{2}\rho_0 \int\int\int ((\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial s})^2)dxdydz,$$

where ρ_0 is the fluid density, can be expressed as a quadratic form as

$$T_f = \frac{1}{2}W^T\Theta W, \;\; \Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}.$$

$W = (v^T, \Omega^t)^T$, Θ_{11} is referred to as the added mass matrix, Θ_{22} as the added inertia matrix, Θ_{12} and Θ_{21} account for cross terms (c.f. [30, 31]).

Assume that the center of gravity (CG) does not coincide with the center of buoyancy (CB) and lies on the e_3^b axis at a distance $l > 0$ (bottom heavy) from the CB i.e $r_g = li_3$ where i_3 denotes a unit vector (in body coordinates) along the e_3^b axis. Also let i_g denote a unit vector (in inertial coordinates) in the direction of gravity, i.e. along the e_3^r axis. Let m be the mass of the vehicle, J_b the inertia matrix for the vehicle. The Lagrangian $L : TSE(3) \to \mathbb{R}$ is then given by

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}(\Omega^T J\Omega + 2\Omega^T Dv + v^T Mv + 2mgl(i_g.Ri_3))$$

where $J = J_b + \Theta_{11}$, $D = ml\widehat{r_g} + \Theta_{12}$ and $M = mI + \Theta_{22}$ (I is the 3×3 identity matrix). In the rest of the discussion the UV is approximated as an ellipsoid and hence $\Theta_{12} = \Theta_{21} = 0$.

Observe that the Lagrangian is invariant under the action of the group

$$G = \{(R, r) \in SE(3) \mid R^T i_g = i_g\} = SE(2) \times \mathbb{R}.$$

and hence the Hamiltonian system on $T^*SE(3)$ (which is also left invariant under the action of $SE(2) \times \mathbb{R}$) can be reduced to a Hamiltonian system on $\mathcal{S}^*$, the dual of the Lie algebra of the semi-direct product $S = SE(3) \times_\rho \mathbb{R}^3$ (see [26] for details). The reduced Hamiltonian on $\mathcal{S}^*$ is

$$\widetilde{H}(\Pi, P, \Gamma) = \frac{1}{2}(\Pi^T A\Pi + 2\Pi^T B^T P + P^T CP - 2mgl(\Gamma.i_3)),$$

where

$$A = (J - DM^{-1}D^T)^{-1}, \quad B = -CD^T J^{-1}, \quad C = (M - D^T J^{-1} D)^{-1},$$
$$\Pi = J\Omega + Dv, \quad P = Mv + D^T\Omega, \quad \text{and } \Gamma = R^T i_g.$$

Choosing

$$B_i = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, \cdots, 6, \quad B_i = \begin{pmatrix} 0 & e_{i-6} \\ 0 & 0 \end{pmatrix}, \quad i = 7, 8, 9.$$

where

$$A_i = \begin{pmatrix} \widehat{e}_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3 \quad A_i = \begin{pmatrix} 0 & \widehat{e}_i \\ 0 & 0 \end{pmatrix}, \quad i = 4, 5, 6$$

as a basis for $\mathcal{S}$ the Lie algebra of S the Lie-Poisson bracket of two differentiable functions G, H on $\mathcal{S}^*$ is given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H$$

where $\mu = (\Pi, P, \Gamma)$ and

$$\Lambda = \begin{bmatrix} \widehat{\Pi} & \widehat{P} & \widehat{\Gamma} \\ \widehat{P} & 0 & 0 \\ \widehat{\Gamma} & 0 & 0 \end{bmatrix}.$$

The Lie-Poisson reduced equations (see [26] for a complete description of reduction procedure) are then given by

$$\dot{\mu}_i = \{\mu_i, \widetilde{H}\}_-(\mu)$$

or explicitly as

$$\dot{\Pi} = \Pi \times (A\Pi + B^T P) + P \times (CP + B\Pi) - mgl\Gamma \times i_3 \quad (25\text{-a})$$
$$\dot{P} = P \times (A\Pi + B^T P) \quad (25\text{-b})$$
$$\dot{\Gamma} = \Gamma \times (A\Pi + B^T P) \quad (25\text{-c})$$

Proposition: The Lie-Poisson reduced Hamiltonian vector field (given by (25)) defined on $\mathcal{S}^*$ is WPPS.

Proof: Choose $V(\Pi, P, \Gamma) = \widetilde{H}(\Pi, P, \Gamma) + \Gamma^T\Gamma$. Observing that V is radially unbounded and that $\dot{V} = 0$ along trajectories of (25) the result follows form theorem 3. $\square$

Remark : In the case of coincident center of gravity and center of buoyancy (i.e. $l = 0$), the Hamiltonian system on $T^*SE(3)$ is left invariant under the $SE(3)$ action of rotations and translations, and we can derive a set of reduced Lie-Poisson equations on $se(3)^*$. Choosing

$$A_i = \begin{pmatrix} \widehat{e}_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3 \qquad A_i = \begin{pmatrix} 0 & \widehat{e}_i \\ 0 & 0 \end{pmatrix}, \quad i = 4, 5, 6$$

as the basis for $se(2)$ the structure matrix $[\Lambda(\mu)]_{ij} = -\sum_{k=1}^{6} c_{ij}^k \mu_k$ is given by

$$\Lambda(\mu) = \Lambda(\Pi, P) = \begin{bmatrix} \widehat{\Pi} & \widehat{P} \\ \widehat{P} & 0 \end{bmatrix}$$

where $\Pi = J\Omega$ and $P = Mv$. The Lie-Poisson reduced equations are given by

$$\dot{\Pi} = \Pi \times (A\Pi) + P \times CP \quad (26\text{-a})$$
$$\dot{P} = P \times A\Pi \quad (26\text{-b})$$

Choosing $V(\Pi, P) = \widetilde{H} = \frac{1}{2}(\Pi^T A\Pi + P^T CP)$, the reduced Hamiltonian, we observe that the Lie-Poisson reduced Hamiltonian vector field defined on $se(3)^*$ is WPPS.

4.3.1 Controllability of Reduced Dynamics

Assuming that we have three controls u_1, u_2, u_3 such that u_1 and u_2 provide a pure torques and u_3 provides a pure force. The Lie-Poisson reduced dynamics with controls for the underwater vehicle with coincident CB and CG are

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2P_3 + u_1 \tag{27-a}$$

$$\dot{\Pi}_2 = \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3P_1 + u_2 \tag{27-b}$$

$$\dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1P_2 \tag{27-c}$$

$$\dot{P}_1 = \frac{P_2\Pi_3}{I_3} - \frac{P_3\Pi_2}{I_2} + u_3 \tag{27-d}$$

$$\dot{P}_2 = \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3} \tag{27-e}$$

$$\dot{P}_1 = \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1} \tag{27-f}$$

Proposition: The Lie-Poisson reduced dynamics, defined by (27), of the underwater vehicle with coincident center of buoyancy and center of gravity are controllable if $I_1 \neq I_2$.

Proof: Observing that

$$det(g_1, g_2, g_3, [[f, g_1], g_2], [[f, g_2], g_3], [[[f, g_2], [f, g_3]], g_1]) = \frac{I_1 - I_2}{I_1 I_2^2 I_3} \neq 0$$

if $I_1 \neq I_2$ where

$$f = \begin{pmatrix} \frac{I_2-I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2-m_3}{m_2 m_3}P_2P_3 \\ \frac{I_3-I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3-m_1}{m_3 m_1}P_3P_1 \\ \frac{I_1-I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1-m_2}{m_1 m_2}P_1P_2 \\ \frac{P_2\Pi_1}{I_3} - \frac{P_3\Pi_2}{I_2} \\ \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3} \\ \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1} \end{pmatrix}, \quad g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

i.e. $\dim(\text{span}\mathcal{L}_{\{f,g_1,g_2,g_3\}})(p) = 6, \quad \forall p \in se(3)^*$, and that f is WPPS the result follows from theorem 2. $\square$

5 Conclusions and Future Work

We have discussed the controllability of the Lie-Poisson reduced dynamics of a class of mechanical systems and present an approach to conclude controllability. The Lie-Poisson reduced dynamics and controllability of hovercraft, spacecraft and underwater vehicle are discussed.

Current research includes the design of feedback laws to stabilize the origin of the reduced system. We are also studying the use of periodic controls to generate loops in the base space, and thereby steer in the fiber. Some future research includes control laws to steer hovercraft and hybrid architectures for generation of "scripts" for obstacle avoidance and navigation.

6 Acknowledgments

The authors would like to thank Naomi E. Leonard for discussions on underwater vehicles and David L. Elliott for comments and suggestions.

Appendix

Accessibility and Controllability

We briefly review definitions and related theorems on accessibility and controllability of smooth affine nonlinear systems,

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \tag{28}$$

where $x = (x_0, \cdots, x_n)$ are local coordinates for a smooth manifold M and $u = (u_1, \cdots u_m) \in U \subset \mathbb{R}^m$. We follow the development in [19] (see historical references therein). It is assumed that -

(i) The input space U is such that the set of associated vector fields of (28) $\mathcal{F} = \{f(x) + \sum_{i=1}^{m} g_i(x)u_i | (u_1, \cdots u_m) \in U\}$ contains the vector fields $f, g_1, \cdots, g_m$.

(ii) The set of admissible controls consists of piecewise constant functions which are piecewise continuous from the right.

Let $R^V(x_0, T)$ be the reachable set from x_0 at time $T > 0$, following trajectories which remain for $t \leq T$ in the neighborhood V of x_0. Let $R^V(x_0, \leq T) = \bigcup_{0 \leq t \leq T} R^V(x_0, t)$

Definition: The system (28) is locally accessible from x_0 if $R^V(x_0, \leq T)$ contains a non-empty open set of M for all neighborhoods V of x_0 and all $T > 0$. If this holds for any $x_0 \in M$ the the system is called locally accessible.

Let $L(x) = \{\text{span}X(x) \mid X \text{ vector field in } \mathcal{L}, x \in M\}$ where $\mathcal{L}$, the accessibility Lie algebra, is the smallest subalgebra of the Lie algebra of vector fields on M that contains $f, g_1, \cdots, g_m$. The system is said to satisfy the accessibility Lie algebra rank condition (LARC) if

$$L(x) = T_x M, \ \forall x \in M \tag{29}$$

Theorem 5 *If $dim L(x) = n \ \ \forall x \in M$ then the system system (28) is locally accessible.*

Definition: The system (28) is said to be locally strongly accessible from x_0 if for any neighborhood V of x_0 the set $R^V(x_0, T)$ contains a non-empty open set for any $T > 0$ sufficiently small.

Let $\mathcal{L}_0$ be the smallest Lie subalgebra which contains $g_1, \cdots, g_m$ and satisfies $[f, X] \in \mathcal{L}_0, \forall X \in \mathcal{L}_0$ and $L_0(x) = \{\text{span}X(x) \mid X \text{ vector field in } \mathcal{L}_0, x \in M\}$. The system is said to satisfy the strong accessibility Lie algebra rank condition if

$$L_0(x) = T_x M, \ \forall x \in M \tag{30}$$

Theorem 6 *If $dim L_0(x_0) = n$, then the system (28) is locally strongly accessible from x_0.*

Let $x(t, 0, x_0, u)$ denote the solution of (28) at time $t \geq 0$ for a particular input function $u(\cdot)$ and initial condition $x(0) = x_0$.

Definition: The system (28) is called controllable if for any two points x_1, x_2 in M there exists a finite time T and an admissible function $u : [0, T] \to U$ such that $x(t, 0, x_1, u) = x_2$.

Theorem 7 *(Chow)[32] The nonlinear system*

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i, \quad u = (u_1, \cdots, u_m) \subset U \tag{31}$$

is controllable if the accessibility LARC is satisfied.

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