CONTROL PROBLEMS OF HYDRODYNAMIC TYPE *

P.S. Krishnaprasad† and **Vikram Manikonda**‡

† *Department of Electrical Engineering Institute for Systems Research*
University of Maryland, College Park, MD 2074
‡ *Scientific Systems Co. Inc. 500 West Cummings Park, Suite 3000*
Woburn, MA 01801

Abstract: It has been known for some time that the classical work of Kirchhoff, Love, and Birkhoff on rigid bodies in incompressible, irrotational flows provides effective models for treating control problems for underwater vehicles. This has also lead to a better appreciation of the dynamics of such systems. In this paper, we develop results based on geometric mechanics and center manifold theory to solve controllability and stabilization questions for a class of under-actuated left invariant mechanical systems on Lie groups that include approximate models of underwater vehicles and surface vehicles. We also provide numerical evidence to capture the global properties of certain interesting feedback laws.

Keywords: Symmetry, Lie-Poisson Reduction, Underactuated Systems, Controllability, Stabilization, Hovercraft, Underwater Vehicles

1 INTRODUCTION

In this paper we present results related to the controllability and stabilization of a class of under-actuated mechanical systems with symmetry. We consider systems with configuration space G, a Lie group, and G-invariant forced dynamics on the cotangent bundle T^*G.

This research is motivated by issues related to the control of under-actuated hovercraft and underwater vehicles. It can be shown that simplified models of these systems satisfy the above assumption of G-invariance (see Section 2). For example, if an underwater vehicle is modeled as a completely submerged rigid body in an inviscid, incompressible and irrotational fluid of infinite volume, the impulse motion of the body-fluid system can be shown (Lamb, 1945; Birkhoff, 1960) to vary as the momentum of a finite dimensional system under the influence of external forces. Hence, identifying the configuration space of the underwater vehicle with the Lie group $SE(3)$, the dynamics can be modeled on $T^*SE(3)$. The existence of a control law to steer these systems in situations of actuator failure poses an interesting problem. To resolve questions related to controllability we observe that a geometric approach leads to a deeper understanding of the problem. The G-invariance permits the dropping of the dynamics to a lower dimensional space, namely the quotient manifold, T^*G/G. Analysis of the reduced dynamics provides insight into properties of the unreduced dynamics. In (Manikonda and Krishnaprasad, 1997) we presented sufficient conditions (Theorem 3.2) for controllability of the reduced dynamics of these systems. In this paper we extend these results and present sufficient conditions for controllability of the unreduced dynamics (Theorem 3.6 and Theorem 3.10).

In addition to proving results on controllability, in this paper, we also present a general approach (Theorem 4.2), based on center manifold theory, to construct feedback laws to stabilize relative equilibria of mechanical systems with symmetry. Our approach again exploits the geometry of the reduced space and the Lie-Poisson structure to show that under certain hypothesis the fixed points of the reduced dynamics can be shown to belong to an immersed equilibrium manifold (Theorem 4.1). The existence of this equilibrium manifold and controllability are used to design stabi-

*This research was supported in part by a grant from the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012, and by the Army Research Office under the MURI97 Program Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures.

lizing feedback laws.

The paper is organized as follows. In Section 2 following a brief discussion on Lie-Poisson reduction, examples of left-invariant mechanical systems are presented. In Section 3 we present results on controllability. In Section 4 a general approach based on center manifold theory, to construct feedback laws to stabilize relative equilibria are presented. Conclusions and directions for future research are discussed in Section 5.

2 PRELIMINARIES AND EXAMPLES

In this section we present, briefly, the geometric framework used in this paper. For the class of mechanical systems discussed, we assume that the configuration space of these systems can be identified with a Lie group G . We model the dynamics of these systems as controlled Hamiltonian systems on T^*G. Written in the form of an affine nonlinear control system the dynamics take the form

$$\Sigma: \qquad \dot{x} = X_H(x) + \sum_{i=1}^{m} Y_i(x) u_i. \quad (1)$$

In Equation (1) X_H is a Hamiltonian vector field with respect to a Hamiltonian $H : T^*G \to \mathbb{R}$ and the canonical Poisson bracket on T^*G. We further assume that the Hamiltonian H and the control vector fields Y_i are G-invariant i.e. $\forall g \in G$, and $x \in T^*G$, $H \circ T^*L_g(x) = H(x)$ and $T(T^*L_g)Y_i(x) = Y_i(T^*L_g x)$. Here L_g denotes the left action of G on itself and T^*L_g the cotangent lift of L_g.

Since T^*L_g is a free and proper action recall that the dynamics on T^*G project [1] to dynamics on $T^*G/G \cong \mathfrak{g}^*$. In particular the Hamiltonian H, projects to $\widetilde{H}$ defined on $\mathfrak{g}^*$ s.t. $H = \widetilde{H} \circ \pi$, where $\pi : T^*G \to \mathfrak{g}^*$ denotes the projection map, and the Hamiltonian vector field X_H projects to a Hamiltonian vector field $X_{\widetilde{H}}$ on $\mathfrak{g}^*$. Further $X_{\widetilde{H}}$ is Hamiltonian on $\mathfrak{g}^*_-$ w.r.t. to the reduced Hamiltonian $\widetilde{H}$ and the minus Lie-Poisson bracket define on $\mathfrak{g}^*$. Hence on $\mathfrak{g}^*_-$ the reduced dynamics are given by

$$\widetilde{\Sigma}: \qquad \dot{\mu} = X_{\widetilde{H}}(\mu) + \sum_{i=1}^{m} \widetilde{Y}_i(\mu) u_i \quad (2)$$

As we shall see the Lie-Poisson structure of $X_{\widetilde{H}}$ and the geometry of $\mathfrak{g}^*$ play an important role in determining controllability and stability properties of the reduced system.

Example 1: The Jet-Puck - An Underactuated Hovercraft We model a hovercraft as a planar rigid body with a vectored thrust (Manikonda and Krishnaprasad, 1996) and identify its configuration space with $SE(2)$[2]. It is further assumed that the line of action of the force is fixed and does not pass through the center of mass. Observing that the Lagrangian, which is simply the kinetic energy, and body fixed forces, are invariant under the lifted action of $SE(2)$ on $TSE(2)$, the dynamics on $T^*SE(2)$ project to reduced dynamics on $se(2)^*$. Choosing convected linear momentum, (P_1, P_2) and body angular momentum Π as coordinates for $se(2)^*$, the reduced dynamics are given by

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + \alpha u \\ \dot{P}_2 &= -P_1\Pi/I + \beta u \\ \dot{\Pi} &= d\beta u \end{aligned} \quad (3)$$

where $\alpha = \cos\phi, \beta = \sin\phi$. The drift vector field in (3) is a Lie-Poisson vector field with respect to the minus Lie-Poisson structure on $se(2)^*$ and the reduced Hamiltonian

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m}. \quad (4)$$

Example 2: Autonomous Underwater Vehicle An autonomous underwater vehicle can be modeled as a rigid body submerged in an infinitely large volume of incompressible, inviscid and irrotational fluid which is at rest at infinity (cf. (Lamb, 1945; Birkhoff, 1960; Leonard, 1997)). We consider an underwater vehicle with ellipsoidal geometry and assume that the center of mass and center of buoyancy are coincident. Identifying the configuration space with $SE(3)$, one observes that the Lagrangian $L : TSE(3) \to \mathbb{R}$ given by

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}(\Omega^T J \Omega + v^T M v)$$

is SE(3)-invariant and dynamics on $T^*SE(3)$ project to dynamics on $se(3)^*$. Here Ω and v denote the body angular velocity, and the linear velocity components along the body frame. J is the body inertia matrix plus the added inertia matrix due to the flow of the fluid. Similarly M is the mass matrix plus the added mass matrix associated with the fluid. (See (Lamb, 1945; Birkhoff, 1960) for details on modeling rigid bodies in incompressible, inviscid and irrotational flows). Let us assume that we have only one pure force and two torques to control the position and orientation of the underwater vehicle. Choosing $\Pi = \frac{\partial L}{\partial \Omega} = J\Omega$ and $P = \frac{\partial L}{\partial v} = Mv$ as coordinates for $se(3)^*$ the dynamics on $T^*SE(3)$, in terms of coordinates (r, R, Π, P), are given by

$$\dot{r} = RM^{-1}P \quad (5)$$
$$\dot{R} = R\widehat{J^{-1}\Pi} \quad (6)$$
$$\dot{\Pi} = \Pi \times J^{-1}\Pi + P \times M^{-1}P + U_1 \quad (7)$$
$$\dot{P} = P \times J^{-1}\Pi + U_2 \quad (8)$$

where $U_1 = (u_1, u_2, 0)^T$ and $U_2 = (u_3, 0, 0)^T$. Equations (7-8) correspond to the reduced dynamics on $se(3)^*$.

Remark: In the rest of the discussion we will assume that

$J = diag(I_1, I_2, I_3)$ and $M = diag(m_1, m_2, m_3)$.

[1]Due to limitations on space we do not discuss the reduction process in detail. The reader is referred to (Marsden and Ratiu, 1994) and references therein for details.

[2]In the rest of the discussion an element of $SE(n)$, $n = 2, 3$ is given by the pair (R, r) where $R \in SO(n)$ and $r \in \mathbb{R}^n$ is a vector from the origin of the inertial frame to the origin of the body frame.

3 CONTROLLABILITY

Proving controllability of affine nonlinear control systems, where the linearization is not controllable, is in general a difficult task. Important contributions in this area have been due to Bonnard, Lobry, Crouch, Byrnes and others (Jurdjevic and Kupka, 1981; Crouch and Byrnes, 1986; Lobry, 1974)

Of particular interest in the current setting is following result due to (Lobry, 1974; Lian *et al.*, 1994) where weak positive Poisson stability (WPPS) of the drift vector field is used to conclude controllability. Recall that a vector field X on a manifold M is called weakly positively Poisson stable if the set of points $p \in M$ such that for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$ such that $\phi_t^X(V_p) \bigcap V_p \neq \emptyset$ (where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$) is dense in M.

Theorem 3.1 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad u = (u_1, \cdots u_m) \in U \subset \mathbb{R}^m$$

where U contains $\{u \mid |u_i| \leq M_i \neq 0, i, \cdots, m\}$ is such that f is a weakly positively Poisson stable vector field, then the system is controllable if the accessibility Lie algebra rank condition (LARC) is satisfied.

While proving WPPS of the drift vector field can be difficult, in a setting where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field, in (Manikonda and Krishnaprasad, 1997) we prove the following result.

Theorem 3.2 *Let G be a Lie group that acts on itself by left (right) translations. Let $H : T^*G \to \mathbb{R}$ be a left (right)-invariant Hamiltonian. Then,*

(i) If G is a compact group, the coadjoint orbits of $\mathfrak{g}^* = T^*G/G$ are bounded and the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS.

(ii) If G is a noncompact group then the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS if there exists a function $V : \mathfrak{g}^* \to \mathbb{R}$ such that $V(\mu)$ is bounded below, $V(\mu) \to \infty$ as $\|\mu\| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.

Here $\widetilde{H}$ is the induced Hamiltonian on the quotient manifold $\mathfrak{g}^* = T^*G/G$ and $\{\cdot,\cdot\}_{-(+)}$ is the induced minus (plus) Lie-Poisson bracket on the quotient manifold $\mathfrak{g}^* = T^*G/G$.

In our present setting of Lie-Poisson reduced dynamics, WPPS conditions in Theorem 3.1 can be verified whenever the hypotheses of Theorem 3.2 hold. Once WPPS of the drift vector field has been established Theorem 3.1 can be used to conclude controllability. In (Manikonda and Krishnaprasad, 1997) we prove the following:

Proposition 3.3 *The reduced jet-puck dynamics defined by (3) are controllable if* $\sin\phi \neq 0$.

Proposition 3.4 *The Lie-Poisson reduced dynamics of the underwater vehicle with coincident center of buoyancy and center of gravity, defined by (7-8), are controllable if $I_1 \neq I_2$.*

If the symmetry group G is compact we now show (Theorem 3.6) that the above hypotheses are sufficient to conclude that the unreduced drift vector field X_H too is WPPS. To prove this we need the following lemma- (see (Manikonda, 1997) for the proof of the lemma and Theorem 3.6).

Lemma 3.5 *Let G be a compact Lie group whose action $\Phi : G \times M \to M$ on a manifold M is free. Let $\pi : M \to M/G$ denote the projection map. Then $D = \pi^{-1}(\widetilde{D})$ is compact iff $\widetilde{D} \subset M/G$ is compact i.e the projection map π is a proper map.*

Theorem 3.6 *Let G be a compact Lie group whose action on a Poisson manifold M is free and proper. A G-invariant Hamiltonian vector field X_H defined on a manifold M is WPPS if there exists a function $V : M/G \to \mathbb{R}$ that is proper, bounded below and $\dot{V} = 0$ along trajectories of the projected vector field $X_{\widetilde{H}}$ defined on M/G.*

As mentioned earlier, having concluded the WPPS nature of the Hamiltonian vector field, if the Hamiltonian control system on M satisfies the LARC, then from Theorem 3.1 controllability can be concluded. The conclusion on the controllability of the unreduced dynamics where G is compact is similar in spirit to that of (Martin and Crouch, 1984).

In the present setting of hovercraft and underwater vehicles we observe that though $SE(n)$, $n = 2, 3$ is not a compact group, it is a semidirect product, i.e. $G = H \times_\rho V$ where $H = SO(n)$ is compact and $V = \mathbb{R}^n$ is a vector space. For semidirect products one observes that $G/V \cong H$. Hence reduction of G-invariant dynamics can be performed in two stages. First by V, to obtain dynamics on $H \times \mathfrak{g}^*$, and then by H to obtain the reduced dynamics on $\mathfrak{g}^*$. Hence under appropriate LARC assumptions we can conclude the reduced dynamics on $H \times \mathfrak{g}^*$ are controllable iff the Lie-Poisson reduced dynamics on T^*G/G are controllable.

Applying these results to the examples discussed earlier we have the following results.

Proposition 3.7 *The reduced dynamics of the jet-puck defined on $SO(2) \times se(2)^*$ are controllable if* $\sin\phi \neq 0$.

Proposition 3.8 *The reduced dynamics (6-8) of the underwater vehicle with coincident center of mass and center of buoyancy, defined on $SO(3) \times \mathfrak{se}(3)^*$ are controllable if $I_1 \neq I_2$.*

In the setting where the symmetry group is non-compact, under additional assumptions of equilibrium controllability, reduced space controllability is sufficient to conclude controllability on T^*G.

Before we prove this result we define equilibrium controllability, a concept introduced in (Lewis and Murray, 1996). Consider a mechanical system with a G-invariant Hamiltonian and G-invariant forces. Assume that the Hamiltonian is quadratic and projects to $\widetilde{H} = \mu^T \mathbb{I}^{-1}\mu, \quad \mu \in \mathfrak{g}^*$, where $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is the inertia tensor. Then the dynamics on T^*G can be written in the form

$$\dot{g} = g\mathbb{I}^{-1}\mu \tag{9}$$

$$\dot{\mu} = \Lambda(\mu)\nabla\widetilde{H} + \sum_{i=1}^{m} f^i u_i. \tag{10}$$

Here $\Lambda(\mu)$ is the Lie-Poisson tensor defined on $\mathfrak{g}^*$.

Definition 3.9 The system (9-10) is equilibrium controllable if for any $(g_1, 0)$, $(g_2, 0)$ there exists a time $T > 0$ and an admissible input $u : [0, T] \to U$ such that the solution $(g(t), \mu(t))$ of (9-10) with initial conditions $(g(0), \mu(0)) = (g_1, 0)$ satisfies $(g(T), \mu(T)) = (g_2, 0)$.

Theorem 3.10 *If the dynamics of a mechanical system given by (9 -10) are such that*
(i) the system is equilibrium controllable, and
(ii) the reduced dynamics (10) are controllable, then the system is controllable.

Proof: We need to show that there exists a $T > 0$ and an admissible control $u : [0, T] \to U$ such that given any (g_1, μ_1) and (g_f, μ_f) the solution $(g(t), \mu(t))$ satisfies $(g(0), \mu(0)) = (g_1, \mu_1)$ and $(g(T), \mu(T)) = (g_f, \mu_f)$. Using the properties (i) and (ii) we construct such a control.

Assume that there exists a state $(g_3, 0)$ and an admissible control u', such that u' will steer the system from $(g_3, 0)$ to (g_f, μ_f) in finite time.(The existence of such a $(g_3, 0)$ and u' is shown later.) The problem is now reduced to finding a control to steer the system from (g_1, μ_1) to $(g_3, 0)$ which is done as follows.

Let $g(t, t_0, g_0, \mu(t))$ denote the the solution of (9) at $t > t_0$ for a particular curve $\mu(t) \in \mathfrak{g}^*$ and initial condition g_0. Similarly let $\mu(t, t_0, \mu_0, u(t))$ denote the the solution of (10) at $t > t_0$ for a particular input u and initial condition μ_0, and let $\zeta(t, t_0, (g_0, \mu_0), u)$ denote the solution of (9-10) at $t > t_0$ for a particular input u and initial condition (g_0, μ_0).

1. Since the reduced dynamics are controllable there exists a $T_1 > 0$ and a control u_1 such that $\zeta(T_1, 0, (g_1, \mu_1), u_1) = (g_2, 0)$ (for some g_2).

2. Since the dynamics are equilibrium controllable there exists a $T_2 > T_1 > 0$ and a control u_2 such that $\zeta(T_2, T_1, (g_2, 0), u_2) = (g_3, 0)$.

3. Finally applying u' we have $\zeta(T_3, T_2, (g_3, 0), u') = (g_f, x_f)$.

The existence of $(g_3, 0)$ and u' is shown as follows. Find u_3 and T_3' such that $\mu(T_3', 0, 0, u_3) = \mu_f$. Existence of such a control follows from the reduced space controllability of (10). Apply the control u_3 to (9-10) with initial condition $\mu(0) = 0$ and arbitrary $g(0) = g_3'$. Then $\zeta(T_3', 0, (g_3', 0), u_3) = (g_4, \mu_f)$ where g_4 need not be equal to g_f. Let $g(t, 0, g_3', \mu(t))$ denote the solution to (9) where $\mu(t) = \mu(t, 0, 0, u_3)$. Let $R \in G$. Then by left invariance $\bar{g} = Rg(t, 0, g_3', \mu(t))$ is a solution to (9-10). Choose R such that $\bar{g}(T_3') = Rg(T_3', 0, g_3', \mu(t)) = g_f$, i.e $R = g_4^{-1} g_f$ and hence $\bar{g}(t) = g_4^{-1} g_f g(t, 0, g_3', \mu(t))$. Again from left invariance it implies that $\bar{g}(T_3', 0, g_4^{-1} g_f g_3', \mu(t)) = g_f$ or equivalently $\zeta(T_3', 0, (g_4^{-1} g_f g_3', 0), u_3) = (g_f, \mu_f)$. Hence choose $g_3 = g_4^{-1} g_f g_3'$, $u' = u_3$ and $T_3 = T_2 + T_3'$. ■

Given G-invariant dynamics, controllability of reduced space can be verified using Theorem 3.1 and Theorem 3.2. Equilibrium controllability can be verified using results in (Lewis and Murray, 1996; Bullo and Lewis, 1996) where the following sufficient conditions for equilibrium controllability are presented.

Let $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}; \eta \mapsto [\xi, \eta]$ denote the adjoint map and ad_ξ^* denote its dual. Let

$$\dot{g} = g\xi \tag{11}$$

$$\mathbb{I}\dot{\xi} = \mathrm{ad}_\xi^* \mathbb{I}\xi + \sum_{i=1}^{m} f^i u_i \tag{12}$$

define G-invariant dynamics on TG.[3] Define the *symmetric product* $\langle \cdot : \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : \xi, \eta \mapsto \langle \xi : \eta \rangle$ as

$$\langle \xi : \eta \rangle = -\mathbb{I}^{-1}(\mathrm{ad}_\xi^* \mathbb{I}\eta + \mathrm{ad}_\eta^* \mathbb{I}\xi) \tag{13}$$

Let $B = \{b_1, \cdots, b_m\} \subset \mathfrak{g}$ (a left invariant distribution on G) denote the input subspace. In the present setting $b_i = \mathbb{I}^{-1} f^i$. Let $\overline{Lie}_\mathfrak{g}(\mathcal{B})$ and $\overline{Sym}_\mathfrak{g}(\mathcal{B})$ denote the involutive and symmetric closure of $\mathcal{B}$ in $\mathfrak{g}$. A symmetric product is *bad* if it contains an even number of each of the vectors in $\mathcal{B}$. A symmetric bracket is *good* if it is not bad.

Theorem 3.11 *(Lewis and Murray, 1996; Bullo and Lewis, 1996) The system (11-12) is equilibrium controllable if rank*$(\overline{Lie}_\mathfrak{g}(\overline{Sym}_\mathfrak{g})(\mathcal{B})) = $ *dim*(G) *and every bad symmetric product can be written as a linear combination of good symmetric products of lower degree.*

We now apply Theorem 3.10 to the autonomous underwater vehicle with coincident center of mass and center of buoyancy. Controllability of reduced dynamics follows from Proposition 3.4. Equilibrium controllability of the above dynamics can now be verified using Theorem 3.11. Details regarding the relevant symmetric bracket calculations can be found in (Manikonda, 1997)

Proposition 3.12 *The unreduced dynamics (5-8) of the autonomous under water vehicle with coincident center of mass and center of buoyancy, defined on $T^*SE(3)$ (or equivalently $TSE(3)$) are controllable if $I_1 \neq I_2$.*

Remark: In the case of the unreduced jet-puck dynamics as we have only one input, every non

[3] In (Lewis and Murray, 1996; Bullo and Lewis, 1996) it is assumed that the dynamics evolve on TG. Setting $\mu = \mathbb{I}\xi$, the two formulations (9-10) and (11-12) are equivalent.

trivial second order symmetric bracket is bad. Hence sufficient conditions for equilibrium controllability are not satisfied and hence we cannot conclude controllability of unreduced dynamics.

4 DISSIPATIVE FEEDBACK CONTROL

In this section we study stabilization of fixed points of the reduced dynamics. These fixed points give rise to relative equilibria, i.e. trajectories that are group orbits in the unreduced phase space. For example, steady translations and rotations correspond to relative equilibria for the underwater vehicle. The energy-Casimir method serves (cf. (Bloch *et al.*, 1992; Bloch and Marsden, 1990)) as a good tool to study stability of equilibria corresponding to dynamics of Lie-Poisson type. Having identified unstable equilibria, various approaches have been adopted to stabilize them. In (Bloch *et al.*, 1992; Bloch and Marsden, 1990; Leonard, 1996) feedback laws have been chosen such that the closed loop system is still Hamiltonian with respect to a Poisson structure defined on the quotient manifold. The advantage of choosing such a feedback law is that the closed loop system again lends itself to stability analysis using the energy-Casimir method or similar techniques. We refer to these controls as *Hamiltonian feedback controls.* In this section we present constructive feedback laws to stabilize relative equilibria of Hamiltonian systems using dissipative control laws. *We define a control law to be dissipative if the divergence of the closed loop system is less than zero.* The approach is based on the observation that, under certain hypothesis the fixed points of the Lie-Poison reduced dynamics can be shown to belong to an immersed equilibrium manifold. The existence of this equilibrium manifold and controllability is used to construct stable center manifolds. The main ideas behind this approach are described below.

Consider the Lie-Poisson reduced dynamics on $\mathfrak{g}^*$ given by

$$\dot{\mu} = f(\mu) + \sum_{i}^{m} g_i(\mu)u_i \qquad \mu \in \mathfrak{g}^*. \qquad (14)$$

Exploiting the symplectic foliation of $\mathfrak{g}^*$ by coadjoint orbits we make the following observation (see (Manikonda, 1997) for proof).

Theorem 4.1 *Let μ_e be an equilibrium point of (14) such that there exists a neighborhood V of μ_e in which the Poisson tensor $\Lambda(\mu)$ has constant rank. Then in V there exists an immersed submanifold $\mathcal{E}$ such that for all $\mu \in \mathcal{E}$, $X_{\widetilde{H}}(\mu) = 0$. Further locally there exist coordinates $(y_1, \cdots, y_r, z_1, z_{n-r})$ such that $z = 0$ on $\mathcal{E}$.*

We call $\mathcal{E}$ the equilibrium submanifold. Lets assume that $\mathcal{E}$ is of dimension k. Choose coordinates (y, z) in a neighborhood V of μ_e such that $z = 0$ on $\mathcal{E}$. In these coordinates $\mu_e = (y_e, 0)$ and(14) can be written as

$$\dot{y} \;=\; A_2^1 z + \widetilde{f}^1(y,z) + \sum_{i=1}^{m} b_i^1 u_i + \sum_{i=1}^{m} \widetilde{g}^1(y,z)u_1 \quad (15)$$

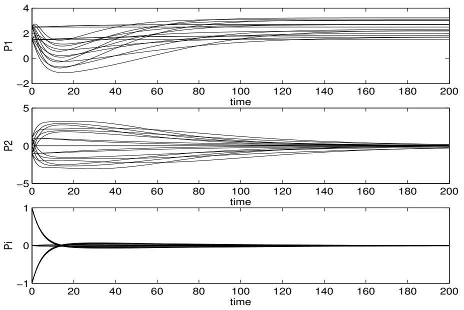

Figure 1: Stabilizing dissipative feedback laws for the Hovercraft. $\mu_e = (2,0,0), \lambda_1 = \lambda_2 = -0.1$

$$\dot{z} \;=\; A_2^2 z + \widetilde{f}^2(y,z) + \sum_{i=1}^{m} b_i^2 u_i + \sum_{i=1}^{m} \widetilde{g}^2(y,z)u_1 \quad (16)$$

where $A_2^1 = \frac{\partial f^1(y,z)}{\partial z}\,|_{(y_0,0)}$ and $A_2^2 = \frac{\partial f^2(y,z)}{\partial z}\,|_{(y_0,0)}$. We refer to (16) as the transverse dynamics. Now if the linearized transverse dynamics are controllable it is easy to show (cf. (Manikonda, 1997; Zenkov *et al.*, 1997)) that one can find a feedback law, such that $z = 0$ is a stable center manifold (Carr, 1981) or equivalently μ_e is a stable (in the sense of Lyapunov) equilibrium point of the closed loop system.

Theorem 4.2 *Under the assumption that (14) has an equilibrium submanifold $\mathcal{E}$, there exists a class of state feedback laws $u_\lambda(\mu) = K_\lambda z + \phi_\lambda(z)$, with $\phi_\lambda(0) = 0$, such that $(y_0, 0) \in \mathcal{E}$, $y_0 \neq 0$ is a stable equilibrium of the closed loop system if the linearized transverse dynamics (16) are stabilizable. Further for all trajectories $(y(t), z(t))$ of the closed closed loop system sufficiently close to the origin*

$$(y(t), z(t)) \to (c, 0) \qquad \textit{as } t \to \infty$$

i.e the closed loop system is asymptotically stable in z and stable in y.

Using this approach we find a class of linear feedback laws to stabilize unstable relative equilibria for the hovercraft and underwater vehicle.

Proposition 4.3 *: The class of feedback laws, parameterized by λ_1, λ_2 given by*

$$u_{\lambda_1,\lambda_2} = \frac{\lambda_1\lambda_2}{P_1^0\gamma}P_2 - \left(\frac{(\lambda_1+\lambda_2)}{\gamma} + \frac{\lambda_1\lambda_2\beta I}{P_1^0\gamma}\right)\Pi, \quad \lambda_1, \lambda_2 > 0 \qquad (17)$$

stabilize the equilibrium $(0, P_1^0, 0)$ of (3) for any $P_1^0 \neq 0$

Remark: (i) If $P_1^0 < 0$ then the divergence of the closed loop system is less than zero for any choice of $\lambda_1, \lambda_2 > 0$, making the closed loop system dissipative. (ii) If $P_1^0 > 0$ then $\lambda_1, \lambda_2 > 0$ can be chosen such that the closed loop system is dissipative.

Figure 1 shows the trajectories of the closed loop system with a dissipative feedback law for various initial conditions in the neighborhood of the equilibrium.

We now construct linear feedback laws to stabilize the equilibrium solution $x_e = (0, 0, \Pi_3^0, 0, 0, P_3^0), P_3^0 \neq 0$ for the underwater vehicle. We assume that $m_3 < m_1$. Recall that this is an unstable relative equilibria (Lamb, 1945). To compare our results with Hamiltonian feedback laws lets assume that in (7-8) $U_1 = (u_1, u_2, u_3)$ and $U_2 = (0, 0, 0)$. [4]

Proposition 4.4 *There exists a class of state feedback laws of the form* $u_i = \sum_1^5 \alpha_i z_i + \phi_i(z), \phi_i(0) = 0$, *where* $z = (z_1, \cdots z_5) = (\Pi_1, \Pi_2, \Pi_3, P_1, P_2)$ *such that the equilibrium* $x_e = (0, 0, 0, \Pi_3^0, 0, P_3^0)$, $P_3^0 \neq 0$ *is a locally stable equilibrium of the closed loop system (7-8).*

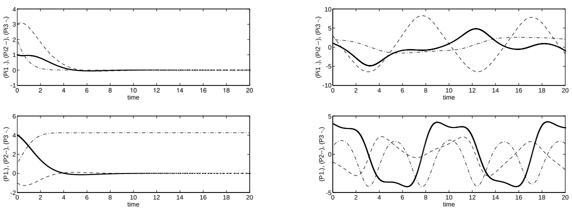

Figure 2: (Left):Dissipative feedback law for the AUV. (Right): Stabilizing Hamiltonian Feedback Law

In Fig. 2 (left) we stabilize the unstable relative equilibrium $(0, 0, 0, 0, 0, 1)$ using a dissipative feedback law. (In the simulations $m_3 << m_1$). Observe that the states $\Pi(t), P_1(t)$ and $P_2(t)$ are asymptotically stable. (Compare the results with Fig. 2 (right) where a Hamiltonian feedback law has been chosen to stabilize the same equilibrium). Since the divergence of the closed loop system is less than zero one might conjecture that under assumptions of boundedness of solutions and absence of limit cycles the closed loop system is globally stable, i.e. globally, trajectories converge to the stable manifold ("attractor"). Analytical results for the examples discussed did seem to indicate this.

5 CONCLUSIONS

In this paper we presented results related to controllability and stabilization of a class of nonlinear left-invariant mechanical systems with symmetry. Our results on controllability provide a manageable tool to check for controllability of a wide class of mechanical systems including hovercraft and underwater vehicles. We also presented an approach, based on center manifold techniques, to design feedback laws to stabilize relative equilibria.

Future directions of research include designing constructive control laws to steer in T^*G, stabilization of the unreduced dynamics and in showing global stability of the closed loop system under the dissipative feedback laws.

[4]In general, any set of control vector fields may be chosen as long as hypotheses of Theorem 4.2 are met.

References

Birkhoff, G. (1960). *Hydrodynamics: A study in logic, fact, and similitude.* 2nd ed.. Princeton University Press.

Bloch, A.M. and J.E. Marsden (1990). Stabilization of rigid body dynamcis by the energy-Casimir method. *Systems and Control Letters* **14**, 341–346.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and Sanchez de Alvarez (1992). Stabilization of rigid body dynamics by internal and external torques. *Automatica* **28**, 745–756.

Bullo, F. and A.D. Lewis (1996). Configuration controllability of mechanical systems on Lie groups. In: *Symposium on Mathematical Theory of Networks and Systems.*

Carr, J. (1981). *Applications of Center Manifold Theory.* Vol. 35 of *Applied Mathematical Sciences.* Springer Verlag.

Crouch, P.E. and C.I. Byrnes (1986). Local accessibility, local reachability, and representaitions of compact groups. *Math. Systems Theory* **19**, 43–65.

Jurdjevic, V. and I. Kupka (1981). Control systems subordinated to a group action: Accessibility. *Journal of Differential Equations* **39**(2), 186–211.

Lamb, H. (1945). *Hydrodynamics.* 6th ed.. Dover.

Leonard, Naomi E. (1996). Geometric methods for robust stabilization of autonomous underwater vehicles. In: *Proc. of 1996 Symp. on Autonomous Underwater Technology.* IEEE. pp. 470–476.

Leonard, Naomi E. (1997). Stability of a bottom-heavy underwater vehicle. *Automatica* **33**(3), 331–346.

Lewis, Andrew D. and Richard M. Murray (1996). Configuration controllability of simple mechanical control systems. *SIAM Journal on Control and Optimization* **35**(3), 766–790.

Lian, Kuang-Yow, Li-Sheng Wang and Li-Chen Fu (1994). Controllability of spacecraft systems in a central graviational field. *IEEE Transactions on Automatic Control* **39**(12), 2426–2441.

Lobry, C. (1974). Controllability of nonlinear systems on compact manifolds. *SIAM Journal of Control* **12**(1), 1–4.

Manikonda, Vikram (1997). Control and Stabilization of a Class of Nonlinear Systems with Symmetry. PhD thesis. University of Maryland at College Park.

Manikonda, Vikram and P.S. Krishnaprasad (1996). Dynamics and controllability of a planar rigid body with a thruster. In: *Proc. of the 30th Annual Conference on Information Sciences and Systems.* Princeton University. pp. 161–166.

Manikonda, Vikram and P.S. Krishnaprasad (1997). Controllability of Lie-Poisson dynamics. In: *Proc. of American Control Conference.* American Automatic Council, Philadelphia. pp. 2203–2207.

Marsden, Jerrold E. and Tudor S. Ratiu (1994). *Introduction to Mechanics and Symmetry.* Springer-Verlag.

Martin, L. San and P.E. Crouch (1984). Controllbility on principal fiber bundles with compact structure group. *Systems & Control Letters* **5**, 35–40.

Zenkov, Dmitry V., Anthony M. Bloch and J.E. Marsden (1997). The energy-momentum method for the stability of nonholonomic systems. (Preprint).