

INTRINSIC MODELS FOR
INFINITE DIMENSIONAL LINEAR SYSTEMS

A thesis presented

by

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To Mary

PREFACE

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SYNOPSIS

Realization theory studies the relation between internal (based on a dynamical model) and external (based on input-output relations) description of systems. It also provides constructive ways (algorithmic if possible) of building simple (economical) models and investigates the relations between equally simple internal models. For linear, time invariant systems, with finite dimensional state spaces, realization theory is well understood and has become by now standard. Its methods and results are used in design, synthesis, identification, and filtering.

In this dissertation a realization theory is developed for linear, time invariant systems, with infinite dimensional state spaces. Previous work in this area has been concentrated on very general systems (the state space is a locally convex Hausdorff topological vector space), rendering thus intractable a detailed study of the relation between characteristic properties of the input-output law and its models.

Here we focus our study on systems, which have state spaces with the structure of a Hilbert space. The input-output laws we consider are given by the standard convolution integral, whose kernel is the weighting pattern and characterizes the external description of the system. We study systems for which the infinitesimal state transition operator is bounded as well as systems for which it is unbounded but generates a C_0 -semigroup. Many important classes of systems governed by linear partial differential equations or by integrodifferential equations are thus included in the category of systems we investigate. On one hand our motivation

comes from a desire to understand engineering problems where the assumption of finite dimensionality is too restrictive and on the other hand we want to see the finite dimensional theory as a part of a larger picture.

In addition to obtaining realizability criteria covering the basic cases, we prove the existence of canonical realizations with spectral properties closely reflecting the properties of the singularities of the transfer functions and the existence of universal dynamical models for the class of systems considered. Our approach is based on the close relation between the transfer functions considered here and the Hardy spaces of complex functions on disks or half-planes, and on certain facts from operator theory and the theory of semigroups of operators.

Applications of the results presented in this dissertation to filtering problems with nonrational spectra, to the implementation of optimal controllers by distributed parameter systems, to systems involving delays and in understanding several ad hoc modeling methods in engineering based on modal approximation suggest themselves as significant and interesting topics for further research. The application of the universal models in studying problems in physics, and more generally the connections of our theory with established physical theories (e.g. quantum mechanics) need further investigation. Finally, further research is needed for an effective construction of simple models for systems with dynamics of special character (e.g. the state transition operators form an analytic semigroup, or a group, or are compact).

CHAPTER I

INTRODUCTION

1.1 The Problem of Realization in Engineering and Physics

The analysis of many engineering and industrial processes, engineering devices as well as the investigation of problems in physics are essentially studies of the properties of particular systems; where by a system we mean an entity which accepts inputs and produces outputs. The inputs (controls) are those specific variables which we can vary, while the outputs are those variables which we can only observe.

There are two generally accepted ways of analyzing a system. The first is based on the development of the input-output relations which account best for a large part of experimental input-output data. Then one proceeds to a detailed study of these relations and of their implications. When this is done we say that we have an external description of the system. The second is based on the construction of a dynamical model for the internal structure of the system with the help of a set of parameters, the so called "internal" variables (or states); the requirement being that input-output relations derived using this internal model must agree with experimental data. When we can construct such a model we say that we have an internal description of the system.

In many problems in design, synthesis and implementation one is given performance specifications (which constitute the input-output relations) and is asked to construct a model which will satisfy

these specifications as close as possible. The same problem appears in the simulation of engineering and physical processes by analog computers.

The problem of realization is that of constructing internal models for given external descriptions. An important requirement is that the model should account for the observed data (the input-output relations) in as simple a way as possible. Realization theory studies the relation between internal and external description of systems, constructive ways of building simple models and the relations between equally simple internal models. Its importance rests largely on two facts: a) a simple dynamical model permits easier and more complete study of the properties of the system and on the other hand indicates simple ways of synthesis, b) knowing the relations between equally simple models, gives the means to conclude if an existing model (which might come from physical theories as well) is as simple as possible and on the other hand permits the construction of simple models particularly suited for the study of certain questions about the behaviour of the system.

1.2 Motivation and Contents

The input-output relations we are considering in this thesis are linear and time invariant. The input and output spaces are appropriate function spaces which are closed under time translation, slicing and concatenation. The input spaces is denoted by I.S. and the output space by O.S. We will be mainly concerned with the scalar input, scalar output

case. Time invariance means that the input-output map commutes with time translations. Linearity means that the input-output map is linear.

Suppose that the input space contains $\mathcal{D}(-\infty, \infty)$ (the space $C_0^\infty(-\infty, \infty)$ endowed with the Schwartz topology [21]) and that the output space contains $\mathcal{E}(-\infty, \infty)$ (the space $C^\infty(-\infty, \infty)$ endowed with the Schwartz topology). These are general assumptions about the input and output spaces. Then using the Schwartz kernel theorem of distribution theory [21], Matsuo [52] has shown, that a continuous, time invariant linear input-output map has the representation

$$y = T*u \tag{1}$$

where u is the input function, y the output function and $*$ denotes convolution in the sense of distributions (see [21]). Similar results can be found in [56]. T is generally a distribution and is called the weighting pattern [2] of the system. Kalman and Hautus [50], Kamen [51], also studied realization theory in the context of distributions. The generality of their approach however does not permit a detailed study of the relation between the weighting pattern (which characterizes the external description) and its models.

We will assume that (1) can be written in the usual integral convolution form

$$y(t) = \int_0^t T(t-\sigma)u(\sigma) d\sigma \tag{2}$$

where T is a real valued function with properties such that

the integral description in (2) is well defined. Moreover we will assume that the Laplace transform of T exists, and we will denote it by \tilde{T} . \tilde{T} is usually called the transfer function.

The internal dynamical models we consider are described by the equations

$$\left. \begin{aligned} \frac{d}{dt} x(t) &= Ax(t) + Bu(t) \\ y(t) &= C[x(t)] \end{aligned} \right\} (\Sigma)$$

where $x(t) \in \mathcal{H}$ a Hilbert space, usually separable, called the state space, $u(t) \in \mathcal{U}$ a Hilbert space, $y(t) \in \mathcal{Y}$ a Hilbert space. The input (resp. output) space is a space of \mathcal{U} -valued (resp. \mathcal{Y} -valued) functions (e.g. $L^2((0,\infty); \mathcal{U})$, $L^p((0,\infty); \mathcal{U})$, $C((0,\infty); \mathcal{Y})$). A is the infinitesimal generator of a C_0 -semigroup of bounded operators $\{e^{At}\}$ (see sec. 1.3) on \mathcal{H} . $B: \mathcal{U} \rightarrow \mathcal{H}$ is a bounded linear operator. $C: \mathcal{H} \rightarrow \mathcal{Y}$ is a linear operator which is either bounded, or is defined on $\mathcal{D}_0(A)$ (the domain of A) and $\|C[x]\| \leq k(\|Ax\| + \|x\|)$ for some constant k , for $x \in \mathcal{D}_0(A)$. In the case of scalar inputs, scalar outputs B becomes just an element $b \in \mathcal{H}$, and C a linear functional c on \mathcal{H} .

So the problem of constructing an internal model like (Σ) given the external description (2) of the system is reduced in our case to the following: Given a weighting pattern T on $[0, \infty)$ which is real valued, find a Hilbert space \mathcal{H} and $[A, b, c]$ so that the input-output relation of (Σ) starting at the state zero agrees with the one given by (2). This is equivalent with the problem of finding \mathcal{H} and $[A, b, c]$ as above, in order to express T as $T(t) = c[e^{At}b]$ or to express its Laplace transform \tilde{T} as $c[(Is-A)^{-1}b]$ in some appropriately defined region of the complex plane.

We consider several distinct, but related cases.

The first centers around the existence of realizations $[A,b,c]$ with A a bounded operator on \mathcal{H} , b an element of \mathcal{H} and c a bounded linear functional on \mathcal{H} . We call such triples bounded realizations. In this case the state transition operators $\{e^{At}\}$ are continuous in the uniform operator topology on \mathcal{H} .

We call a triple $[A,b,c]$ a regular realization if A is the infinitesimal generator of a strongly continuous semigroup of bounded operators $\{e^{At}\}$ on \mathcal{H} , (see sec. 1.3), b is an element of \mathcal{H} and c is a bounded linear functional on \mathcal{H} .

In both cases above the output can also be expressed, as is well-known as the inner product of $x(t)$ with some element of \mathcal{H} which is uniquely determined by the functional c , and which we denote also by c ; i.e. we shall write $y(t) = c[x(t)] = \langle c, x(t) \rangle$.

We also consider cases where A is the infinitesimal generator of a strongly continuous semigroup of bounded operators on \mathcal{H} , b is restricted to belong to the domain of A (written $\mathcal{D}_0(A)$) but c is a linear functional defined on $\mathcal{D}_0(A)$ and such that $|c(x)| \leq k(|Ax| + \|x\|)$ for all $x \in \mathcal{D}_0(A)$ and some constant k . Such realizations will be called balanced realizations. They have important properties not shared by regular realizations (see sec. 2.5).

When \mathcal{H} is finite dimensional the problems concerned with realization are well understood and the theory has become by now standard (see [2], [42], [43]).

Our motivation for concentrating on models like (Σ) comes from a desire to understand many engineering problems involving transmission lines, elastic deformations, systems involving delays, moving fluids and related matters, where the assumption of finite dimensionality is too restrictive. Systems governed by linear partial differential equations, including stochastic systems are in the category described in (Σ) . On the other hand we want to see the finite dimensional results themselves as a part of a larger picture. Our motivation for introducing the notion of a balanced realization comes from systems governed by partial differential equations with observations on the boundary of the domain of definition (see example 1 of sec. 2.4, and [39] p. 200).

The external description and equivalently the weighting pattern T can appear in several different ways. For example we may have already a dynamical model for the system which is very complicated; a fact which might prevent further study. However by looking at the input-output relation we can find via realization theory a simple model which is amenable to detailed study. In certain optimal control problems, e.g. of systems involving delays with quadratic cost criteria, the optimal control turns out to be a convolution of the state (see [44]), i.e. is given by a formula similar to (2) where y is replaced by u_0 the optimal control and u by x , the state of the system. The kernel of this convolution does not have rational Laplace transform and hence we cannot implement the optimal control by a finite dimensional system. It is possible however to implement the controller by an infinite

dimensional linear system. In several other problems in optimal control of systems governed by partial differential equations, with quadratic criteria the optimal control turns out to be a spatial convolution of the state. For example for the diffusion equation on the line, with distributed control and quadratic criteria the optimal control is of the form $u_0 = \int_{-\infty}^{\infty} w(z-z_1)x(t,z_1)dz_1$. The kernel w , usually does not have rational Laplace transform. The problem of implementing the optimal controller can be formulated as a problem in infinite dimensional realization theory, where the role of time is played now by the variable z . In problems in filtering, one is given a covariance matrix, and the problem is to construct a dynamical system with white noise as input, whose output has as covariance matrix the given one. This also can be formulated as a realization problem (see [45]). To date however the assumption is that the elements of the covariance matrix have rational Laplace transforms, which is equivalent to assuming that the model is finite dimensional. Infinite dimensional realization theory will provide ways to solve this problem in the case of nonrational spectra.

Our approach to these problems is based on two facts:

a) the set of transfer functions which admit bounded realizations is related in a close way to the space of complex functions analytic and square integrable on the disk $|s| < 1$ and the set of transfer functions which admit balanced or regular realizations is related in a close way to functions analytic and square integrable on a half-plane.

b) It is a fact from operator theory that the unilateral shift is a universal model for bounded operators (c.f. [9]) (see sec. 2.8 for a precise statement) and from the theory of semigroups of bounded operators that the translation semigroup is a universal model for C_0 -semigroups (c.f. [39]) (see sec. 2.8 for a precise statement).

It is a) that permits the construction of models whose infinitesimal generators have spectral properties reflecting the properties of the singularities of \tilde{T} . This requirement is essential from the engineering point of view, and is used as a guide for modeling in many problems (see for example the work of Carlin and Noble in [46], [47] in modeling of wave guide systems). On the other hand b) permits the construction of very simple universal dynamical models for the class of weighting patterns considered.

The study of systems like (Σ) begun by A.V. Balakrishnan [31]. W. Helton [14] also investigates some questions of this type but emphasizes a different class of ideas. Closest to our approach is that of Paul A. Fuhrmann [13] who analyzes bounded realizations in discrete time.

In the first half of chapter II we characterize the class of transfer functions which admit bounded realizations and investigate the relation between the spectral properties of the infinitesimal generators and the properties of the singularities of the transfer functions. In the second half of chapter II we give a similar study for transfer functions which admit balanced or regular realizations. In this chapter we prove the equivalence of balanced and regular realizations and in the final section we analyze the importance of the

universal dynamical models.

In the first section of chapter III we give a brief review of the theory of controllability and observability for infinite dimensional systems. In sections 3.3, 3.4, 3.5 we study canonical bounded realizations, we introduce the notion of S-minimal realizations and we classify transfer functions according to the properties of their singularities. In the last section of chapter III we undertake a similar study for canonical regular realizations.

Some of the results described here represent joint work with R.W. Brockett and have been previously reported in [48], [49].

1.3 Notation and Some Mathematical Background

The real numbers are denoted by \mathbb{R} ; the complex numbers by \mathbb{C} . \mathcal{H} denotes usually a Hilbert space. $B(\mathcal{H})$ denotes the Banach algebra of bounded operators on \mathcal{H} . When $M_n, n=0,1,\dots$ are subspaces of \mathcal{H} , $\bigvee_{n=1}^{\infty} M_n$ denotes the subspace spanned by M_n .

Following [18], [21] a one parameter family $\Phi(t)$ of bounded operators on \mathcal{H} form a C_0 semigroup if

- i) $\Phi(t+s) = \Phi(t)\Phi(s), \quad t, s > 0$
- ii) $\Phi(0) = I$
- iii) $\lim_{t \rightarrow 0^+} \|\Phi(t)x - x\| = 0$

The infinitesimal generator A of the semigroup is defined as

$$Ax = \lim_{h \rightarrow 0^+} \frac{\Phi(h)x - x}{h}$$

whenever the limit exists. The set of x for which the limit exists is the domain of A denoted by $\mathcal{D}_0(A)$ and is dense. Moreover A is closed. Usually we denote $\Phi(t)$ by e^{At} . By the Hille-Yosida Theorem [18] a necessary and sufficient condition that a closed linear operator A with domain $\mathcal{D}_0(A)$ dense, be the infinitesimal generator of a C_0 semigroup, is that there exist positive real numbers M and β such that for every real $\lambda > \beta$, λ is in the resolvent set of A and

$$\| (I\lambda - A)^{-n} \| \leq \frac{M}{(\lambda - \beta)^n} \quad (n=1,2,\dots).$$

If these conditions hold for all

$\lambda > \beta$ then $(Is - A)^{-1}$ exists for all complex s with $\operatorname{Re} s > \beta$ and is

given by $(Is - A)^{-1}x = \int_0^\infty e^{-st} e^{At} x dt$ for all $x \in \mathcal{H}$, $\| (Is - A)^{-n} \| \leq M / (\operatorname{Re} s - \beta)^n$ for $\operatorname{Re} s > \beta$, and $\| e^{At} \| \leq M e^{\beta t}$.

We say that the operators $\Phi(t)$ form a C_0 -group whenever i) above holds for $t, s < 0$ as well. This means that A and $-A$ generate C_0 -semigroups, and hence by the Hille-Yosida Theorem one gets necessary and sufficient conditions for an operator A to generate a C_0 -group ([18], [21]).

We say that the operators $\Phi(t)$ form an analytic semigroup whenever i), ii), iii) above hold for t, s in the sector $|\arg t| < \alpha < \frac{\pi}{2}$ of the complex plane. Properties of analytic semigroups and their infinitesimal generators are described in [18], [53].

By $\ell_2(\mathbf{Z}^+)$ we denote the standard Hilbert space of sequences $\{a_i\}_{i=1}^\infty$ which are square summable. By $\ell_2(\mathbf{Z})$ we denote the space of sequences $\{a_i\}_{i=-\infty}^\infty$ which are square summable.

The open disk of radius ρ is denoted by $\mathcal{D}_\rho = \{s \mid |s| < \rho\}$. We write \mathcal{D} for \mathcal{D}_1 . The boundary of \mathcal{D} , the unit circle, is denoted by \mathcal{T} . By $H^2(\mathcal{D})$ we mean the set of complex valued functions which are

holomorphic in \mathbb{D} and have a Taylor series about zero with square summable coefficients. The space $H^2(\mathbb{D}_\rho)$ is defined by saying that $\psi(s)$ belongs to $H^2(\mathbb{D}_\rho)$ if and only if $\psi(s/\rho)$ belongs to $H^2(\mathbb{D})$. By $L^2(\mathbb{T})$ we mean the set of complex valued functions which are defined and square integrable, in the Lebesgue sense, on the unit circle. By $H^2(\mathbb{T})$ we mean the subspace of $L^2(\mathbb{T})$ of functions with vanishing negative Fourier coefficients. $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are related by the fact that for any function in $H^2(\mathbb{D})$ the radial limits from within the disk $\lim_{r \rightarrow 1} \psi(re^{i\theta}) = \phi(\theta)$ exist for almost all θ and give an element ϕ of $H^2(\mathbb{T})$. This correspondence is, moreover, one to one and onto so that $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are closely related indeed. In fact the Fourier coefficients of ϕ are the Taylor coefficients of ψ . In addition, $H^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle \psi_1, \psi_2 \rangle = \sum_{n=0}^{\infty} \bar{\alpha}_n \beta_n$$

where $\psi_1(s) = \sum_{n=0}^{\infty} \alpha_n s^n$ and $\psi_2(s) = \sum_{n=0}^{\infty} \beta_n s^n$

This makes $H^2(\mathbb{D})$, $H^2(\mathbb{T})$ and $\ell_2(\mathbb{Z}^+)$ isomorphic as Hilbert spaces with the isomorphisms defined by

$$(a_0, a_1, a_2, \dots) \leftrightarrow \sum_{i=0}^{\infty} a_i s^i \leftrightarrow \sum_{n=0}^{\infty} a_n e^{in\theta}$$

We denote by Π_ρ^+ the half-plane $\text{Re } s > \rho$. We understand by $H^2(\Pi_\rho^+)$ the space of functions which are analytic in Π_ρ^+ and square integrable along vertical lines in Π_ρ^+ such that

$$\sup_{x>0} \int_{-\infty}^{+\infty} |\psi(x+iy)|^2 dy \leq M < \infty$$

The relationship between $H^2(\mathbb{D})$ and $H^2(\Pi^+)$ is this: $\phi(\cdot) \in H^2(\Pi^+)$ if and only if ψ defined by

$$\psi(s) = \frac{1}{s-1} \phi\left(\frac{s+1}{s-1}\right)$$

belongs to $H^2(\mathbb{D})$. (See Hoffman [7] page 130).

We would like to recall some of the facts from Fourier Transform theory that involve $H^2(\Pi^+)$ and especially the Paley-Wiener theorem. We denote by \mathbb{I} the imaginary axis in the complex plane. It is well known that the Fourier transform

$$g(t) \xrightarrow{\mathfrak{F}} \int_{-\infty}^{\infty} e^{-i\omega t} g(t) dt = G(i\omega)$$

is a unitary map between $L_2(-\infty, \infty)$ and $L_2(\mathbb{I}, \frac{d\omega}{2\pi})$. Consider $L_2(0, \infty)$ as the subspace of $L_2(-\infty, \infty)$ of functions which vanish on $(-\infty, 0)$, and $L_2(-\infty, 0)$ as the subspace of $L_2(-\infty, \infty)$ of functions which vanish on $(0, \infty)$. Then obviously $L_2(-\infty, 0) = L_2(0, \infty)^\perp$ in $L_2(-\infty, \infty)$. Moreover if we let $H^2(\mathbb{I}) = \mathfrak{F} L_2(0, \infty)$ and $\tilde{H}^2(\mathbb{I}) = \mathfrak{F} L_2(-\infty, 0)$ we see that $H^2(\mathbb{I})^\perp = \tilde{H}^2(\mathbb{I})$. $H^2(\mathbb{I})$ consists exactly of the boundary values of the elements of $H^2(\Pi^+)$ (which exist for almost all ω). Moreover if \mathcal{L} denotes the Laplace transform

$$g(t) \xrightarrow{\mathcal{L}} \int_0^{\infty} e^{-st} g(t) dt = G(s) \text{ for } g \in L_2(0, \infty)$$

$$f(t) \xrightarrow{\mathcal{L}} \int_{-\infty}^0 e^{-st} f(t) dt = F(s) \text{ for } f \in L_2(-\infty, 0)$$

the Paley-Wiener theorem says that $H^2(\Pi^+) = \mathcal{L} L_2(0, \infty)$. If we let Π^- denote the half-plane $\operatorname{Re} s < 0$, then also $H^2(\Pi^-) = \mathcal{L} L_2(-\infty, 0)$. Moreover $\tilde{H}^2(\mathbb{I})$ consists exactly of the boundary values of the elements of $H^2(\Pi^-)$. The relation between $H^2(\Pi^+)$ and $H^2(\Pi^-)$ is simple. A function $f(s)$ belongs to $H^2(\Pi^+)$ if and only if $\overline{f(-\bar{s})}$ belongs to $H^2(\Pi^-)$. Then obviously we see that $\tilde{H}^2(\mathbb{I}) = \overline{H^2(\mathbb{I})}$.

The spaces $H^2(\mathcal{D})$, $H^2(\mathbb{T})$, $H^2(\Pi^+)$, $H^2(\mathbb{I})$ are called Hardy spaces.

CHAPTER II

LINEAR REALIZABILITY THEORY

2.1 Examples of Systems with Uniformly Continuous State-Transition Operators

In this section we give some examples of systems where the state-transition operators are uniformly continuous, i.e. the operator A is bounded.

The first group of examples come from systems governed by parabolic and certain hyperbolic partial differential equations with constant coefficients, where the spatial domain is infinite or semi-infinite, after semi-discretization with uniform spatial mesh (see Birkhoff and R.S. Varga [23]).

Example 1: Consider the system

$$\left. \begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) x(t, z) &= g(z)u(t) \\ y(t) &= \int_0^{\infty} f(z)x(t, z) dz \end{aligned} \right\} \quad (1)$$

where $x(t, \cdot) \in L_2(0, \infty)$ and is such that $\frac{\partial^i}{\partial z^i} x(t, \cdot)$ ($i=1,2$) exist and belong to $L_2(0, \infty)$ for all t ; $x(\cdot, z)$ is twice differentiable; $f(\cdot) \in L_2(0, \infty)$; $g(\cdot), \frac{\partial}{\partial z} g(\cdot), \frac{\partial^2}{\partial z^2} g(\cdot) \in L_2(0, \infty)$; $x(t, z) = 0$ for $z \in (-\infty, 0)$ and the initial conditions $x(0, \cdot)$ and $\frac{\partial}{\partial t} x(0, \cdot)$ are given.

If we let $\frac{\partial}{\partial t} x(t, z) = \xi(t, z)$ we can write (1) as follows

$$\left. \begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x(t,z) \\ \xi(t,z) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial z^2} & 0 \end{bmatrix} \begin{bmatrix} x(t,z) \\ \xi(t,z) \end{bmatrix} + \begin{bmatrix} 0 \\ g(z) \end{bmatrix} u(t) \\ y(t) &= \int_0^\infty [f(z), 0] \begin{bmatrix} x(t,z) \\ \xi(t,z) \end{bmatrix} dz \end{aligned} \right\} \quad (2)$$

If we introduce a uniform spatial mesh of length h , let

$$\begin{aligned} x_m(t) &= x(t, mh) \\ \xi_m(t) &= \xi(t, mh) \\ g_m &= g(mh) \\ f_m &= f(mh) \end{aligned} \quad m = 0, 1, 2, \dots$$

and approximate the second spatial derivative with the standard central difference scheme and the integral with the usual Riemann sum, we obtain from (2) the equations

$$\frac{d}{dt} \begin{bmatrix} x_0(t) \\ \xi_0(t) \\ x_1(t) \\ \xi_1(t) \\ \vdots \\ x_{m-1}(t) \\ \xi_{m-1}(t) \\ x_m(t) \\ \xi_m(t) \\ x_{m+1}(t) \\ \xi_{m+1}(t) \\ \vdots \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} 0 & h^2 & 0 & 0 & 0 & 0 & \dots \\ -2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & h^2 & 0 & 0 & \dots \\ 1 & 0 & -2 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & h^2 & 0 & 0 & 0 & 0 & \dots \\ \dots & -2 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & h^2 & 0 & 0 & \dots \\ \dots & 1 & 0 & -2 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & h^2 & \dots \\ \dots & 0 & 0 & 1 & 0 & -2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_0(t) \\ \xi_0(t) \\ x_1(t) \\ \xi_1(t) \\ \vdots \\ x_{m-1}(t) \\ \xi_{m-1}(t) \\ x_m(t) \\ \xi_m(t) \\ x_{m+1}(t) \\ \xi_{m+1}(t) \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ g_0 \\ 0 \\ g_1 \\ \vdots \\ 0 \\ g_{m-1} \\ 0 \\ g_m \\ 0 \\ g_{m+1} \\ \vdots \end{bmatrix} u(t) \quad (3)$$

$$y(t) = h \sum_{m=0}^{\infty} f_m x_m(t)$$

where now $\{x_m(t)\}_{m=0}^{\infty}$, $\{f_m\}_{m=0}^{\infty}$, $\{g_m\}_{m=0}^{\infty}$ belong to $\ell_2(\mathbb{Z}^+)$. Clearly the operator A in (3) above belongs to $B(\ell_2(\mathbb{Z}^+))$.

Note: Many physical processes can be modeled by (1) or (and) its n-dimensional version e.g. transmission lines, wave propagation, waveguide systems, linearized elastic deformations.

Example 2: Consider the system

$$\left. \begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} \right) x(t, z) &= g(z)u(t) \\ y(t) &= \int_0^{\infty} f(z)x(t, z) dz \end{aligned} \right\} \quad (4)$$

where $x(t, \cdot) \in L_2(0, \infty)$ and is such that $\frac{\partial^i}{\partial z^i} x(t, \cdot)$ ($i=1,2$) exist and belong to $L_2(0, \infty)$ for all t ; $x(\cdot, z)$ is differentiable; $g(\cdot)$, $\frac{\partial}{\partial z} g(\cdot)$, $\frac{\partial^2}{\partial z^2} g(\cdot)$ and $f(\cdot)$ belong to $L_2(0, \infty)$; $x(t, z) = 0$ for $z \in (-\infty, 0)$ and $x(0, \cdot)$ is given.

Proceeding as in example 1, introducing a semi-discretization scheme we can write (4) in the form:

$$\frac{d}{dt} \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_{m-1}(t) \\ x_m(t) \\ x_{m+1}(t) \\ \vdots \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 1 & -2 & 1 & 0 & \dots \\ \dots & 0 & 1 & -2 & 1 & \dots \\ \dots & 0 & 0 & 1 & -2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_{m-1}(t) \\ x_m(t) \\ x_{m+1}(t) \\ \vdots \end{bmatrix} + \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{m-1} \\ g_m \\ g_{m+1} \\ \vdots \end{bmatrix} u(t) \quad (5)$$

$$y(t) = h \sum_{m=0}^{\infty} f_m x_m(t)$$

where as before the sequences $x_m(t)$, f_m , g_m belong to $\ell_2(\mathbb{Z}^+)$.

Again it is obvious that the operator A in (5) belongs to $B(\ell_2(\mathbb{Z}^+))$.

Note: Many physical phenomena and industrial as well as engineering problems can be modeled by (4) and several different versions of it or (and) its n-dimensional analogue e.g. problems in heating furnaces, diffusion processes, chemical processes.

Remark 1: The same semi-discretization process, applied to Cauchy problems with differential operators of the form

$$\frac{\partial x_j}{\partial t} = \sum_{k=1}^n p_{jk}(D_1, \dots, D_r) x_k; \quad D_i = \frac{\partial}{\partial x_i}$$

where p_{jk} are polynomials, will give systems governed by ordinary (but operational) differential equations in Banach spaces (in general) with the operator A (the so called infinitesimal transition operator) being bounded. Also the same technique works for similar linear problems with variable (but time invariable) coefficients. However, it is shown in [23] §5, that it is for problems with constant coefficients that the semi-discrete schemes work best. Namely it is shown that for any linear Cauchy problem with constant coefficients which is "well-set" in the sense of Petrowsky, one can by using central differences obtain semi-discretizations having arbitrarily high order of consistency which are also numerically stable.

Remark 2: In the examples above, we assumed distributed control and distributed observation. The same technique, of course, will work with boundary control and boundary observation which are more close to practical situations. As a matter of fact the semi-discretization procedure can be used effectively to study problems like these which at first glance look rather complicated.

The second group of examples come from systems governed by certain particular classes of partial differential equations.

Example 3: Consider the system

$$\left. \begin{aligned} -\frac{\partial}{\partial t} x(t,z) + \frac{\partial}{\partial t} \frac{\partial}{\partial z} x(t,z) - \frac{\partial}{\partial z} x(t,z) - x(t,z) &= b(z)u(t) \\ y(t) &= x(t,0) \end{aligned} \right\} \quad (6)$$

where $x(t, \cdot) \in \mathcal{D}_0\left(\frac{\partial}{\partial z}\right) = \{h \in L_2(0, \infty) \text{ such that } h \text{ is locally absolutely continuous and } \frac{\partial}{\partial z} h \in L_2(0, \infty)\}$; $x(\cdot, z)$ and $\frac{\partial}{\partial z} x(\cdot, z)$ differentiable; $b(\cdot) \in L_2(0, \infty)$. By the properties of the operator $\frac{\partial}{\partial z}$ with domain $\mathcal{D}_0\left(\frac{\partial}{\partial z}\right)$ in $L_2(0, \infty)$ (see sec. 2.6 of this thesis, or [25] p. 38, or [11] p. 150) we know that $(I - \frac{\partial}{\partial z})^{-1}$ is an everywhere defined bounded operator on $L_2(0, \infty)$ (since 1 is in the resolvent set of $\frac{\partial}{\partial z}$). If we let

$$\xi(t, z) = \left(\frac{\partial}{\partial z} - I\right)x(t, z)$$

we can write (6) as follows:

$$\left. \begin{aligned} \frac{\partial}{\partial t} \xi(t, z) &= \left(\frac{\partial}{\partial z} + I\right) \left(\frac{\partial}{\partial z} - I\right)^{-1} \xi(t, z) + b(z)u(t) \\ y(t) &= \left(\frac{\partial}{\partial z} - I\right)^{-1} \xi(t, z) \Big|_{z=0} \end{aligned} \right\} \quad (7)$$

But the operator

$$\left(\frac{\partial}{\partial z} + I\right)\left(\frac{\partial}{\partial z} - I\right)^{-1}$$

is bounded (see [11]) p. 151) and obviously many properties of the system (6) can be deduced from the study of the properties of system (7).

For a complete description of the process involved in the above case, as well as for an abstract exposition of these ideas, the reader is referred to the book by Sz-Nagy-Foias [11] p. 141.

Example 4: Consider the system

$$\left. \begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial z} x(t,z) + \frac{\partial}{\partial t} x(t,z) - \frac{\partial}{\partial z} x(t,z) &= b(z)u(t) \\ y(t) &= x(t,1) \end{aligned} \right\} \quad (8)$$

where $x(t, \cdot) \in L_2(0,1)$, is absolutely continuous and $x(t,0) = 0$; $x(\cdot, z)$ and $\frac{\partial}{\partial z} x(\cdot, z)$ are differentiable $b(\cdot) \in L_2(0,1)$. Considering as domain of $-\frac{\partial}{\partial z}$ the subspace $\mathcal{D}_0(-\frac{\partial}{\partial z}) = \{h \in L_2(0,1) \text{ such that } h \text{ is absolutely continuous and } h(0) = 0\}$, we see that $(\frac{\partial}{\partial z} + I)^{-1}$ is a bounded operator on $L_2(0,1)$.

If we let $\xi(t,z) = (\frac{\partial}{\partial z} + I)x(t,z)$ we can write (8) as follows:

$$\left. \begin{aligned} \frac{\partial}{\partial t} \xi(t,z) &= \frac{\partial}{\partial z} (\frac{\partial}{\partial z} + I)^{-1} \xi(t,z) + b(z)u(t) \\ y(t) &= (\frac{\partial}{\partial z} + I)^{-1} \xi(t,z) \Big|_{z=1} \end{aligned} \right\} \quad (9)$$

It is easy to verify that the operator

$$\frac{\partial}{\partial z} (\frac{\partial}{\partial z} + I)^{-1} : L_2(0,1) \rightarrow L_2(0,1)$$

is bounded (indeed is given by $\xi(t,z) \mapsto \xi(t,z) - \int_0^z e^{-(z-\sigma)} \xi(t,\sigma) d\sigma$.

Again we can use system (9) to study system (8).

Example 5: The same idea carries over to second order systems like

$$\left. \begin{aligned} \frac{\partial^2}{\partial t^2} x(t,z) + \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial z} x(t,z) - x(t,z) &= b(z)u(t) \\ y(t) &= x(t,1) \end{aligned} \right\} \quad (10)$$

where $x(t, \cdot) \in L_2(0,1)$, is absolutely continuous and $x(t,0) = 0$;

$x(\cdot, z)$ and $\frac{\partial}{\partial z} x(\cdot, z)$ are twice differentiable; $b(\cdot) \in L_2(0,1)$.

Similarly as above $(\frac{\partial}{\partial z} + I)^{-1}$ is bounded and if we let $(\frac{\partial}{\partial z} + I)x(t,z) = \xi(t,z)$

we can write (10) as follows

$$\left. \begin{aligned} \frac{\partial^2}{\partial t^2} \xi(t,z) &= (\frac{\partial}{\partial z} + I)^{-1} \xi(t,z) + b(z)u(t) \\ y(t) &= (\frac{\partial}{\partial z} + I)^{-1} \xi(t,z) \Big|_{z=1} \end{aligned} \right\} \quad (11)$$

Denote $\frac{\partial}{\partial t} \xi(t,z)$ by $\phi(t,z)$ and obtain from (11):

$$\left. \begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} \xi(t,z) \\ \phi(t,z) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ (\frac{\partial}{\partial z} + I)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \xi(t,z) \\ \phi(t,z) \end{bmatrix} + \begin{bmatrix} 0 \\ b(z) \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} (\frac{\partial}{\partial z} + I)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi(t,z) \\ \phi(t,z) \end{bmatrix} \Big|_{z=1} \end{aligned} \right\} \quad (12)$$

Other examples for systems with uniformly continuous state-transition operators can be found in problems of queuing theory, and in systems governed by certain classes of integro-differential equations (see equation (9) in example 4 above).

2.2 The Class of Weighting Patterns which Admit Bounded Realizations

In this section we characterize the class of weighting patterns which admit bounded realizations.

Recall (see sec. 1.2) that the problem is the following:

Let $T : [0, \infty) \rightarrow \mathbb{R}^1$ be a continuous function of time. When can it be written as

$$T(t) = \langle c, e^{At} b \rangle$$

where $b, c \in \mathcal{H}$ (a Hilbert space) and $A : \mathcal{H} \rightarrow \mathcal{H}$ is bounded?

As is well known such a representation is possible for T with \mathcal{H} finite dimensional if and only if T is of exponential order and its transform

$$\tilde{T}(s) = \int_0^{\infty} e^{-st} T(t) dt; \quad \text{Re } s > \sigma_0$$

is rational. In the present case A is bounded; $\{e^{At}\}$ defines a uniformly continuous semigroup of operators (see [1], page 626), and since b , and c belong to \mathcal{H} we have

$$\langle c, e^{At} b \rangle \leq \|b\| \cdot \|c\| \cdot M \cdot e^{\|A\| |t|}, \quad t \in \mathbb{R}^1$$

where $\|e^{At}\| \leq M e^{\|A\| |t|}$, and the norms are as usual. Thus the class we are looking for includes only functions of exponential order. Moreover since A is bounded, $\langle c, e^{At} b \rangle$ is an entire function.

The following two theorems characterize in the time and frequency domain the set of realizable input-output maps.

Theorem 1: $T : [0, \infty) \rightarrow \mathbb{R}^1$ has a bounded realization if and only if T is an entire function of exponential order.

and also that $\tilde{T}(\infty) = 0$. Hence $\tilde{T}(s) = \langle c, b \rangle s^{-1} + \langle c, Ab \rangle s^{-2} + \langle c, A^2 b \rangle s^{-3} + \dots$ for $|s| > \|A\|$.

Theorem 2: The function $T : [0, \infty) \rightarrow \mathbb{R}^1$ has a bounded realization if and only if the Laplace Transform \tilde{T} of T is analytic at infinity and vanishes there.

Proof: The necessity follows clearly from the above. For the sufficiency since \tilde{T} is analytic at infinity and vanishes there, it has a power series expansion

$$\tilde{T}(s) = \sum_{i=0}^{\infty} a_i s^{-(i+1)} \quad \text{for } |s| > \gamma$$

for some finite γ . Then for $k > \gamma$ we have that the sequence

$$\left\{ \frac{|a_i|}{k^i} \right\}_{i=0}^{\infty} \in \ell_2(\mathbb{Z}^+).$$
 So take again

$$A = k \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & 0 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

$$b = \{1, 0, 0, \dots\}$$

$$c = \{a_0, \frac{a_1}{k}, \frac{a_2}{k^2}, \dots\}$$

and this completes the proof.

Remark 1: It is apparent from the above that if T has any bounded realization, then it can be realized by a multiple of the unilateral shift in $\ell_2(\mathbb{Z}^+)$ or by a multiple of the bilateral shift in $\ell_2(\mathbb{Z})$. To see the latter take

is implied by physical and asymptotic conditions. Therefore if these conditions exclude the possibility of considering the branch cuts for \tilde{T} in the finite plane, then \tilde{T} has no bounded realization.

Using several classical results on entire functions of exponential order we can describe the set of singularities of their Laplace transforms in more detail (we follow R. Boas [25] ch. 5).

Let $f(t)$ (t regarded as a complex variable) be an entire function of exponential order (exponential type in Boas's terminology). The indicator function of f is defined as

$$h(\theta) = \limsup_{r \rightarrow \infty} r^{-1} \log |f(re^{i\theta})| ; \theta \in [0, 2\pi]$$

Obviously $h(\theta)$ is finite or $-\infty$. Moreover h is continuous on $(0, 2\pi)$ and if $\theta_1 < \theta_2 < \theta_3$ with $\theta_2 - \theta_1 < \pi$, $\theta_3 - \theta_2 < \pi$ $h(\theta)$ satisfies

$$h(\theta_1)\sin(\theta_3 - \theta_2) + h(\theta_2)\sin(\theta_1 - \theta_3) + h(\theta_3)\sin(\theta_2 - \theta_1) \geq 0$$

Let K be a compact convex set in the complex plane \mathbb{C} . The supporting function of K is defined as

$$k(\phi) = \max_{s \in K} \operatorname{Re}(se^{-i\phi}); \quad \phi \in [0, 2\pi]$$

Pictorially this is illustrated in the following figure 1:

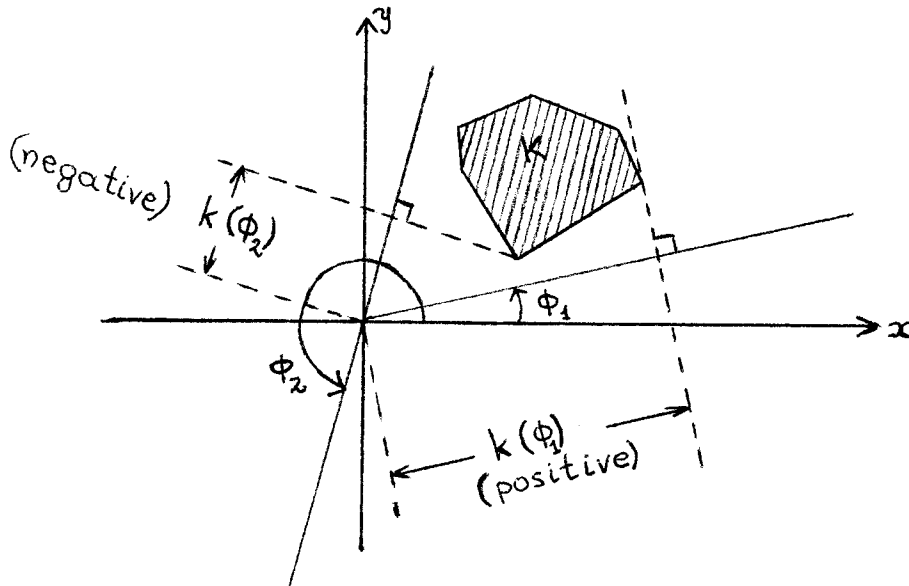


Figure 1: Illustrating the derivation of $k(\phi)$ for the set K

A function k on $(0, 2\pi)$ is the supporting function of some nonempty compact convex set if and only if (c.f. [25] p. 71)

i) k is continuous

ii) for $\phi_1 < \phi_2 < \phi_3$, with $\phi_2 - \phi_1 < \pi$, $\phi_3 - \phi_2 < \pi$

we have

$$k(\phi_1)\sin(\phi_3 - \phi_2) + k(\phi_2)\sin(\phi_1 - \phi_3) + k(\phi_3)\sin(\phi_2 - \phi_1) \geq 0$$

It is obvious from this fact and the properties of the indicator function of an entire function f of exponential order, that any indicator function is a supporting function for some compact convex set. The question which arises naturally is how this set is determined by f , and if it has any other relation with the properties of f . The answer is given by the following theorem.

Theorem 3: ([25] p. 73) Let f be an entire function of exp. order, h its indicator function and \tilde{f} its Laplace transform. If C_f is the smallest compact convex set outside which \tilde{f} is regular, the supporting function of C_f is $h(-\theta)$.

In other words h is the supporting function of a compact convex set, the indicator diagram of f , and C_f is the reflection of the indicator diagram with respect to the real axis, and is called the conjugate indicator diagram.

This theorem therefore gives a more precise description of the position of singularities of \tilde{T} , than the one given in Remark 3 above.

Now an extreme point of a convex set K is a boundary point which is not an interior point of a line segment belonging to the boundary of K . We have then that the extreme points of C_f are singular points of \tilde{f} (c.f. [25] p. 75).

A very important for us case is when the conjugate indicator diagram becomes a line segment. This will be explained in the following section. An example of this is the Bessel function of zeroth order J_0 for which we have $C_{J_0} = [-i, i]$.

2.3 Spectral Properties of the Infinitesimal Generators for Bounded Realizations in Relation with Classical Function Theory

In this section we derive some qualitative properties of the spectrum of an infinitesimal generator A which realizes T , and study its relation to the set of singularities of \tilde{T} .

We begin by discussing the relation $\tilde{T}(s) = \langle c, (Is-A)^{-1}b \rangle$ for

$\operatorname{Re} s > \gamma$, (where γ is large enough) in more detail, and drawing some conclusions from it. Since A here is a bounded operator the point at infinity is in the resolvent set of A , denoted $\rho(A)$. We denote by $\rho_0(A)$ the connected component of $\rho(A)$ containing the point at infinity, by $\sigma(A)$ the spectrum of A and by $\sigma_0(A)$ the complement in \mathbb{C} (the complex plane) of $\rho_0(A)$. The function $\langle c, (Is-A)^{-1}b \rangle$ is obviously analytic for $|s| > \|A\|$ and thus for $\operatorname{Re} s > \|A\|$. But T is of exponential order, say $|T(t)| \leq Me^{\sigma_0 t}$, and so \tilde{T} is analytic for $\operatorname{Re} s > \sigma_0$ (and also for $|s| > \sigma_0$), (see [20] p. 95). Since $T(t) = \langle c, e^{At}b \rangle$ we see that $\sigma_0 \leq \|A\|$. So from the equation $\tilde{T}(s) = \langle c, (Is-A)^{-1}b \rangle$ valid for $\operatorname{Re} s > \|A\|$ we deduce by analytic continuation that \tilde{T} is analytic for all $s \in \rho_0(A)$. If we let

$$\sigma(\tilde{T}) = \{s \in \mathbb{C} \mid \tilde{T} \text{ is not analytic at } s\}$$

we deduce that for any bounded realization $[A, b, c]$ of T we must have

$$\sigma(\tilde{T}) \subseteq \sigma_0(A)$$

This relation will be referred in the sequel as the "spectral inclusion property." For example the function $T(t) = e^{1/2 t}$ can obviously be

realized by the unilateral shift as above. Then for A being the

unilateral shift we have $\sigma(A) = \mathbb{D} \cup \mathbb{T}$ (i.e. the closed disk). But

\tilde{T} has just a pole at $s = \frac{1}{2}$. Consider now $T(t) = \sum_{n=0}^{\infty} \frac{t^{2^n-1}}{(2^n-1)!}$. Then

we know that $\tilde{T}(s) = \sum_{n=0}^{\infty} s^{-2^n}$ has \mathbb{T} as its natural boundary ([19]).

Obviously we can realize T using the construction of Theorems 1 or 2

of section 2.2 with any $k > 1$.

Realizations using infinitesimal generator A whose spectral properties reflect as close as possible the properties of the singularities of \tilde{T} are important (see sec. 1.2). We will study this problem in the next chapter in some detail, and we will exhibit there a class of transfer functions for which this can be achieved.

Remark 1: The realization constructed in Theorems 1 and 2 of section 2.2 uses the operator kU , where U is the unilateral shift on $\ell_2(\mathbb{Z}^+)$. The spectrum of kU is the closed disk of radius k , and hence its resolvent set is connected. The value of k we used is big enough so that the singularities of \tilde{T} are included in the disk of radius k . In other words we used an operator with spectrum large enough to include all the singularities of \tilde{T} . It is therefore of interest to know how small k can be taken for a given realizable weighting pattern T . It follows from a theorem in Widder [20], page 95, that if σ_0 is the exponential order (exponential type in Widder's terminology) of the entire function T then \tilde{T} will be analytic for $|s| > \sigma_0$ and will vanish at infinity, and conversely. Hence k in Theorems 1 and 2 of section 2.2 must satisfy $k \geq \sigma_0$.

The connectedness of the resolvent set of the infinitesimal generator A has important implications as far as the relationship to frequency response methods for system identification is concerned. The values of \tilde{T} for s purely imaginary are often empirically determined by letting $u(t) = \sin \omega t$ and looking at the periodic solution which results. If the periodic component of the response is $M(\omega) \sin(\omega t + \phi(\omega))$ then

$$\tilde{T}(i\omega) = Me^{i\phi(\omega)}$$

However, if the domain of analyticity of \tilde{T} is such that the entire imaginary axis does not belong to a single component then there is no way that experimental data taken in different components can be pieced together and we must regard the system as consisting of several unrelated parts.

We close this section by showing how representation theorems from classical function theory can be employed to construct bounded realizations.

For any entire function T of exponential order we have the Pólya representation which is given by the following theorem.

Theorem 1 ([25] p. 74): Let C be any contour containing the conjugate indicator diagram C_T of T , and let \tilde{T} be its Laplace transform.

Then

$$T(t) = (2\pi i)^{-1} \int_C \tilde{T}(s) e^{ts} ds$$

As follows from the discussion in section 2.2 and from Remark 1 above we can take always as C the circle $|s| = \rho$ provided $\rho > \sigma_0$, where σ_0 is the exponential order (type) of T . Let \mathbb{T}_ρ denote this cycle and consider $L_2(\mathbb{T}_\rho; \frac{d\theta}{2\pi})$. Then from the above formula we have

$$T(t) = \int_0^{2\pi} e^{\rho e^{i\theta}} t_{\rho e^{i\theta}} \tilde{T}(\rho e^{i\theta}) \frac{d\theta}{2\pi}$$

Obviously, if $h(\rho e^{i\theta}) = \rho e^{i\theta} \tilde{T}(\rho e^{i\theta})$, we have that $h \in L_2(\mathbb{T}_\rho; \frac{d\theta}{2\pi})$, and the operator

$$A : L^2(\mathbb{T}_\rho; \frac{d\theta}{2\pi}) \rightarrow L^2(\mathbb{T}_\rho; \frac{d\theta}{2\pi}) \tag{1}$$

$$g(\rho e^{i\theta}) \xrightarrow{A} \rho e^{i\theta} g(\rho e^{i\theta})$$

is bounded. Hence by taking A as above, $b = h$, $c = d$, where $d(\rho e^{i\theta}) \equiv 1$, and $\mathcal{H} = L^2(\mathbb{T}_\rho; \frac{d\theta}{2\pi})$ we have

$$T(t) = \langle d, e^{At} h \rangle$$

and hence we have constructed a bounded realization for f. We also note that for A as in (1) we have $\sigma(A) = \mathbb{T}_\rho$.

As we mentioned in section 2.2 an important case is when C_T becomes a line segment. Hence if we can shrink the contour in the Pólya representation so that it coincides with C_T we will get a representation for the function as an integral over a line segment. The most elegant and generally useful such case is whenever T belongs to L^2 on the real axis and is given by the Paley-Wiener Theorem (see Boas [25] p. 103):

"The entire function T is of exponential order (type) σ_0 and belongs to L^2 on the real axis if and only if

$$T(t) = \frac{1}{2\pi} \int_{-\sigma_0}^{\sigma_0} e^{i\omega t} \phi(i\omega) d\omega$$

where $\phi \in L^2([-i\sigma_0, i\sigma_0]; \frac{d\omega}{2\pi})$ "

Here ϕ is the Fourier transform of T (see sec. 1.3), and we have also that the conjugate indicator diagram of T is the segment $[-i\sigma_0, i\sigma_0]$ (see [25], p. 108).

Hence if we let \mathcal{H} to be $L_2([-i\sigma_0, i\sigma_0]; \frac{d\omega}{2\pi})$, $b = \phi$, $c = d$, where

$d(i\omega) \equiv 1$ for $i\omega \in [-i\sigma_0, i\sigma_0]$, and A be multiplication by $i\omega$ on $L_2([-i\sigma_0, i\sigma_0]; \frac{d\omega}{2\pi})$ we get from the above

$$T(t) = \langle d, e^{At} \phi \rangle$$

and hence a bounded realization for T . Moreover for this realization $\sigma(A) = [-i\sigma_0, i\sigma_0]$ which is the smallest compact convex set which contains the singularities of T .

We summarize these results in the following table.

Representation theorem from classical function theory	Class of weighting patterns	Bounded realization on \mathcal{H}	Spectrum of A	Has A connected resolvent?
Pólya	entire exp. order	$L_2(\mathbb{T}_\rho; \frac{d\theta}{2\pi})$ where $\rho > \sigma_0$	\mathbb{T}_ρ	NO
Paley-Wiener	entire exp. order L^2 on the real axis	$L_2([-i\sigma_0, i\sigma_0]; \frac{d\omega}{2\pi})$ where σ_0 is the exp. order	$[-i\sigma_0, i\sigma_0] = C_T$	YES

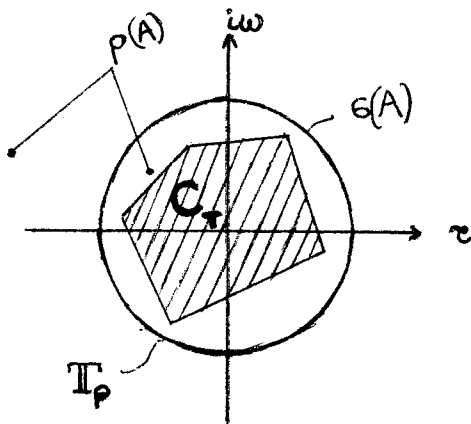


Fig. 2. Illustrating the relation between the conjugate indicator diagram C_T of A , for the realization obtained using Pólya's repres. theorem

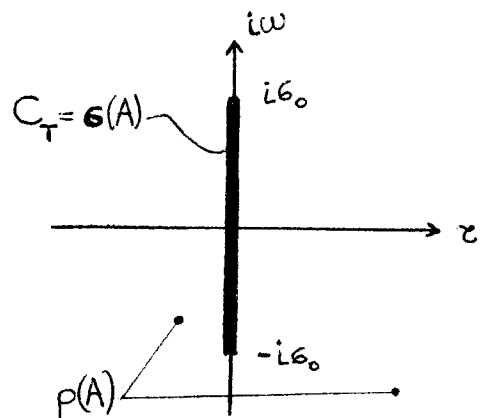


Fig. 3. Illustrating the relation between the conjugate indicator diagram C_T and the spectrum of A , for the realization obtained using Paley-Wiener theorem.

2.4 Examples of Systems with Strongly Continuous State Transition Operators

In this section we give examples of systems where the state transition operators form a C_0 semigroup of bounded linear operators on a Hilbert space \mathcal{H} (see sec. 1.3).

The examples 1,2 of section 2.1 are relevant here as well (before we apply semi-discretization of course).

Example 1 ([26] p. 3-5): Consider a continuous heating furnace. The control variable is the temperature of the heating medium $u(t,z)$; $0 \leq z \leq L$, $0 \leq t \leq T$. The material, which is to be heated, is moving with velocity $v(t)$. We let $x(t,z)$ represent the temperature distribution of the material, and we assume that the material is thin enough in the transverse direction. Then the heat transfer is described by the equations:

$$b \frac{\partial x(t,z)}{\partial t} + bv \frac{\partial x(t,z)}{\partial z} + x(t,z) = u(t,z) \quad (1)$$

with the boundary condition $x(t,0) = f(t)$. Here $b(t,z)$ is a function describing the thermophysical properties of the material, namely it gives the time constant associated with heating a thin layer of the material at time t , when placed at the position given by z in the furnace. Usually then one measures the temperature of the material at the exit of the furnace:

$$y(t) = x(t,L) \quad (2)$$

and this gives the output of the system. To simplify matters we assume that $v(t) = v$, $b(z,t) = b(z)$ and $u(t,z) = g(z)\xi(t)$, $f(t) = 0$. This type

of control is actually close to practical situations, where the method of heating is such that the spatial dependence of the temperature distribution of the heating medium in the furnace is fixed (known apriori from the geometry of the device) while the time dependence consists of variation of the amplitude according to the controls.

Then (1), (2) give the system

$$\left. \begin{aligned} \frac{\partial}{\partial t} x(t,z) + v \frac{\partial}{\partial z} x(t,z) + \frac{1}{b(z)} x(t,z) &= \frac{g(z)}{b(z)} \xi(t) \\ y(t) = x(t,L); x(t,0) &= 0 \end{aligned} \right\} \quad (3)$$

Where $x(t, \cdot)$ and $\frac{\partial}{\partial z} x(t, \cdot)$ belong to $L_2(0,L)$; g/b and $\frac{\partial}{\partial z} (g/b)$ also are elements of $L_2(0,L)$. It is easy to see that the operator $A = v \frac{\partial}{\partial z} + \frac{1}{b}$ with domain $\mathfrak{D}_0(A) = \{h \in L_2[0,L] \text{ such that } \frac{\partial}{\partial z} h \text{ exists and belongs to } L_2[0,L] \text{ and } h(0) = 0\}$, generates a C_0 semigroup on $L_2[0,L]$ ([1] p. 630). Moreover if we define c on $\mathfrak{D}_0(A)$ to be just evaluation at L we have obviously $y(t) = c[x(t,z)]$. It is quite reasonable to assume that $g(0) = 0$ (since we have already assumed that $x(t,0) = 0$) and it is then easy to verify that the realization $[A, g/b, c]$ is a balanced realization (see Sec. 1.2).

Example 2: Many systems (including the equations for probability distributions in stochastic systems) are governed by parabolic equations of the form

$$\left. \begin{aligned} \frac{\partial}{\partial t} x(t, \underline{z}) &= Ax(t, \underline{z}) + b(\underline{z})u(t) \\ y(t) &= \int_{\Omega} c(\underline{z})x(t, \underline{z})d\underline{z} \end{aligned} \right\} \quad (4)$$

where

$$A = \sum_{i,j}^n \frac{\partial}{\partial z_i} (a_{ij}(\underline{z}) \frac{\partial}{\partial z_j}) \quad (5)$$

and $\sum_{i,j=1}^n a_{ij}(\underline{z}) \xi_i \xi_j \geq \alpha (\xi_1^2 + \dots + \xi_n^2)$; $\alpha > 0$, $\xi_i \in \mathbb{R}$, almost everywhere in Ω , which is an open bounded subset in \mathbb{R}^n . Moreover a_{ij} are continuous, with continuous partial derivatives of first order in $\bar{\Omega}$. For the definition of weak partial derivatives we refer to [18] p. 11. Then usually $x(t, \cdot)$ belongs to $H^{2,2}(\Omega)$. (This Sobolev space consists of the elements of $L_2(\Omega)$ which have weak partial derivatives of order ≤ 2 also in $L_2(\Omega)$, is endowed with the usual Sobolev norm ([18] p. 14) and is a Hilbert space). $x(\cdot, \underline{z})$ belongs to $L_2[0, T]$. The initial condition $x(0, \underline{z}) = x_0(\underline{z})$ is given in Ω , while we have to satisfy certain boundary conditions on $\Sigma = \partial\Omega$ which is assumed to be of class C^2 .

In the most general situation the inputs (or controls) can be applied through the boundary conditions as well, and through a bounded operator B mapping the space of controls \mathcal{U} which is a Hilbert space, into $H^{2,2}(\Omega)$. However here we restrict for simplicity to the case of one dimensional controller and so B takes the form of an element b of $H^{2,2}(\Omega)$, as in equation (4) above. Similar remarks are valid for the observation process, and here again we assume distributed observation (some sort of weighted observation through the spatial domain) which is given via a linear bounded functional on $H^{2,2}(\Omega)$.

It is standard in the theory of partial differential equations [18], that A in (5) with domain $\mathcal{D}_0(A) = H^{2,2}(\Omega) \cap H_0^{2,1}(\Omega)$ generates a C_0 semigroup. (Here $H_0^{2,1}(\Omega)$ denotes the subspace of $H^{2,1}(\Omega)$ consisting

of functions with compact support in Ω). In fact e^{At} in this case is an analytic semigroup (see sec. 1.3 and [18] p. 101). For more general evolution equations of the parabolic type which lead to C_0 semigroups we refer to [18] and [27].

Example 3: Another important class of systems are those governed by hyperbolic equations of the form

$$\left. \begin{aligned} \frac{\partial^2}{\partial t^2} x(t, \underline{z}) &= Ax(t, \underline{z}) + b(\underline{z})u(t) \\ y(t) &= \int_{\Omega} c(\underline{z})x(t, \underline{z})dz \end{aligned} \right\} \quad (6)$$

where A and b are as in equations (4), (5) above, $x(t, \cdot)$ belongs to $H^{2,2}(\Omega)$, $x(\cdot, \underline{z})$ and $\frac{\partial}{\partial t} x(\cdot, \underline{z})$ belong to $L_2[0, T]$. The initial conditions $x(0, \underline{z})$ and $\frac{\partial}{\partial t} x(0, \underline{z})$ are given in Ω . Similar remarks as before (Example 2) are valid about controls and observations. In this case the evolution operators form a group (see sec. 1.3). Many physical phenomena can be described by equations similar to (6) (e.g. small elastic deformations, wave propagation, electromagnetic systems). For more general evolution equations of the hyperbolic type which give rise to C_0 semigroups we refer to [27].

Example 4: In this example we describe a class of systems governed by functional differential equations of the retarded type. Since we want to have a Hilbert space as state space we follow [29], [30] instead of the standard approach described in [28]. Here $M^2([-r, 0]; \mathbb{R}^n)$ denotes the Hilbert space obtained as the quotient of the space of measurable and

square integrable functions

$$h : [-r, 0] \rightarrow \mathbb{R}^n$$

by its linear subspace consisting of all elements for which the semi-norm

$$\|h\|_M^2 = (|h(0)|^2 + \int_{-r}^0 |h(t)|^2 dt)^{1/2}$$

vanishes. $AC^2([-r, 0]; \mathbb{R}^n)$ denotes the Hilbert space of absolutely continuous functions with square integrable derivatives

$$h : [-r, 0] \rightarrow \mathbb{R}^n$$

with the norm

$$\|h\|_{AC^2} = (|h(0)|^2 + \int_{-r}^0 \left|\frac{dh}{dt}\right|^2 dt)^{1/2}$$

Moreover let $L^2_{loc}([0, \infty); \mathbb{R}^n)$ denote the Fréchet space [21] of \mathbb{R}^n -valued measurable functions on $[0, \infty)$ which are square integrable on every compact subset of $[0, \infty)$, and $AC^2_{loc}([0, \infty); \mathbb{R}^n)$ denote the Fréchet space of all absolutely continuous \mathbb{R}^n -valued functions on $[0, \infty)$ with derivatives in $L^2_{loc}([0, \infty); \mathbb{R}^n)$. Let $\theta_1, \dots, \theta_N$ be reals and $-r = \theta_N < \dots < \theta_1 < \theta_0 = 0$.

Consider the hereditary differential system

$$\left. \begin{aligned} \frac{dx}{dt}(t) &= L(x(t)) + bu(t) \quad \text{for } t \in [0, \infty] \\ y(t) &= \langle c, x(t) \rangle_{\mathbb{R}^n} \\ \text{and } x(t) &= \phi(t) \text{ for } t \in [-r, 0]. \end{aligned} \right\} \quad (7)$$

where

$$L(x(t)) = A_0 x(t) + \sum_{i=1}^N A_i \left\{ \begin{array}{l} x(t+\theta_i), t+\theta_i \geq 0 \\ \phi(t+\theta_i), t+\theta_i < 0 \end{array} \right\} + \\ + \int_{-r}^0 K(\theta) \left\{ \begin{array}{l} x(t+\theta), t+\theta > 0 \\ \phi(t+\theta), t+\theta < 0 \end{array} \right\} d\theta$$

$x \in AC_{loc}^2([0, \infty); \mathbb{R}^n)$, $u \in L_{loc}^2[0, \infty)$, $\phi \in M^2([-r, 0]; \mathbb{R}^n)$, A_0, A_i are $n \times n$ matrices and $K(\theta)$ is an $n \times n$ matrix with elements in $L^\infty(-r, 0)$ and $b, c \in \mathbb{R}^n$.

Under these assumptions it is shown in [29], [30], that the system (7) can be described by an evolution equation in $M^2([-r, 0]; \mathbb{R}^n)$. The state of the system is defined as follows:

$$\hat{x}(t)(\theta) = \begin{cases} x(t+\theta), t + \theta \geq 0 \\ \phi(t+\theta), t + \theta < 0 \end{cases} \quad (8)$$

for $\theta \in [-r, 0]$ and $t \in [0, \infty)$

Then the map $t \rightarrow \hat{x}(t)$ generates a C_0 semigroup $S(t)$ of bounded operators on $M^2([-r, 0]; \mathbb{R}^n)$ with the following properties [29]:

(P₁) $t \rightarrow S(t)h$ is continuous for all $h \in M^2([-r, 0]; \mathbb{R}^n)$

(P₂) $S(0) = I$

(P₃) For $t \geq r$, $S(t)$ is compact

(P₄) The operator A defined by

$$(Ah)(\theta) = \begin{cases} A_0 h(0) = \sum_{i=1}^N A_i h(\theta_i) + \int_{-r}^0 K(\theta) h(\theta) d\theta & \text{for } \theta = 0 \\ \frac{dh}{d\theta}(\theta) & \text{otherwise} \end{cases}$$

is the infinitesimal generator for the semigroup $S(t)$ and has domain $\mathcal{D}_0(A) = AC^2([-r,0]; \mathbb{R}^n)$. If we let now $\hat{b}, \hat{c} \in M^2([-r,0]; \mathbb{R}^n)$ be defined as

$$\hat{b}(\theta) = \begin{cases} b & \text{if } \theta = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{c}(\theta) = \begin{cases} c & \text{if } \theta = 0 \\ 0 & \text{otherwise} \end{cases}$$

we can write actually the system (7) in the form

$$\left. \begin{aligned} \frac{d\hat{x}(t)}{dt} &= A\hat{x}(t) + \hat{b}u(t) \\ y(t) &= \langle \hat{c}, \hat{x}(t) \rangle \end{aligned} \right\} \quad (10)$$

$M^2([-r,0]; \mathbb{R}^n)$

and $\hat{x}(0) = \phi$

Since obviously systems involving delays are included in this class, this is a very important class of systems with strongly continuous state transition operators.

2.5 Balanced and Regular Realizations and their Equivalence

In case of a regular realization $b \in \mathcal{H}$ and c is a bounded linear functional on \mathcal{H} . Then the observation procedure (i.e. $y(t) = c[x(t)]$) is somehow restricted since we cannot have point evaluations, or point evaluations of derivatives as $c(\cdot)$. Moreover since b is just an element of \mathcal{H} we can regard in general the equation

$$\frac{d}{dt} x(t) = Ax(t) + bu(t) \quad (1)$$

only in the weak sense. On the other hand in a regular realization the properties of b and c are symmetric, a fact which has some implications on the desired duality in systems theory.

In case of a balanced realization $b \in \mathcal{D}_0(A)$ and c is a linear functional defined on $\mathcal{D}_0(A)$ and such that $|c(x)| \leq k(|Ax| + |x|)$ for all $x \in \mathcal{D}_0(A)$ and some constant k . Here we can regard equation (1) in the strong sense. Moreover we can allow point evaluations, or point evaluations of derivatives as $c(\cdot)$; (for example with A being $\frac{\partial^2}{\partial z^2}$ on $L_2[0, \infty)$ and $c(\cdot)$ being $\frac{\partial}{\partial z}(\cdot)|_0$, or with A being $\frac{\partial}{\partial z}$ on $L_2[0, \infty)$ and c being evaluation at 0). However in this case c and b do not have symmetric properties.

Remark 1: If c is a closed linear functional on \mathcal{H} with $\mathcal{D}_0(A) \subseteq \mathcal{D}(c)$ then c satisfies the conditions stated above in the case of a balanced realization. To see this we have that $\mathcal{D}_0(A)$ with the norm $\|x\|_1 = \|Ax\| + \|x\|$ becomes a Banach space since A is closed. Then the restriction of c to $\mathcal{D}_0(A)$ is a closed linear operator, defined everywhere and hence by the closed graph theorem is bounded. Hence $\exists k$ such that

$$|c(x)| \leq k\|x\|_1 = k(\|Ax\| + \|x\|)$$

for all $x \in \mathcal{D}_0(A)$.

The following Theorem proves that in our setting (more specifically when the state space is a Hilbert space) the class of weighting patterns which admit balanced realizations is identical with the class of weighting patterns which admit regular realizations.

Theorem 1: A weighting pattern $T(\cdot)$ has a balanced realization

if and only if it has a regular one. Moreover the infinitesimal generators in the two cases can be taken to be the same.

Proof: Suppose $T(\cdot)$ has a regular realization. Then $\exists c_1, b_1$ elements of \mathcal{H} , and a linear operator A generating a C_0 semigroup e^{At} on \mathcal{H} such that

$$T(t) = \langle c_1, e^{At} b_1 \rangle$$

By the Hille-Yosida theorem there exist a positive real number β such that for every real $\lambda > \beta$, λ is in the resolvent set of A . Pick such a $\lambda > 1$. Then $(\lambda I - A)^{-1}$ is an everywhere defined bounded operator, since A is closed (see [21] p. 209). Let

$$b = (\lambda I - A)^{-1} b_1$$

Then $b \in \mathcal{D}_0(A)$ and $b_1 = (\lambda I - A)b$. Hence

$$T(t) = \langle c_1, e^{At} (\lambda I - A)b \rangle = \langle c_1, (\lambda I - A)e^{At} b \rangle$$

Define the linear functional $c(\cdot)$ via

$$c(x) = \langle c_1, (\lambda I - A)x \rangle \quad \text{for } x \in \mathcal{D}_0(A)$$

Then $|c(x)| \leq \|c_1\| \|\lambda x - Ax\| \leq \|c_1\| (\lambda \|x\| + \|Ax\|) \leq \lambda \|c_1\| (\|Ax\| + \|x\|)$

Therefore $T(t) = c[e^{At} b]$ and this is a balanced realization.

Conversely, assume that $T(\cdot)$ has a balanced realization. Then $\exists A, b, c$ as in the definition of a balanced realization so that $T(t) = c[e^{At} b]$. Consider $\mathcal{D}_0(A)$ with the inner product $\langle x, y \rangle_A = \langle Ax, Ay \rangle + \langle x, y \rangle$ for $x, y \in \mathcal{D}_0(A)$. This inner product induces the norm $\|x\|_A = (\|Ax\|^2 + \|x\|^2)^{1/2}$. Since A is closed $\mathcal{D}_0(A)$ is complete under the norm $\|\cdot\|_A$ and hence it is

is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_A$. For $x \in \mathcal{D}_0(A)$ we have

$$|c(x)| \leq k(|\|Ax\| + \|x\||) \leq 2k(\|Ax\|^2 + \|x\|^2)^{1/2} = 2k\|x\|_A$$

So $c(\cdot)$ is a bounded linear functional on the Hilbert space $\mathcal{D}_0(A)$ (with the above inner product). Hence by the Riesz representation theorem there exists $d \in \mathcal{D}_0(A)$ such that

$$c(x) = \langle d, x \rangle_A \quad \text{for all } x \in \mathcal{D}_0(A).$$

$$\text{Hence } T(t) = \langle Ad, Ae^{At}b \rangle + \langle d, e^{At}b \rangle.$$

Since the space we are working with is a Hilbert space, A^* generates also a C_0 semigroup which is exactly $(e^{At})^*$. Hence if we pick a real $\lambda > \beta$ (β from the Hille-Yosida theorem) then both $(\lambda I - A)^{-1}$ and $(\lambda I - A^*)^{-1}$ are everywhere defined bounded operators.

We have then

$$\begin{aligned} T(t) &= \langle Ad, (A - \lambda I)e^{At}b \rangle + \lambda \langle Ad, e^{At}b \rangle + \langle d, e^{At}b \rangle \\ &= \langle Ad, e^{At}(A - \lambda I)b \rangle + \langle \lambda Ad + d, e^{At}b \rangle \end{aligned}$$

If we let $(A - \lambda I)b = b_1 \in \mathcal{H}$, then $b = (A - \lambda I)^{-1}b_1$. Therefore

$$\begin{aligned} T(t) &= \langle Ad, e^{At}b_1 \rangle + \langle \lambda Ad + d, e^{At}(A - \lambda I)^{-1}b_1 \rangle \\ &= \langle Ad + (A^* - \lambda I)^{-1}(\lambda Ad + d), e^{At}b_1 \rangle \end{aligned}$$

Let $c_1 = Ad + (A^* - \lambda I)^{-1}(\lambda Ad + d)$, then

$$T(t) = \langle c_1, e^{At}b_1 \rangle$$

and obviously $[A, b_1, c_1]$ is a regular realization for $T(\cdot)$.

The last statement in the theorem is obvious from the above construction.

This theorem motivates the following definition.

Definition: A weighting pattern $T(\cdot)$ is realizable iff it has a balanced realization.

In the next section we will make full use of the above theorem in our effort to characterize the class of realizable weighting patterns. In some cases it will be easier to construct balanced realizations, in other regular ones. It is this theorem which will make possible a more detailed study of this class, and it is the non-existence of such a theorem which obstructed previous efforts on this problem (see for example [31]).

2.6 On the Class of Realizable Weighting Patterns

We give first a preliminary description of the realizable weighting patterns.

Theorem 1: A necessary condition for T to be realizable is to be continuous and of exponential order. A sufficient condition is that it be locally absolutely continuous (i.e. absolutely continuous, on each bounded closed interval) and that $\dot{T}(\cdot)$ (which then exists as an a.e. defined function) be of exponential order (i.e. $\text{ess. sup } |\dot{T}(t)| \leq Ke^{\alpha t}$ for some positive K, α).

Proof: (Necessity) Since T has a balanced realization, and hence by Theorem 1 of section 2.5 it has a regular one, $T = \langle c, e^{At} b \rangle$. Since e^{At} is strongly continuous we get that T is continuous. Since

$\|e^{At}\| \leq Me^{\beta t}$ by the Hille-Yosida Theorem we get that T is of exponential order.

(Sufficiency): Let T be as in the hypothesis. Then for large enough σ , $e^{-\sigma t} \dot{T}(t) \in L_2(0, \infty)$. Hence the function $e^{-\sigma t} T(t)$ is in $L_2(0, \infty)$, it is locally absolutely continuous and its derivative belongs to $L_2[0, \infty)$. Take as b the function $e^{-\sigma t} T(t)$, and as Hilbert space the space $L_2(0, \infty)$. The differentiation operator $A = \frac{\partial}{\partial z}$ on $L_2(0, \infty)$ is a closed operator with domain dense, generates the semigroup of left translations (restricted to $[0, \infty)$ of course) and its spectrum is the closed left half plane (i.e. $\sigma(A) = \{s \in \mathbb{C} \mid \operatorname{Re} s \leq 0\}$) (see [18]). Its domain consists of elements of $L_2(0, \infty)$ which are locally absolutely continuous and their derivatives belong also to $L_2(0, \infty)$. Consider as c the linear functional whose action on a function f is described by

$$c[f] = f(0) \quad (\text{i.e. evaluation at } 0)$$

Then c is defined on $\mathfrak{D}_0(A)$. Moreover for $x \in \mathfrak{D}_0(A)$ we have:

$$\begin{aligned} |c(x)|^2 &= |x(0)|^2 \leq \int_0^\infty 2|x(z)| |\dot{x}(z)| dz \\ &\leq \int_0^\infty |x(z)|^2 dz + \int_0^\infty |\dot{x}(z)|^2 dz \end{aligned}$$

So $|c(x)| \leq (\|Ax\| + \|x\|)$. Hence b, c satisfy our requirements. Now $c[e^{At}b] = c[e^{-\sigma(t+z)}T(t+z)] = e^{-\sigma t}T(t)$ and therefore $T(t) = c[e^{(A+\sigma I)t}b]$ and this is a balanced realization.

From the equation $T(t) = \langle c, e^{At}b \rangle$ we get via Laplace transform the equation

$$\tilde{T}(s) = \langle c, (Is-A)^{-1}b \rangle \quad \text{for } \operatorname{Re} s > \beta$$

where the β comes from the Hille-Yosida Theorem.

The corresponding (to Theorem 1) conditions in the complex domain are described below.

Theorem 2: A necessary condition for T to be realizable is that its Laplace transform \tilde{T} belongs to $H^2(\Pi_\rho^+) \cap H^\infty(\Pi_\rho^+)$ for some $\rho > 0$. A sufficient condition is that $\tilde{T} \in H^2(\Pi_\rho^+)$ and $(s\tilde{T}-T(0)) \in H^2(\Pi_\rho^+)$ for some $\rho > 0$.

Proof: This is an immediate consequence of Theorem 1, the Paley-Wiener Theorem [7] and the Hille-Yosida Theorem.

Example: The delayed step whose transform is e^{-s}/s is not realizable. whereas the delayed ramp e^{-s}/s^2 is realizable.

Remark 1: Suppose T is continuous and of exponential order.

Let $b \in L_2(0, \infty)$ be the function $e^{-\sigma t}T(t)$ where σ is large enough. Let $c_\lambda \in L_2(0, \infty)$ be the function $c_\lambda(z) = \frac{2}{\pi} \frac{\lambda}{\lambda^2+z^2}$ and A be the differentiation operator. Then by Theorem 9.9 in [19] we have $e^{-\sigma t}T(t) = \lim_{\lambda \rightarrow 0} \langle c_\lambda, e^{At}b \rangle = \lim_{\lambda \rightarrow 0} \int_0^\infty e^{-\sigma(t+z)} T(t+z) c_\lambda(z) dz$. Hence $T(t) = \lim_{\lambda \rightarrow 0} \langle c_\lambda, e^{(A+\sigma I)t}b \rangle$. Therefore $T(\cdot)$ is the pointwise limit of a one parameter family of realizable functions.

In order to give some better sufficient conditions for realizability we need the following well-known result [22] from the theory of $H^p(\Pi^+)$ functions: If $F \in H^p(\Pi^+)$ $1 \leq p < \infty$, then it is represented by the proper Cauchy integral of its boundary values. That is for $\operatorname{Re} s > 0$ we have the representation

$$F(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{F(i\omega)}{s-i\omega} d(i\omega)$$

Theorem 3: Let $T \in L_2(0, \infty)$ and continuous. If $\tilde{T}(i\omega) = \overline{F_1(i\omega)} F_2(i\omega)$, where F_1, F_2 belong to $H^2(\mathbb{I})$, then T is realizable.

Proof: Certainly $\tilde{T}(i\omega) \in L_1(\mathbb{I}; \frac{d\omega}{2\pi})$. Hence since $T \in L_2(0, \infty)$ we have that

$$T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{T}(i\omega) e^{i\omega t} d\omega \quad \text{a.e.}$$

But since both sides are continuous the equality holds everywhere. So $T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F_1(i\omega)} e^{i\omega t} F_2(i\omega) d\omega$. But this equality says that if we take as Hilbert space $H^2(\mathbb{I})$, as b the function F_2 , as c the function F_1 and as A the operator induced on $H^2(\mathbb{I})$ by multiplication by $i\omega$ followed by restriction on $H^2(\mathbb{I})$ we have

$$T(t) = \langle c, e^{At} b \rangle$$

(where the inner product is that of $L^2(\mathbb{I}, \frac{d\omega}{2\pi})$). Hence $[A, b, c]$ is a regular realization for T , and by Theorem 1 of section 2.5 T is realizable.

Let us note that since the Fourier transform is a unitary map between $L_2(0, \infty)$ and $H^2(\mathbb{I})$, and multiplication by $e^{i\omega t}$ on $H^2(\mathbb{I})$ followed by projection on $H^2(\mathbb{I})$, corresponds to left translation on $L_2(0, \infty)$ restricted on $L_2(0, \infty)$, we can give also a realization of T in $L_2(0, \infty)$ by the left translation semigroup. Indeed if we let

$$f_1 = \mathcal{F}^{-1}(F_1), \quad f_2 = \mathcal{F}^{-1}(F_2)$$

and e^{At} = left translation semigroup restricted to $L_2(0, \infty)$ we have $T(t) = \langle f_1, e^{At} f_2 \rangle$. Moreover we can give a realization in terms of the

right translation semigroup on $L_2(0, \infty)$ since we have also

$$T(t) = \langle f_2, e^{A^*t} f_1 \rangle$$

with A, f_1, f_2 as above. (Note that $L_2(0, \infty)$ is invariant under right translations).

Note: If T satisfies the conditions of Theorem 3, then by the Paley-Wiener Theorem $\tilde{T} \in H^2(\Pi^+)$. Hence

$$\tilde{T}(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tilde{T}(i\omega) d(i\omega)}{s-i\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{F_1(i\omega)} F_2(i\omega) d}{s-i\omega}$$

and we could have used this approach in the proof.

Corollary 3.1: Suppose T is continuous and of exponential order. If for some α the function $T_1(t) = e^{-\alpha t} T(t)$ satisfies the conditions of Theorem 3 then T is realizable.

Proof: Of course if α is bigger than the exponential order of T then $e^{-\alpha t} T(t)$ belongs to $L_2(0, \infty)$. So we really have to check for the factorization only. Now, if T_1 satisfies Theorem 3 then $T_1(t) = \langle c, e^{At} b \rangle$. So $T(t) = \langle c, e^{(A+\alpha I)t} b \rangle$.

Remark 2: The conditions of Corollary 3.1 are weaker than those of Theorem 1. To see this observe first of all that continuity is required in both. Theorem 1 implies that for large enough σ the function $e^{-\sigma t} T(t) = T_1(t)$ belongs to $L_2(0, \infty)$, is locally absolutely continuous and its derivative belongs to $L_2(0, \infty)$. Hence $\tilde{T}_1(i\omega)$ and $i\omega \tilde{T}_1(i\omega)$ belong to $H^2(\mathbb{I})$ by the Paley-Wiener Theorem. But $(1-i\omega)\tilde{T}_1(i\omega) = G(i\omega)$ also belongs to $H^2(\mathbb{I})$. Hence $\tilde{T}_1(i\omega) = \frac{1}{1-i\omega} G(i\omega) = \frac{1}{1+i\omega} G(i\omega)$ and since

$\frac{1}{1+i\omega}$ belongs to $H^2(\mathbb{I})$ we see that T satisfies the conditions of Corollary 3.1 (that $T(0)$ may be nonzero is immaterial).

We give another sufficient condition for realizability.

Theorem 4: If $T \in L_2(0, \infty)$ is continuous and $T \in H^1(\Pi^+)$ then T is realizable.

Proof: Since T is continuous, belongs to $L_2(0, \infty)$ and $\tilde{T}(i\omega) \in L^1(\mathbb{I}; \frac{d\omega}{2\pi})$ we have that

$$T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{T}(i\omega) e^{i\omega t} d\omega$$

the equality holding everywhere. We know that $F \in H^1(\Pi^+)$ if and only if $F = F_1 F_2$ where $F_1, F_2 \in H^2(\Pi^+)$ (see [7] p. 134). Hence $\exists F_3, F_2 \in H^2(\Pi^+)$ so that $\tilde{T}(i\omega) = F_3(i\omega) F_2(i\omega) = \overline{F_1(i\omega)} F_2(i\omega)$ where $F_1 = \overline{F_3} \in \tilde{H}^2(\mathbb{I})$

Hence

$$T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F_1(i\omega)} e^{i\omega t} F_2(i\omega) d\omega$$

So by taking as Hilbert space \mathcal{H} the space $L_2(\mathbb{I}; \frac{d\omega}{2\pi})$, as A multiplication by $i\omega$, as c the function F_1 and as b the function F_2 , we get

$$T(t) = \langle c, e^{At} b \rangle$$

and T is realizable.

Again using Fourier transforms we can give in the above case a realization of T on $L_2(-\infty, \infty)$ using the left translation semigroup or the right translation semigroup.

Corollary 4.1: Let T be continuous and of exponential order. If for some α , $\tilde{T} \in H^1(\Pi_\alpha^+)$, then T is realizable.

Proof: By taking σ big enough, then, we can have the function $e^{-\sigma t} T(t)$ to satisfy the conditions of Theorem 4.

This last Theorem indicates some other classes of realizable functions. We need first a standard definition for fractional derivatives in the L_2 -sense, or equivalently for the Sobolev spaces of fractional order.

Definition: Let $0 \leq \gamma \leq 1$. Then $F \in L_2(0, \infty)$ has an L_2 -derivative of fractional order γ if and only if $\exists G \in L_2(0, \infty)$ such that $s^\gamma \tilde{F}(s) = \tilde{G}(s)$, where we always choose the branch of s^γ so that $\text{Re}(s^\gamma) > 0$ for $\text{Re } s > 0$. The space of all those F is usually denoted by H_γ^2 . (\tilde{F}, \tilde{G} are the Laplace transforms of F, G).

Corollary 4.2: If T is continuous and belongs to H_γ^2 for $\frac{1}{2} < \gamma \leq 1$ then T is realizable.

Proof: We have that $\tilde{T}(s)$ and $s^\gamma \tilde{T}(s) = G(s) \in H^2(\Pi^+)$. Hence $\tilde{T}(s) = \frac{1}{s^\gamma} G(s)$. Since for $\frac{1}{2} < \gamma \leq 1$ and for all $\alpha > 0$ we have trivially that $\frac{1}{(s+\alpha)^\gamma} \in H^2(\Pi^+)$ we get finally that for all $\alpha > 0$, $\tilde{T} \in H^1(\Pi_\alpha^+)$ and the result follows from Corollary 4.1.

Finally we have the obvious generalization of Corollary 4.2.

Corollary 4.3: If T is continuous and for some $\alpha > 0$, $e^{-\alpha t} T(t) \in H_\gamma^2$ with $\frac{1}{2} < \gamma \leq 1$, then T is realizable.

Some concluding comments about our results for the class of realizable weighting patterns are in order. We saw above that any realizable T has to be continuous, or if we interpret the integral

$$\int_0^t T(t-\sigma)u(\sigma) d\sigma$$

in the Lebesgue sense, it has to be equal a.e. to a continuous function. There are examples of systems for which the weighting pattern does not have this property. Typical cases are systems governed by partial differential equations with boundary control and boundary observation, which are important for practical applications. We give very briefly a simple but illustrating example.

Consider the following L, C, transmission line equation, where we normalized $L = C = 1$:

$$\frac{\partial}{\partial t} x(t, z) = - \frac{\partial}{\partial z} \psi(t, z)$$

$$\frac{\partial}{\partial t} \psi(t, z) = - \frac{\partial}{\partial z} x(t, z)$$

where $x(t, z)$ denotes the voltage and $\psi(t, z)$ the current along the line

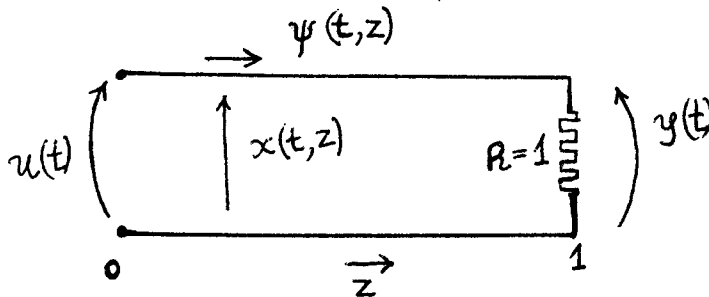


Figure 4: Illustrating a finite length transmission line

Suppose that the line is without load at $t = 0$, i.e. that $x(0, z) = \frac{\partial}{\partial t} x(0, z) = 0$, is of length 1, and is resistively terminated by its characteristic impedance which is 1, i.e. $x(t, 1) = \psi(t, 1)$. Our input is the voltage at 0, and the output is the voltage at 1. So

$$x(t, 0) = u(t)$$

$$y(t) = x(t, 1)$$

The input-output relation is then trivially shown to be

$$y(t) = \int_0^t K(t-\sigma)u(\sigma)d\sigma$$

where K is the delayed step function.

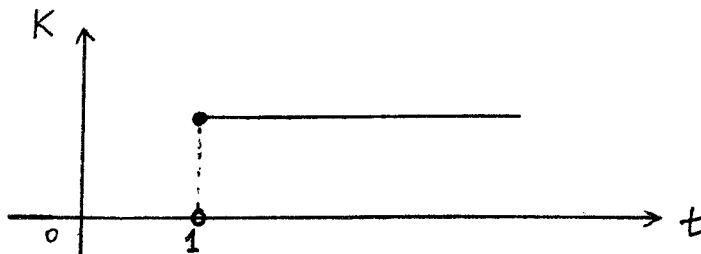


Figure 5: The delayed unit step function

For such systems we cannot construct models (realizations) similar to the ones described in the beginning of this section (balanced or regular).

It seems likely that by considering more general spaces (locally convex Hausdorff topological vector spaces, or spaces of distributions) as state spaces, one can develop a realizability theory to include discontinuous weighting patterns.

2.7 Qualitative Properties of the Spectra of the Infinitesimal Generators for Balanced Realizations

Suppose that $[A, f, g]$ is a balanced realization for T . Then by Theorem 1 of section 2.5, we have a regular realization $[A, b, c]$ and hence we can write

$$\tilde{T}(s) = \langle c, (Is-A)^{-1}b \rangle$$

for $\operatorname{Re} s > \gamma$ where γ is large enough. On the other hand since T is realizable we know it is of exp. order, say σ_0 . Hence \tilde{T} is analytic in $\operatorname{Re} s > \sigma_0$. Moreover from Theorem 1 of sec. 2.5 we have that $\sigma_0 \leq \beta$ and by the Hille-Yosida Theorem the function $\langle c, (Is-A)^{-1}b \rangle$ is analytic in $\operatorname{Re} s > \beta$. Let $\rho_0(A)$ be the connected component of $\rho(A)$ which contains the half-plane $\operatorname{Re} s > \beta$. Then by analytic continuation we see that \tilde{T} is analytic for all $s \in \rho_0(A)$. So again (as in the bounded case) we arrive at the conclusion, that for any realization $[A,b,c]$ of T we must have again the "spectral inclusion property"

$$\sigma(\tilde{T}) \subseteq \sigma_0(A)$$

where $\sigma_0(A)$ is the complement of $\rho_0(A)$ in \mathbb{C} .

For reasons similar to the ones discussed in section 2.3 for bounded realizations, a very significant property of an A realizing a weighting pattern T , is to have connected resolvent set.

Remark 1: The realization constructed in Theorem 4 of sec. 2.6, has as infinitesimal generator the differentiation operator on $L_2(-\infty, \infty)$, whose spectrum is just the imaginary axis. So in this model we do not have connected resolvent set. On the other hand in the realization of Theorem 3 of sec. 2.6, we do have connected resolvent set, but instead the spectrum becomes very large (in fact it becomes the closed left half-plane).

Remark 2: We see from the spectral inclusion property and from the Hille-Yosida-Theorem that the singularities of \tilde{T} , for any realizable

T , are in some left half-plane. In most of our constructions we used operators with spectrum large enough (in fact with spectrum some left half-plane) to include the singularities of a large class of realizable functions.

A deeper study of the relation between the singularities of \tilde{T} and the spectra of the infinitesimal generators which realize it will be given in section 6 of chapter III. However the results in this case (i.e. when A is unbounded) are less general and more of a special character than the corresponding ones for bounded realizations. This is due mainly to the fact that when A is bounded e^{At} is an analytic (in t) group of bounded transformations, while for A unbounded e^{At} can be analytic (for t in a sector) but not a group (c.f. diffusion processes), or a group but not analytic (c.f. the evolution operators in the wave equation). These restrictions on A have obvious reflections on its spectral properties. Namely when A generates an analytic semigroup (see sec. 1.3) its spectrum is contained in a sector $\frac{\pi}{2} + \theta \leq \arg s \leq \frac{3\pi}{2} - \theta$, $\theta > 0$, of the complex plane [18], and when A generates a group (see sec. 1.3) its spectrum is contained in a strip $-\alpha \leq \operatorname{Re} s \leq \alpha$, $\alpha > 0$, of the complex plane [21]. On the other hand a realization $[A, b, c]$ with A having any of the above properties, has additional physical properties. For example when A generates a group this implies that the system is time reversible. In sec. 3.1 (see Fig. 7) we will discuss other implications. Now it is obvious that if such an A realizes T , the spectral properties of A will give some more information about T . For example if A generates an analytic semigroup, T will be analytic in a sector $-\alpha < \arg t < \alpha$ and of

exponential order in this sector, and then results from classical function theory about this class of functions can be probably used to produce realizations (see sec. 2.3 where similar ideas were developed for bounded realizations).

So it appears that in this case a more effective study of the relation between $\sigma(\tilde{T})$ and $\sigma(A)$ can be done under additional assumptions on A (or equivalently on T).

2.8 Universal Models

We saw in the constructions of sections 2.2 and 2.6 that certain operators can realize a very large class of realizable weighting patterns. Namely in the case of bounded realizations any realizable weighting pattern can be realized by a multiple of the unilateral shift on $\ell_2(\mathbf{Z}^+)$, and in the case of balanced realizations a very large class can be realized by the differentiation operator plus a multiple of the identity on $L_2(0, \infty)$.

This fact is not a mere coincidence and is rather strongly suggested by certain general results in operator theory and in the theory of semi-groups of bounded operators.

Now the scalar multiple (in the case of the unilateral shift) and the addition of a multiple of the identity (in the case of the differentiation operator) have no essential significance as far as realization theory is concerned. Hence we can say that the unilateral shift and the differentiation operator are universal models for the infinitesimal transition operators in the case of bounded and balanced realizations respectively.

The purpose of this section is to relate these facts to similar ones in operator theory. G.C. Rota [9] was the first to notice that the unilateral shift can be thought as a universal model for contractions (bounded operators of norm less than one) on Hilbert spaces, and hence for bounded operators modulo a scaling factor. We give a very brief discussion of this fact following [40]. Let \mathcal{H} be a Hilbert space and A a contraction on \mathcal{H} such that $A^{*n} \rightarrow 0$. We let D_{A^*} denote the operator $(I-AA^*)^{1/2}$, \mathcal{D}_{A^*} denote the closure of $D_{A^*}\mathcal{H}$ and $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$ the Hilbert space of sequences $\mathcal{X} = \{x_n\}_0^\infty$ with $x_n \in \mathcal{D}_{A^*}$ and $\|\mathcal{X}\|^2 = \sum \|x_n\|^2 < \infty$. Since $\|x\|^2 = \|D_{A^*}x\|^2 + \|A^*x\|^2$, and therefore $\|x\|^2 = \sum_0^\infty \|D_{A^*}A^{*j}x\|^2$ we can embed \mathcal{H} in $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$ via the mapping

$$i : \mathcal{H} \rightarrow \ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$$

$$x \mapsto \mathcal{X} = \{D_{A^*}x, D_{A^*}A^*x, D_{A^*}A^{*2}x, \dots\}$$

Denoting by U the unilateral shift

$$\{x_0, x_1, \dots\} \xrightarrow{U} \{0, x_0, x_1, \dots\}$$

on $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$ we see that A^* is unit. equiv. to U^* restricted on $i(\mathcal{H})$ (the image of \mathcal{H} under i). Now obviously $i(\mathcal{H})$ is invariant under U^* in $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$ and we conclude that A is unitarily equivalent to the restriction of the right shift (on $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$) on a left translation invariant subspace of $\ell_2(\mathbb{Z}^+; \mathcal{D}_{A^*})$ (namely $i(\mathcal{H})$). Let N be a Hilbert space and consider the unilateral shift U on $\ell_2(\mathbb{Z}^+; N)$. The dimension of N is usually called the multiplicity of the unilateral shift U on

$\ell_2(\mathbb{Z}^+; N)$. Here \mathcal{D}_{A^*} can be finite or infinite dimensional and hence in general the unilateral shift of infinite multiplicity is the universal model for bounded operators. It is in this sense that the universality of the shift in operator theory must be understood. This simple fact was the beginning of a systematic study of contractions (and generally bounded operators) on Hilbert spaces by many authors (Halmos [10], Helson [8], L. DeBranges [41], Ronvjak [40], Sz-Nagy - C. Foias [11]) and lead to a complete study which is presented in the book by Sz-Nagy and C. Foias, "Harmonic Analysis of Operators in Hilbert Spaces," [11].

In the case of balanced realizations the universal model is the differentiation operator which generates the translation semigroup (left or right is immaterial). The corresponding theorem in the theory of semigroups of bounded operators is given in Lax and Phillips [39] p. 67 and states: "A strongly continuous semigroup of contractions on \mathcal{H} , which is asymptotically stable (i.e. $e^{At}x \rightarrow 0$ as $t \rightarrow \infty$ for all x in \mathcal{H}) can be represented isometrically as the left translation semigroup on an appropriate left translation invariant subspace of $L_2(0, \infty; N)$ where N is some auxiliary Hilbert space. Obviously a similar result holds when the semigroup is equibounded, i.e. $\|e^{At}\| \leq M$, and asymptotically stable, with the difference that the representation is not isometric. The requirement that the semigroup be asymptotically stable, is not restrictive since for any C_0 -semigroup $\|e^{At}\| \leq Me^{\beta t}$ and hence by multiplication by an appropriate exponential we can make any semigroup asymptotically stable (the same reasoning was used above,

when we called the differentiation operator the universal model).

There is however a difference between these analogies when A is bounded and when A is unbounded. When A is bounded (bounded realizations) the unilateral shift of multiplicity one is an actual universal model (realizes every realizable weighting pattern modulo a scalar factor). When A is unbounded (balanced realizations) however we were not able to show (c.f. section 2.6) that the differentiation operator on $L_2(0, \infty)$ (modulo an additive multiple of the identity) realizes any realizable weighting pattern. We did show that it realizes a very large class of realizable weighting patterns. Using the above stated result from Lax and Phillips it is very easy to see that any realizable weighting pattern, does have a realization via the translation semigroup (modulo an unimportant exponential factor) but on $L_2(0, \infty; N)$; where N may be an infinite dimensional Hilbert space. It is in this sense that the universality of the translation semigroup must be understood. The difference of course is, that in the bounded case we were able to prove that it suffices to take the corresponding N to be one dimensional, whereas in the unbounded case this does not appear to be true. Certainly, a very interesting question which we were not able to answer is the following: What are the relations between the dimension of N , needed in the realization of a weighting pattern T via the translation semigroup on $L_2(0, \infty; N)$, and T itself?

When we study finite dimensional realizations only, it is quite obvious that there are no universal models. The existence of universal models is rather strongly related to the infinite dimensionality of the

state space for the systems considered here.

These models can be used effectively to construct realizations with special properties as we will see in chapter III. They also can be employed to answer several questions about physical properties of the systems under consideration. An example of the latter is given in Lax and Phillips', "Scattering Theory," where the so called translation representation is used to study quite effectively several questions about physical phenomena described by the wave equation.

CHAPTER III

CONTROLLABILITY, OBSERVABILITY AND CANONICAL REALIZATIONS

3.1 Summary of Controllability and Observability Theory for Infinite Dimensional Systems

Here we give a brief review of the development of the theory of controllability and observability for linear infinite dimensional systems. We assume the state space to be a Hilbert space, but all the results can be stated immediately for reflexive Banach spaces and with a little care about adjoints to non-reflexive Banach spaces. The systems under consideration are described by the usual equations:

$$\left. \begin{aligned} \frac{d}{dt} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \right\} (\Sigma)$$

where the notation is as in sec. 1.2.

We denote by R_t the subspace of \mathcal{H} which consists of elements that can actually be reached from the zero state in time t . In other words

$$R_t = \{x \in \mathcal{H} \text{ s.t. } \exists u(\cdot) \in I.S. \text{ and } x = \int_0^t e^{A(t-\sigma)} Bu(\sigma) d\sigma\}$$

Obviously $R_{t_1} \subseteq R_{t_2}$ for $t_1 \leq t_2$ and we let $R = \bigcup_{t>0} R_t$.

Definition: A realization (Σ) is controllable whenever R is dense in \mathcal{H} .

Remark 1: Here we do not require that we can actually reach every state from the zero state in finite time, as we do in the finite dimensional case. Instead we require that we can actually reach (from the zero state)

infinite time some state in every neighborhood of any state. In case of one dimensional controls the requirement that we can reach in finite time every state implies finite dimensional state space (see [32]).

We recall that if e^{At} is a C_0 -semigroup on a Hilbert space \mathcal{H} , then $(e^{At})^*$ is also. The infinitesimal generator of $(e^{At})^*$ is A^* [24].

Theorem 1 (Fattorini): (Σ) is controllable if and only if

$$\bigcap_{t>0} \ker(B^*e^{A^*t}) = \{0\}.$$

Proof: Fattorini [4] p. 393.

We give a more geometric form to Theorem 1. For this we need the following definitions.

Definition: Let $F : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator and $M \subset \mathcal{H}$ a subspace (not necessarily closed). We say that M is a cyclic subspace of F if the span $\bigvee_{n=0}^{\infty} F^n(M)$ is dense in \mathcal{H} . When M is one-dimensional, say is spanned by the vector g , g is called a cyclic vector of the operator F .

Definition: Let e^{At} be a C_0 -semigroup on \mathcal{H} with generator A . A subspace $M \subset \mathcal{H}$ is a cyclic subspace of the semigroup e^{At} if the span $\bigvee_{t \geq 0} e^{At}(M)$ is dense in \mathcal{H} . Whenever M is one-dimensional, say generated by the vector g , g is called a cyclic vector of the semigroup e^{At} .

Remark 2: When A is bounded a subspace $M \subset \mathcal{H}$ is a cyclic subspace of A if and only if it is a cyclic subspace of e^{At} .

Using the above terminology we have

Corollary 1.1: (Σ) is controllable if and only if $R(B)$ (the range of B) is a cyclic subspace of the semigroup e^{At} .

Corollary 1.2: Suppose A is bounded, and we have one dimensional controls (i.e. $Bu(t) = bu(t)$, where $b \in \mathcal{H}$). Then (Σ) is controllable if and only if b is a cyclic vector of A .

It is well known that when A, B are finite dimensional, (Σ) is controllable if and only if $\text{rank}(A^{n-1}B, \dots, AB, B) = n$, where n is the dimension of \mathcal{H} . Hence the above conditions are generalizations of this condition.

Theorem 2 (Fattorini) [39]: (Σ) is controllable if and only if $\text{Ker } B^*(I\lambda - A^*)^{-1} = \{0\}$ for all $\lambda \in \rho_0(A)$, ($\rho_0(A)$ is defined in sec. 2.7).

Proof: Apply Laplace transform to Theorem 1.

Corollary 2.1: (Σ) is controllable if and only if $R(B)$ is a cyclic subspace of $(I\lambda - A)^{-1}$ for some $\lambda \in \rho_0(A)$.

Remark 3: There are several other notions related to controllability. We give a very brief discussion here and refer the reader to [35], [36], for details. We denote by C_t the subspace of \mathcal{H} which consists of the states we can drive to the zero state in time t . Let $C = \bigcup_{t \geq 0} C_t$.

In the finite dimensional case the chart in Figure 6 indicates that all conditions are equivalent for (Σ) . This is mainly due to the fact that in this case e^{At} is an analytic group of bounded operators.

In the present case however this chart is false. For example for a system governed by the wave equation (sec. 2.1, example 1) with boundary and distributed control, or with boundary control only the condition $\bar{R} = \mathcal{H}$ can be true, while the condition $\bar{R}_t = \mathcal{H}, \forall t$, is false (see [37]). For a system governed by the diffusion equation (sec. 2.1, example 2) the condition $\bar{C}_t = \mathcal{H}$, for some t (or $\bar{C} = \mathcal{H}$) can be true, while the condition

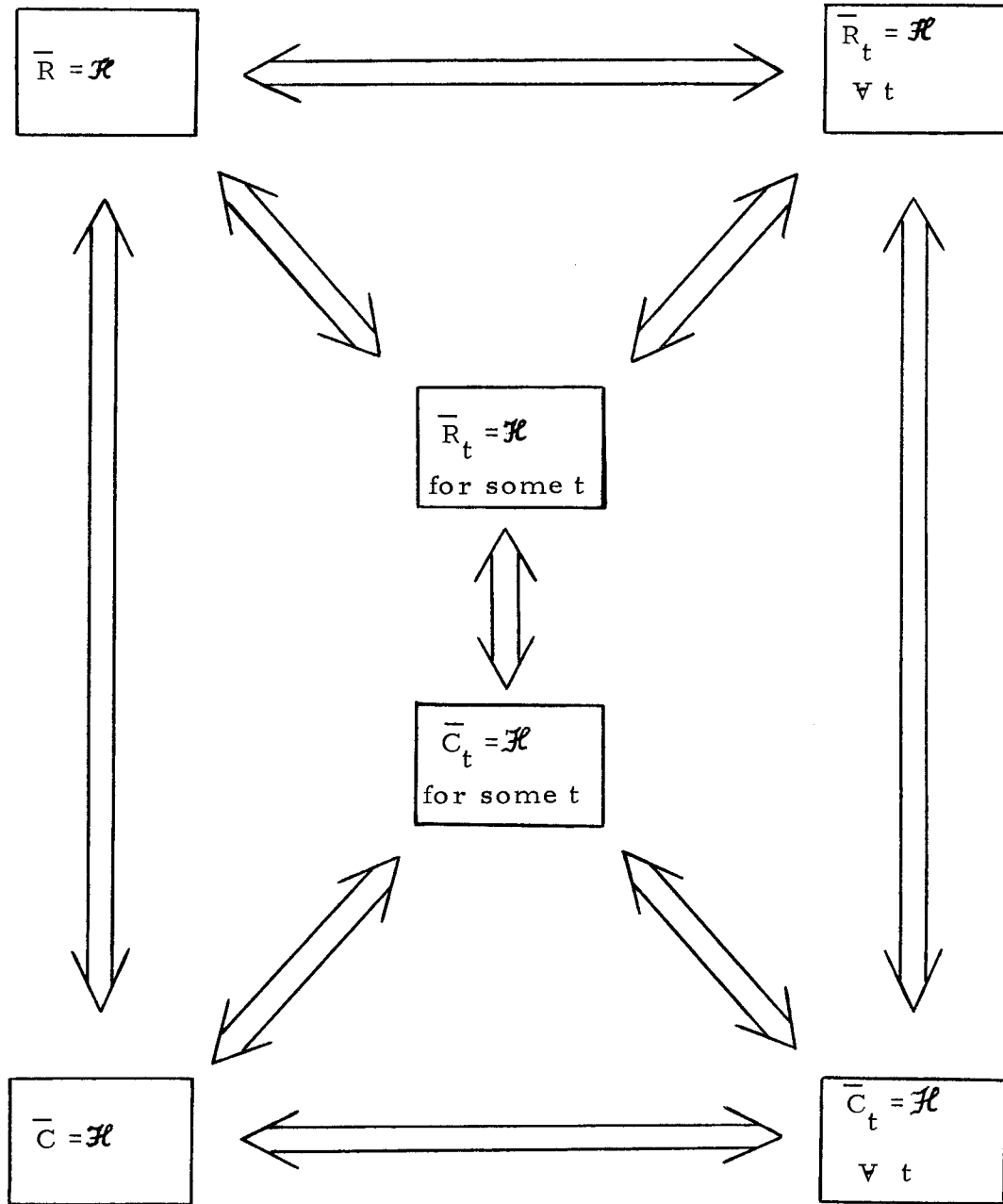


Figure 6: Equivalence of various controllability conditions for finite dimensional systems.

$\bar{R}_t = \mathcal{H}$, for some t (resp. $\bar{R} = \mathcal{H}$) are false (see [28]). The chart in Figure 7 indicates that under additional assumptions about the state-transition operators $\{e^{At}\}$, some of the conditions are equivalent.

We come now to the discussion of observability.

Definition: Two states x_1, x_2 are indistinguishable if whenever we start from x_1 or x_2 the outputs of the system are identical, no matter what input is used.

Definition: (Σ) is observable whenever there is no pair of indistinguishable states. When C is a bounded linear operator we have the following criteria for observability which are the duals of the criteria for controllability.

Theorem 3: (Σ) is observable if and only if $\bigcap_{t \geq 0} \ker(Ce^{At}) = \{0\}$.

Corollary 3.1: (Σ) is observable if and only if $R(C^*)$ is a cyclic subspace of the semigroup e^{A^*t} .

Corollary 3.2: When A is bounded and $Cx(t) = \langle c, x(t) \rangle$, $c \in \mathcal{H}$, (Σ) is observable if and only if c is a cyclic vector of A^* .

Theorem 4: (Σ) is observable if and only if $\ker C(I\lambda - A)^{-1} = \{0\}$ for all $\lambda \in \rho_o(A)$.

Corollary 4.1: (Σ) is observable if and only if $R(C^*)$ is a cyclic subspace of $(I\lambda - A^*)^{-1}$ for some $\lambda \in \rho_o(A^*)$.

Comments similar to the ones in Remark 3 above are relevant here as well.

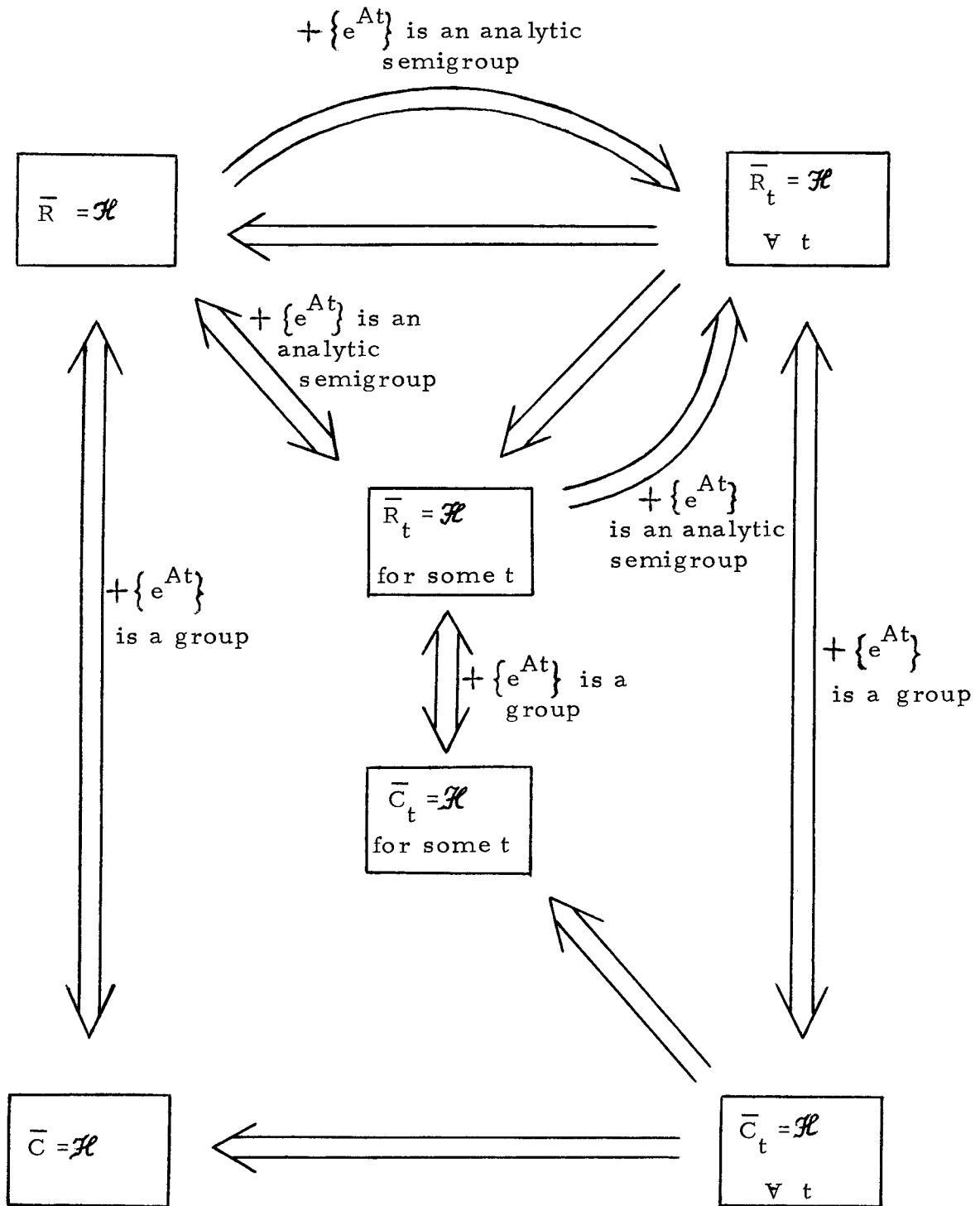


Figure 7: Relations between the properties of the state transition operators and the various controllability conditions for infinite dimensional systems.

Definition: A balanced realization is observable if and only if $x \in \mathcal{D}_0(A)$ and $C[e^{At}x] = 0$ for $t \geq 0$, imply $x = 0$.

Remark 4: This definition implies that any two states that can be reached exactly are distinguishable. It is motivated by Theorem 3 and the discussion above.

For the rest of this chapter we will assume one dimensional inputs and outputs.

3.2 Canonical Realizations

It is obvious that if a weighting pattern has one realization it has many. If a weighting pattern T has a finite dimensional realization, then it also has a realization whose state space has minimal dimension; such a realization is called minimal. Minimal realizations are desirable for many reasons, but mainly because they give the simplest internal models for a given external description (input-output map). The dimension of the state space provides a very good measure of the simplicity of a realization for the finite dimensional case. However in our case since we have infinite dimensional spaces the dimension of the state space is not useful as a measure of simplicity.

Definition: A realization is canonical whenever is controllable and observable (according to the definitions given in sec. 3.1).

We avoid the term minimal because many of the implications of this term are absent in the present setting.

As is well known (see [2], pages 105-115) a finite dimensional system is minimal if and only if it is controllable and observable (canonical).

Moreover any two minimal realizations differ by a change of basis in the state space and the spectral properties of A in any minimal realization are uniquely determined by the weighting pattern. Namely let $\tilde{T} = q/p$ where p, q polynomials with no common factor. The eigenvalues of A (i.e. $\sigma(A)$) in any minimal realization are exactly the set $\sigma(\tilde{T})$ multiplicities counted, or equivalently they are the zeros of p multiplicities counted, and p is the minimal polynomial of A . Here the situation is much more complicated. It happens that a canonical realization is much more loosely specified by the weighting pattern.

3.3 Canonical Bounded Realizations and the Hardy Class on Disks

In this section we study canonical realizations for the class of weighting patterns described in sec. 2.2. According to sec. 2 and 1 of this chapter a bounded realization $[A, b, c]$ is canonical whenever b is a cyclic vector for A , and c is a cyclic vector for A^* . The first question which arises naturally is: If a weighting pattern T has a bounded realization does it have a bounded canonical one?

The following theorem gives a construction of a canonical bounded realization starting from a given one.

Theorem 1: Let $T(t) = \langle c, e^{At} b \rangle$. Let M be the closure of the linear span of $c, A^*c, A^{*2}c, \dots$, in \mathcal{H} (a sep. Hilbert space.) Let P_M be the orthogonal projection on M . Then (i) $T(t) = \langle c, e^{P_M A P_M t} P_M b \rangle$.

Let now N be the closure of the linear span of $P_M b, \dots, (P_M A P_M)^i P_M b, \dots$

in M and let P_N be the orthogonal projection on N . Then (ii)
 $T(t) = \langle P_N c, e^{(P_N A P_N)t} P_M b \rangle$. Moreover N is the closure of the linear span of $P_M b, \dots, (P_N A P_N)^i P_M b, \dots$ and the closure of the linear span of $P_N c, \dots, (P_N A P_N)^{*i} P_N c$; i.e. $[P_N A P_N, P_M b, P_N c]$ is a canonical realization of T with state space N .

Proof: It is obvious that M is the smallest closed subspace of \mathcal{H} which contains c and is invariant under A^* . Hence M^\perp is invariant under A . Hence $A(I-P_M)x \in M^\perp, \forall x \in \mathcal{H}$. So $P_M A(I-P_M)x = 0, \forall x \in \mathcal{H}$. Hence

$$P_M A = P_M A P_M \quad (1)$$

Using (1) we get $(P_M A P_M)^i P_M b = P_M A^i b$.

Hence $\langle c, e^{P_M A P_M t} P_M b \rangle = \langle c, P_M e^{At} b \rangle = \langle c, e^{At} b \rangle = T(t)$ and this proves (i). Similarly N is the smallest closed subspace of M which contains $P_M b$ and is invariant under $P_M A P_M$. Then for every $x \in \mathcal{H}$, $(I-P_N)P_M A P_M P_N x = 0$. So

$$P_N A P_N = P_N P_M A P_M P_N = P_M A P_M P_N = P_M A P_N \quad (2)$$

Using (1), (2) we obtain

$$\begin{aligned} (P_N A P_N)^i P_M b &= P_M A P_N (P_N A P_N)^{i-1} P_M b = P_M A P_N P_N A P_N (P_N A P_N)^{i-2} P_M b = \\ &= P_M A P_M A P_N (P_N A P_N)^{i-2} P_M b = P_M A^2 P_N (P_N A P_N)^{i-2} P_M b = P_M A^i P_N P_M b = \\ &= P_M A^i P_M b = (P_M A P_M)^i P_M b \end{aligned} \quad (3)$$

So

$$\begin{aligned} \langle P_N c, e^{(P_N A P_N)t} P_M b \rangle &= \langle c, P_N e^{(P_N A P_N)t} P_M b \rangle = \langle c, e^{(P_M A P_M)t} P_M b \rangle = \\ &= \langle c, P_M e^{At} b \rangle = \langle c, e^{At} b \rangle = T(t). \end{aligned}$$

and this proves (ii).

From (2) we get

$$(P_N A P_N)^{*i} P_N c = P_N A^{*i} c \quad (4)$$

The first assertion in the last statement is proved, by (3) i.e. by the fact that $(P_N A P_N)^i P_M b = (P_M A P_M)^i P_M b$, for all i . The second is an easy consequence of (4) and of the cyclicity of c for A^* .

Let us focus our study to weighting patterns T which are realizable by the unilateral shift itself (i.e. we can take $k = 1$ in the construction of Theorems 1 or 2 of sec. 2.2, or equivalently that $\frac{1}{s} \tilde{T}(\frac{1}{s}) \in H^2(\mathbb{D})$). In this way the identification of the important issues becomes very simple. This does not diminish the generality of the discussion in any essential way, since we can reduce the general case by a simple change of variable, to the above case. Indeed if we define

$$\tilde{T}_k(s) = \sum_{i=0}^{\infty} \frac{a_i}{k^i} s^{-(i+1)} = k \tilde{T}(ks) = \mathcal{L} [T(\frac{t}{k})], \text{ (where } \mathcal{L} \text{ denotes Laplace transform), then } \tilde{T}_k \text{ satisfies the above for some finite } k > 0.$$

Some important questions, which arise naturally are the following. It is obvious that the realization given by Theorem 1 and 2 of sec. 2.2 is controllable. Also we know that the spectrum of the unilateral shift is the closed unit disk. Given a weighting pattern T , how simple can the spectrum of the infinitesimal generator A of a realization be? How small can the spectrum be? If we take a canonical realization $[A, b, c]$ is the spectrum of A uniquely determined by T ? How all canonical realizations of a given T are related to each other? When can we make the resolvent set of the infinitesimal generator A connected?

An immediate observation, which gives, however, some indication of the interplay of the notions described in these questions is the following: We can realize any such T by the bilateral shift. Such a realization is obviously non-canonical. On the other hand since the spectrum of the bilateral shift is just \mathbb{T} , the spectrum can be considered as simple. However the resolvent set is not connected.

Given a weighting pattern T we have the 'shift realization' as described in Theorems 1 and 2 of sec. 2.2.

$$\left. \begin{aligned} \frac{d}{dt} x(t) &= Ux(t) + bu(t) \\ y(t) &= \langle c, x(t) \rangle \end{aligned} \right\} \quad (5)$$

where $x(t) \in \ell_2(\mathbb{Z}^+)$ for all t , $b = \{1, 0, 0, \dots\}$, U is the unilateral shift and $c = \{T(0), T^{(1)}(0), T^{(2)}(0), \dots\}$. Here b is obviously a cyclic vector for U . It is immediately seen as a consequence of Theorem 1 above, that if we let M be the closed linear span of $c, U^*c, \dots, U^{*i}c, \dots$ and P_M the projection on M , then $[P_M U P_M, P_M b, c]$ is a canonical realization of T , with state space M . We can write the 'shift realization' in terms of H^2 functions as follows:

$$\left. \begin{aligned} \frac{d}{dt} x(t, s) &= sx(t, s) + u(t) \\ y(t) &= \int_{\mathbb{T}} s \tilde{T}(s) x(t, s) d\mu(s) \end{aligned} \right\} \quad (6)$$

where $x(t, \cdot) \in H^2$ ($H^2(\mathbb{D})$ or $H^2(\mathbb{T})$). (Compare with [17] where similar equations are used.) Under the isomorphism between $\ell_2(\mathbb{Z}^+)$ and $H^2(\mathbb{D})$ c corresponds to $\frac{1}{s} \tilde{T}(\frac{1}{s})$ which equals with $\overline{s\tilde{T}(s)}$ on \mathbb{T}

(since \tilde{T} has real Taylor coefficients). U corresponds to multiplication by s , U^* corresponds to the mapping: $f(s) \mapsto \frac{f(s)-f(0)}{s}$ on $H^2(\mathbb{D})$.

We usually denote $H^2(\mathbb{D})$ or $H^2(\mathbb{T})$ simply by H^2 .

We need a few well-known facts from the theory of H^2 functions and Toeplitz operators. The reader is referred to [7], [8], and [10] for further details. A non constant function $f \in H^2(\mathbb{D})$ is called inner if $|f(e^{i\theta})| = 1$ a.e. A function $f \in H^2(\mathbb{D})$ is called outer if it is a cyclic vector for the shift in $H^2(\mathbb{D})$. (i.e. the linear span of the functions f, sf, s^2f, \dots is dense in $H^2(\mathbb{D})$).

A Blaschke product is a function of the form

$$B(s) = s^k \prod_{j=1}^{\infty} \frac{a_j - s}{1 - \bar{a}_j s} \cdot \frac{\bar{a}_j}{|a_j|} \quad (7)$$

where k is nonnegative integer and a_j are complex numbers (not necessarily distinct) such that $0 < |a_j| < 1$, $\sum_{j=1}^{\infty} (1 - |a_j|) < \infty$.

A singular function is a function of the form

$$S(s) = \exp\left(- \int_{\mathbb{T}} \frac{e^{i\theta} + s}{e^{i\theta} - s} d\mu(\theta)\right) \quad (8)$$

where μ is any positive finite measure on $[0, 2\pi]$ which is singular with respect to the normalized Lebesgue measure. Every $f \in H^2(\mathbb{D})$ has a factorization $f = \phi \cdot h$ where ϕ is inner and h is outer. The factors are unique up to constant factors of modulus one. Any inner function has a factorization $\phi = cBS$, where c is a constant of modulus one, B is a Blaschke product, and S is a singular function. An inner function is normalized if we choose $c = 1$, or equivalently if we

require the first nonzero Taylor coefficient to be real and positive. Beurling showed that to every closed subspace M of $H^2(\mathbb{D})$ which is invariant under the shift (i.e. under multiplication by s) there corresponds a unique normalized inner function ϕ such that $M = \phi H^2(\mathbb{D})$ and conversely. We have also the corresponding facts for $H^2(\mathbb{T})$.

A Laurent operator on $\ell_2(\mathbb{Z})$ has a matrix representative which is constant on diagonals (i.e. $\alpha_{ij} = a_{i-j}$) and corresponds to multiplication by $\phi \in L^\infty(\mathbb{T})$ on $L_2(\mathbb{T})$, where $\phi(s) = \sum_{i=-\infty}^{\infty} a_i s^i$ and $a_i = \alpha_{i+k, k}$. A Toeplitz operator A , on $\ell_2(\mathbb{Z}^+)$ has a similar matrix representative which is infinite in one only direction and if $P : L^2(\mathbb{T}) \rightarrow H^2(\mathbb{T})$ is the associated orthogonal projection, we have

$$Af = P(\phi \cdot f), \quad \forall f \in H^2(\mathbb{T})$$

The only way the 'shift realization' can be canonical is if c is a cyclic vector for U^* , (i.e. for the backward shift) or equivalently if $\frac{1}{s} \tilde{T}(\frac{1}{s})$ is a cyclic vector for the backward shift on $H^2(\mathbb{D})$. In [5] the authors studied cyclic vectors of the backward shift very extensively. We are going to use some of their results and we refer to [5] for further details. In the rest of this section and the two subsequent sections when we say cyclic (or cyclic vector), non-cyclic (or non-cyclic vector), we always mean for the backward shift on H^2 . There exist many cyclic vectors for the backward shift on $H^2(\mathbb{D})$, as well as non-cyclic ones. The rational functions are non-cyclic. The authors give several ways of constructing cyclic vectors. Any H^2 function with isolated branch points on \mathbb{T} is a cyclic vector and also

any function with lacunary Taylor series and square summable Taylor coefficients is also a cyclic vector.

Since $f(s) \in H^2(\mathbb{D})$ is a cyclic vector for the backward shift if and only if $sf(s)$ is one, we have two cases to consider. Namely the case when $\tilde{T}(\frac{1}{s})$ is a cyclic vector for the backward shift and the case when $\tilde{T}(\frac{1}{s})$ is non-cyclic.

Also from [5] we have that the set of cyclic vectors is dense in $H^2(\mathbb{D})$ as is the set of non-cyclic vectors. However the set of non-cyclic vectors is a set of the first category, whereas the set of cyclic vectors is not. Hence the non-cyclic vectors are somehow much more rare than the cyclic ones. Moreover an element of $H^2(\mathbb{D})$ is non-cyclic if and only if there exist a sequence of rational functions (satisfying special conditions, see [5] Theorem 4.1.1) which converges to it in the $L_2(\mathbb{T})$ norm. A fact which indicates that the non-cyclic case is very much like the rational functions whereas the cyclic situation is new, harder, and potentially more interesting.

We return now to the study of the class of weighting patterns which admit bounded realizations. We describe a partition of this class which is induced by the cyclicity or non-cyclicity of the weighting pattern, in a sense that will be made precise in the sequel.

Recall from section 2.3, that for any function in our class we have the Pólya representation

$$T(t) = (2\pi i)^{-1} \int_C \tilde{T}(s) e^{ts} ds \quad (9)$$

where we can take as C any cycle of radius $k > \sigma_0$, with σ_0 being the

exponential order of T . In sec. 2.3 we used this representation to obtain a realization on $L^2(\mathbb{T}_p; \frac{d\theta}{2\pi})$. Here we show how it can be employed to reconstruct the realizations given in Theorems 1 and 2 of sec. 2.2, now considered on $H^2(\mathbb{T})$ or $H^2(\mathbb{D})$. We take A to be the operator of multiplication by ks restricted on H^2 , as b the function which is identically equal to 1 on $\mathbb{D} \cup \mathbb{T}$, and as c the function $\mathcal{G}\tilde{T}_k$. Where

$$\mathcal{G}\tilde{T}_k(s) \stackrel{\text{def}}{=} \frac{1}{s} \tilde{T}_k\left(\frac{1}{s}\right) \quad (\text{i.e. } \frac{k}{s} \tilde{T}\left(\frac{k}{s}\right)) \quad (10)$$

So we have the generalization of equations (6) above

$$\left. \begin{aligned} \frac{d}{dt} x(t,s) &= ks x(t,s) + u(t) \\ y(t) &= \int_{\mathbb{T}} s \tilde{T}_k(s) x(t,s) d\mu(\theta) \end{aligned} \right\} \quad (11)$$

It is trivial to verify that the equation for the weighting pattern of the system described by (11) is given by (9). Since $h \in H^2$ is a cyclic vector for U^* (the backward shift) on H^2 if and only if it is a cyclic vector for kU^* (for any real finite k) we conclude that the realization described by (8) is canonical if and only if $\mathcal{G}\tilde{T}_k$ is a cyclic vector for the backward shift.

We know (see [20] p. 95) that the radius of convergence for the Laurent series of \tilde{T} is σ_0 . Hence the radius of convergence for the Taylor series of $\mathcal{G}\tilde{T}_k$ is $\frac{k}{\sigma_0}$, which is >1 unless we can take $k = \sigma_0$ in (11) above. Let now Q be the subset of the boundedly realizable transfer functions, for elements of which the set of reals k , such that $\mathcal{G}\tilde{T}_k$ is in $H^2(\mathbb{D})$ is open (i.e. $k > \sigma_0$ in (11) above, as well as in

Theorems 1 and 2 of sec. 2.2). Let G be the complement of Q in the set of boundedly realizable transfer functions (i.e. for elements of G we can have $k = \sigma_0$ in (11)). Now theorem 2.2.4 in [5] states: "If f is holomorphic in $|s| < R$ for some $R > 1$, the f is either cyclic (for the backward shift) or a rational function." An immediate consequence of this is that for elements of Q , $\mathcal{J}\tilde{T}_k$ (from (10)) is either cyclic or a rational function for any admissible k . Then for elements of G $\mathcal{J}\tilde{T}_k$ is either cyclic or non-cyclic but not a rational function; indeed if $\mathcal{J}\tilde{T}_k$ is rational for some k , then obviously $\mathcal{J}\tilde{T}_{\sigma_0}$ is rational and since it has poles on \mathbb{T} it cannot belong to $H^2(\mathbb{D})$ which is a contradiction. In fact only $\mathcal{J}\tilde{T}_{\sigma_0}$ can be non-cyclic.

The motivation which leads us to the following definitions will become clear in sections 3.4 and 3.5.

Definition: An entire function T , of exponential order, say σ_0 , with non-rational Laplace-transform is irreducible if, either $\mathcal{J}\tilde{T}_{\sigma_0} \notin H^2(\mathbb{D})$, or $\mathcal{J}\tilde{T}_{\sigma_0} \in H^2(\mathbb{D})$ and is a cyclic vector for the backward shift.

Definition: An entire function T , of exponential order, say σ_0 , is reducible if, either it has rational Laplace-transform, or $\mathcal{J}\tilde{T}_{\sigma_0} \in H^2(\mathbb{D})$ and is not a cyclic vector for the backward shift.

Similarly we call the Laplace transform \tilde{T} of a reducible (resp. irreducible) weighting pattern T , a reducible (resp. irreducible) transfer function.

Summarizing we have the following partition of the set of boundedly realizable transfer functions, as shown in the picture:

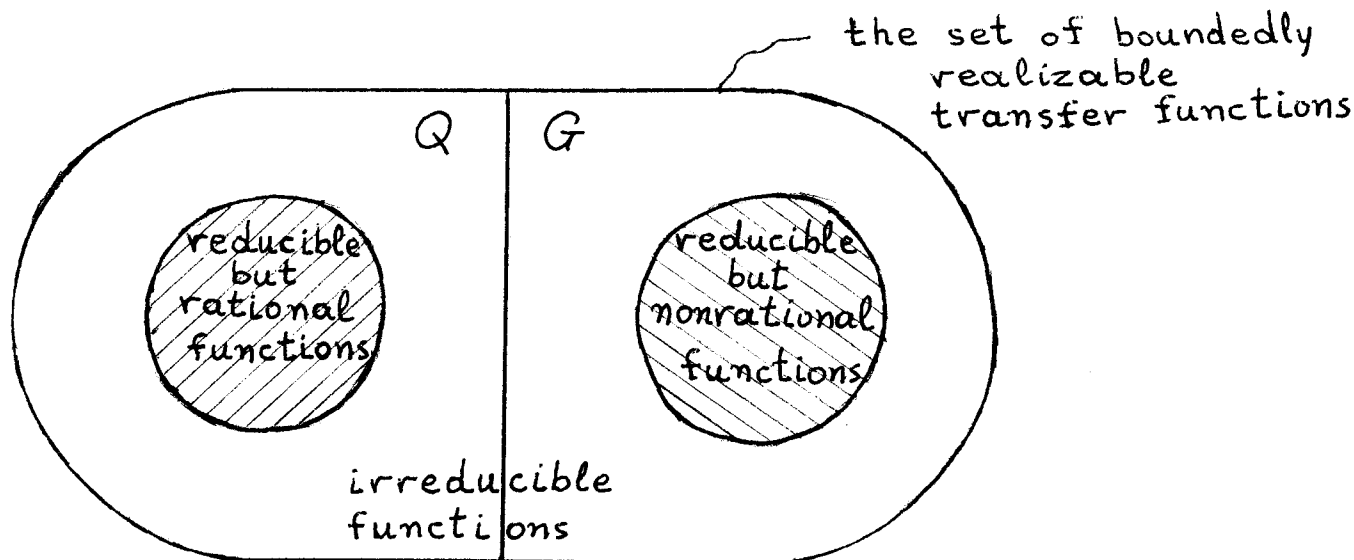


Figure 8: Illustrating a partition of the set of boundedly realizable transfer functions

The interesting subclasses for us are of course the reducible but nonrational transfer functions and the irreducible transfer functions and they will be examined separately in the following two sections.

3.4 Reducible Weighting Patterns and S-Minimal Bounded Realizations

Let a weighting pattern T be reducible, according to the definitions at the end of the previous section but with non-rational Laplace transform. Consider the realization given in (11) of sec. 3.3 (or in Theorems 1 and 2 of sec. 2.2) for T . For $k \neq \sigma_0$, $\tilde{\mathcal{I}}_k$ ((10) of sec. 3.3) is a cyclic vector for the backward shift and so this realization is canonical. However since the spectrum of the infinitesimal generator is $\mathbb{D}_k \cup \mathbb{T}_k$ and $\sigma(\tilde{T})$ may be just a countable set in $\mathbb{D}_k \cup \mathbb{T}_k$, this realization hardly

reflects the properties of the singularities of \tilde{T} . For $k = \sigma_0$, $\mathcal{G}_{\tilde{T}\sigma_0}$ is not a cyclic vector for the backward shift and so the realization given in (11) of sec. 3.3 is not canonical. Hence there is room for improvement, using the reduction described in Theorem 1 of sec. 3.3. We will show that this process leads to a canonical realization with an infinitesimal generator whose spectral properties reflect as close as possible the properties of $\sigma(\tilde{T})$. Moreover we will find many similarities between the class of transfer functions considered in this section and the rational transfer functions.

To simplify our discussion let us take for the moment $\sigma_0 = 1$. So we have a T of exp. order 1 and such that $\frac{1}{s} \tilde{T}(\frac{1}{s})$ is not a cyclic vector for the backward shift.

To proceed we need the following theorem from [5] p. 56.

Theorem 1 ([5]): $f \in H^2(\mathbb{D})$ is non-cyclic if and only if there exists $g \in H^2(\mathbb{D})$ and an inner function ϕ such that $f(e^{i\theta}) = e^{-i\theta} \overline{g(e^{i\theta})} \phi(e^{i\theta})$ a.e. on \mathbb{T} . Moreover if we require that ϕ be normalized and relatively prime to the inner factor of g , then ϕ and g are uniquely determined. In this case the closed subspace generated by $U^{*n}f$, $n = 0, \dots, \infty$ is precisely $(\phi H^2(\mathbb{D}))^\perp$.

The normalized inner function ϕ thus uniquely associated with each non-cyclic (for the backward shift) vector f is called the associated inner function of f .

We see immediately that the subspace M of $\ell^2(\mathbb{Z}^+)$ which is the state space for the canonical realization $[P_M \cup P_M, P_M^b, c]$ derived from

the shift realization using Theorem 1 of 3.3 corresponds to the closed subspace of H^2 generated by $\{U^{*n} \frac{1}{s} \tilde{T}(\frac{1}{s})\}_{n=0}^{\infty}$ which we call also M.

Hence applying Theorem 1 we get that

$$M = (\phi H^2)^\perp$$

where $\phi(e^{i\theta}) \overline{g(e^{i\theta})} = \tilde{T}(e^{-i\theta}) = \overline{\tilde{T}(e^{i\theta})}$ a.e. on \mathbb{T} (since $\tilde{T}(e^{i\theta})$

has real Fourier coefficients), and ϕ and g are uniquely determined by Theorem 1.

We need another theorem now from [6].

Theorem 2 ([6]): Let $K = \phi H^2$ (i.e. K is a closed subspace of H^2 invariant under the shift U). Let $M = (\phi H^2)^\perp$. Then the spectrum of U restricted on M is the set s_ϕ which consists of

- i) all the points in \mathbb{C} with $|\lambda| < 1$, where $\phi(\lambda) = 0$
- ii) all the points in \mathbb{C} with $|\lambda| = 1$, where ϕ is not continuable analytically across the boundary \mathbb{T} of \mathbb{D} at λ .

Using Theorems 1, 2 we see that the spectrum of the infinitesimal generator of the canonical realization $[P_M \cup P_M, P_M^b, c]$ is uniquely determined by T . Namely the spectrum consists of the zeros of ϕ in \mathbb{D} (which coincide with the zeros of the Blaschke product part of ϕ) and the points of \mathbb{T} through which ϕ is not continuable analytically outside the unit circle (which coincide with the union of the support of the measure on \mathbb{T} , which is associated with the singular part of ϕ , and the set of points of \mathbb{T} which are accumulation points of the sequence of zeros of ϕ . (See [7] p. 68-69)).

We have thus proved the following theorem.

Theorem 3: Let T be a given reducible weighting pattern of exp. order 1. Then there exist a canonical realization of T with the spectrum of the infinitesimal generator of the realization being exactly s_ϕ , where ϕ is the associated inner function of $\frac{1}{s} \tilde{T}(\frac{1}{s})$. This realization is constructed, by taking as state space the subspace of H^2 , $M = (\phi H^2)^\perp$, as c the function $\frac{1}{s} \tilde{T}(\frac{1}{s})$, as b the projection of 1 on M (i.e. $1 - \bar{\phi}(0)\phi$) and as A the restriction of the forward shift on M (i.e. $P_M \cup P_M$).

When σ_o is not equal to 1, then we factor $\mathcal{J}\tilde{T}_{\sigma_o}$ as $\mathcal{J}\tilde{T}_{\sigma_o}(e^{-i\theta}) = \frac{1}{g(e^{i\theta})\phi(e^{i\theta})}$, or $\tilde{T}(\sigma_o e^{i\theta}) = \frac{\frac{1}{\sigma_o} g(e^{i\theta})}{\phi(e^{i\theta})}$ since ϕ is inner. Now we let

\tilde{T}_o and \tilde{T}_i be the functions $\tilde{T}_o(s) = \frac{1}{\sigma_o} g(\frac{s}{\sigma_o})$, $\tilde{T}_i(s) = \phi(\frac{s}{\sigma_o})$. Then for every reducible weighting pattern with non-rational Laplace transform we have the factorization

$$\tilde{T}(\sigma_o e^{i\theta}) = \frac{\tilde{T}_o(\sigma_o e^{i\theta})}{\tilde{T}_i(\sigma_o e^{i\theta})}$$

a.e. on \mathbb{T}_{σ_o} (the circle of radius σ_o). We call \tilde{T}_i the associated inner factor of \tilde{T} .

We have now the following obvious corollaries of Theorems 2 and 3.

Corollary 2.1: Any reducible transfer function has a uniquely defined factorization $\tilde{T} = \frac{\tilde{T}_o}{\tilde{T}_i}$ on the circle of convergence of its

Laurent series, where $\sigma_0 \tilde{T}_0(\sigma_0 s)$ and $\tilde{T}_1(\sigma_0 s)$ are the uniquely determined factors of $\mathcal{J}\tilde{T}_{\sigma_0}$ given in Theorem 1 above.

Corollary 3.1: Any reducible weighting pattern has a canonical realization with the spectrum of the infinitesimal generator being the union of the set of zeros of \tilde{T}_1 in \mathbb{D}_{σ_0} and of the set of points of \mathbb{T}_{σ_0} through which \tilde{T}_1 cannot be continued analytically. This realization is constructed by taking as state space $M = (\phi H^2)^\perp$, as A the restriction of $\sigma_0 U$ (where U is the forward shift) on M , as b the projection of 1 on M (i.e. $1 - \bar{\phi}(0)\phi$) and as c the function $\mathcal{J}\tilde{T}_{\sigma_0}$; where ϕ is the associated inner function of $\mathcal{J}\tilde{T}_{\sigma_0}$.

An important case for practical applications is when \tilde{T} has a meromorphic continuation in \mathbb{D}_{σ_0} (for the definition of a meromorphic function in an open subset Ω of \mathbb{C} see [19] p. 260). To simplify the discussion take again $\sigma_0 = 1$. Then $\tilde{T}(s) = \frac{g(s)}{\phi(s)}$ in \mathbb{D} . Since g, ϕ are analytic on \mathbb{D} and ϕ is relatively prime to the inner factor of g , it follows that the singularities of \tilde{T} in \mathbb{D} , are exactly the zeros of ϕ in \mathbb{D} , (with the same multiplicity as well). On the other hand since $e^{-i\theta} \tilde{T}(e^{-i\theta}) \in (\phi H^2(\mathbb{T}))^\perp$ and $e^{-i\theta} \tilde{T}(e^{-i\theta})$ is a non-cyclic vector for the backward shift by assumption, we know ([5], p. 58-59, corollaries 3.1.8 and 3.1.10) that the set of points of \mathbb{T} , through which ϕ is analytically continuable coincides with the set of points of \mathbb{T} , through which $\frac{1}{s} \tilde{T}(\frac{1}{s})$ is analytically continuable. Hence the set of points of \mathbb{T} , through which ϕ is analytically continuable coincides with the set of points of \mathbb{T} through which $\tilde{T}(\frac{1}{s})$ is analytically continuable, which is the same as the set of points of \mathbb{T} , through which \tilde{T} is analytically

continuable (in the reverse direction). So in this case we arrive at the conclusion that the spectrum of $P_M U P_M$, (of Theorem 3) consists of the set of points of \mathbb{D} which are singularities of \tilde{T} and of points of \mathbb{T} through which \tilde{T} cannot be continued analytically. Obviously the last set is what we have defined as $\sigma(\tilde{T})$. Hence we obtain

$$\sigma(\tilde{T}) = \sigma(P_M U P_M)$$

When σ_0 is not 1, the same result holds for the A given in Corollary 3.1 above.

We see that in the above case the "spectral inclusion property" becomes in fact an equality, i.e. the spectrum of the infinitesimal generator of the realization described in Theorem 3 becomes "minimal". This motivates the following definition:

Definition: A canonical realization $[A,b,c]$ of a weighting pattern T is called S-minimal (S from spectrum) if $\sigma(A) = \sigma(\tilde{T})$ (multiplicities counted whenever possible).

We have thus proved the following:

Corollary 3.2: Any reducible transfer function \tilde{T} , which is meromorphic in an open disk, has an S-minimal realization, with A having connected resolvent set. This realization is given by Corollary 3.1.

This last case is very similar to the rational case. Compare our results described in Cor. 2.1, Cor. 3.1, and Cor. 3.2 above with the brief review of the theory for rational transfer functions in sec. 3.2. We can establish another similarity using the functional calculus for contractions described in [11]. Following [11] a contraction A on a

Hilbert space \mathcal{H} is completely non unitary (c.n.u.) if for no non-zero reducing subspace L of A , is the restriction of A to L a unitary operator. Given $v \in H^\infty(\mathbb{T})$ we have $v(s) = \sum_{k=0}^{\infty} c_k s^k$, $|s| < 1$ and we let $v_r(s) = \sum_{k=0}^{\infty} c_k r^k s^k$, $0 < r < 1$, $|s| < 1$. For a contraction A on H we denote by H_A^∞ the set of $v \in H^\infty$ for which $\lim_{r \rightarrow 1-0} v_r(A)$ exists in the strong sense, and for $v \in H_A^\infty$ we define

$$v(A) = s\text{-}\lim_{r \rightarrow 1-} v_r(A)$$

Whenever A is c.n.u. $H_A^\infty = H^\infty$.

A c.n.u. contraction A , such that there exist a nonzero function $v \in H^\infty$, with $v(A) = 0$, is called a C_0 contraction. For every contraction A of class C_0 there exists a minimal function m_A , in the sense that for every $v \in H^\infty$, $v \neq 0$, such that $v(A) = 0$, then v is a multiple of m_A . Moreover m_A is an inner function and is determined up to a constant of modulus 1.

Let s_{m_A} be the set consisting of the zeros of m_A in \mathbb{D} , and of the complement in \mathbb{T} , of the union of the arcs of \mathbb{T} on which m_A is analytic.

Then

$$\sigma(A) = s_{m_A}$$

The operator $P_M^{UP_M}$ of Theorem 3 is of class C_0 and its minimal function is $m_{P_M^{UP_M}} = \phi$, where ϕ is the associated inner function of $\frac{1}{s} \tilde{T}(\frac{1}{s})$.

So the associated inner factor of a reducible transfer function is the minimal function of the infinitesimal generator of the canonical

realization described in Cor. 3.1. Moreover when the transfer function is meromorphic in an open disk, the role of this factor in the realization problem resembles that played by the denominator of a rational transfer function.

The similarity between operators of class C_0 and finite dimensional matrices is further exploited in [15], [16].

We do not have a complete picture for the relation between canonical (resp. S-minimal) realizations of the same weighting pattern T , in the case considered here. It is clear however that we need more assumptions to establish anything like the state space isomorphism theorem of the finite dimensional theory (see sec. 3.2). An easy counterexample* is provided by the realization given in Cor. 3.1 above and any of the realizations of equation (11) of sec. 3.3 with $k \neq \sigma_0$ (for the same T of course). They are both canonical but obviously the corresponding infinitesimal generators are not similar.

3.5 Irreducible Weighting Patterns

This case, is very interesting since it reflects a number of physically interesting phenomena; e.g. transfer functions with branch points and branch cuts. Transfer functions like these arise in systems governed by partial differential equations. Hence an understanding of this case should undoubtedly shed some light towards the realization

*This counterexample was communicated to the author of this thesis by Dr. P.A. Fuhrmann.

problem for distributed systems.

The situation here is more difficult, since the associated inner factor of \tilde{T} which proves so crucial in the case of a reducible transfer function is now trivial. That is, the 'shift realization' and more generally all the realizations based on equation (11) of sec. 3.3 (or Theorems 1 and 2 of sec. 2.2) for irreducible transfer functions are already canonical. However the spectrum of the infinitesimal generators of these realizations is far from being equal to $\sigma(\tilde{T})$, unless we have a pathological transfer function with branch points on a dense subset of \mathbb{T}_{σ_0} . Hence canonical by no means imply S-minimal.

It is apparent from the "spectral inclusion property" that all the points on the branch cuts (if the transfer function has branch points) are included in the spectrum of any infinitesimal generator A with connected resolvent set which realizes the transfer function. However we have not uniquely defined branch cuts. Hence the set $\sigma(\tilde{T})$ is not uniquely determined and consequently there is not a unique "minimal spectrum" for the infinitesimal generators of the realizations. A reasonable expectation is that the spectrum of an S-minimal realization (provided there exists one) will be unique if there are no branch points and otherwise will be unique modulo the branch cuts.

The question we want to answer here is: Can irreducible weighting patterns have S-minimal realizations?

To this end we give an example of a realization for the Bessel function of zeroth order \mathcal{J}_0 which is S-minimal. It is easy to verify that

Hence the above A along with $b = c = [\dots 0 \ 0 \ 1 \ 0 \ 0 \ \dots]$ give a realization for \mathcal{Y}_0 , in $\ell_2(\mathbb{Z})$. That the spectrum of A is exactly $[i, -i]$, is a well known fact from [10]. However this realization is not canonical (it is easy to verify that the vector $[\dots 0010100\dots]$ is orthogonal to $A^i b$ for all i). We are going to use Theorem 1 of sec. 3.3 to reduce the above realization to a canonical one. So let M be the closed linear span of $c, A^*c, A^{*2}c, \dots$ in $\ell_2(\mathbb{Z})$. Then M is A^* invariant. But since $A^* = -A$ it is also A invariant, i.e. M reduces A . Since $b = c$, we get by Theorem 1 sec. 3.3 that $A_1 = A$ restricted to M , b, c is a realization of \mathcal{Y}_0 which is obviously canonical. Let $\lambda \in \rho(A)$. Then $\lambda I - A$ has a bounded inverse. But since M reduces A , $\lambda I - A_1$ has also a bounded inverse. Hence

$$\rho(A) \subseteq \rho(A_1) \tag{1}$$

Therefore $\rho(A_1)$ is connected. Using the spectral inclusion property, the fact that $\sigma(A) = \sigma(\tilde{\mathcal{Y}}_0)$ and (1) we have

$$\sigma(\tilde{\mathcal{Y}}_0) \subseteq \sigma(A_1) \subseteq \sigma(A) = \sigma(\tilde{\mathcal{Y}}_0) \tag{2}$$

So

$$\sigma(A_1) = \sigma(\tilde{\mathcal{Y}}_0) \text{ and } [A_1, b, c] \text{ is an S-minimal realization.}$$

It is easy to display an infinite matrix for A_1 . If we denote by e_n the n th basis vector in $\ell_2(\mathbb{Z})$ it is straightforward to verify that an orthonormal basis for M above, is given by the vectors, $\xi_0 = e_0$

and $\xi_n = \frac{1}{\sqrt{2}} (e_n + (-1)^n e_{-n})$, $n=1,2,\dots$. Evaluating the matrix elements for A_1 , with respect to this basis, we find that the realization $[A_1, b, c]$ above is given explicitly by

$$b = c = (1, 0, 0, 0, \dots)$$

$$A_1 = \frac{1}{2} \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & \dots \\ \sqrt{2} & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & -1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

This example shows that S-minimal realizations can exist for irreducible transfer functions as well. It also shows that there exists no Hilbert space analogue of the finite dimensional state space isomorphism theorem between two canonical realizations of the same T (consider the 'shift realization' and the one constructed above, for \mathcal{Y}_0) unless further assumptions are made (see Helton [14]).

Notice also that nearly the same realization will work for the Bessel function \mathcal{J}_n , where $\tilde{\mathcal{J}}_n(s) = \frac{1}{\sqrt{s^2+1}} \left(\frac{1}{s+\sqrt{s^2+1}} \right)^n$ provided we keep A, b as above and take $c = \{\dots 00100\dots\}$, with the 1 in the nth place.

3.6 Canonical Regular Realizations and the Hardy Class on Half-Planes

In this section we study canonical realizations for the class of weighting patterns described in sec. 2.6 (for the definitions of canonical balanced or regular realizations see sec. 3.1 and 3.2).

The following theorem gives a way to obtain a canonical regular realization for a weighting pattern T, starting from any regular realization of T.

Note: When L is a closed subspace of \mathcal{H} and B a linear operator, $P_L B|_L$ denotes the operator 'B restricted on L,' provided it is well defined.

Theorem 1: Let $[A,b,c]$ be a regular realization of a weighting pattern T on the Hilbert space \mathcal{H} . Let M be the closure of the linear span of the vectors $e^{At}b$ with $t \geq 0$, and P_M the associated orthogonal projection. Then

i) $[P_M A|_M, b, P_M c]$ is a regular realization of T, with state space M.

Let now N be the closed linear span in M of the vectors $P_M e^{A^*t} P_M^c$ with $t \geq 0$, and let P_N be the associated orthogonal projection ($P_N: \mathcal{H} \rightarrow N$).

Then

ii) $[P_N A|_N, P_N b, P_M c]$ is a canonical regular realization of T, with state space N.

Proof: Obviously M is invariant under e^{At} for $t \geq 0$. Hence

$$P_M e^{At} P_M = e^{At} P_M \quad (1)$$

Let $S(t) = P_M e^{At}|_M$. Then because of (1) $S(t_1)S(t_2) = S(t_1+t_2)$ for $t_1, t_2 \geq 0$. We have also that $S(0) = \text{identity on M}$ and that $S(t)$

is strongly continuous on M . Hence $S(t)$, $t \geq 0$, is a C_0 -semigroup on M . By the definition of the infinitesimal generator of a C_0 -semigroup (see sec. 1.3) and (1) it is clear that the infinitesimal generator of $S(t)$ is the operator $P_M A|_M = A|_M$, which has domain dense in M . Then we write in standard notation $S(t) = e^{(P_M A|_M)t}$. Now

$$\langle P_M c, e^{(P_M A|_M)t} b \rangle = \langle c, P_M e^{At} b \rangle = \langle c, e^{At} b \rangle = T(t)$$

Since M, \mathcal{H} are Hilbert spaces we have that

$$(S(t))^* = (P_M e^{At}|_M)^* = P_M e^{A^*t}|_M = e^{(P_M A^*|_M)t}$$

is also a C_0 -semigroup on M with infinitesimal generator $P_M A^*|_M$. Obviously N is invariant under $P_M e^{A^*t}|_M$ for $t \geq 0$ and so

$$P_N P_M e^{A^*t}|_N = P_M e^{A^*t}|_N$$

or

$$S_1(t) = P_N e^{A^*t}|_N = P_M e^{A^*t}|_N \quad (2)$$

Based on (2) we prove as above that $S_1(t)$ is a C_0 -semigroup on N , with infinitesimal generator the operator $P_N A^*|_N$. We write again in standard notation $P_N e^{A^*t}|_N = e^{(P_N A^*|_N)t}$. We have also that

$$P_N e^{At} P_N = P_N e^{At} P_M \quad (3)$$

Hence $P_N e^{At}|_N = (S_1(t))^*$ is a C_0 -semigroup on N with infinitesimal generator $P_N A|_N$. Now

$$\begin{aligned} \langle P_M c, e^{(P_N A|_N)t} P_N b \rangle &= \langle P_M c, P_N e^{At} P_N b \rangle = \\ &= \langle P_M c, P_N e^{At} P_M b \rangle = \langle c, e^{At} b \rangle = T(t) \end{aligned}$$

Moreover if $x \in N$ and $\langle P_M c, e^{(P_N A|_N)t} x \rangle = 0$ for all $t \geq 0$, then it follows using (2) that $\langle P_M e^{A^*t} P_M c, x \rangle = 0$ for $t \geq 0$, and so $x = 0$ by the definition of N . If now $x \in N$ and $\langle e^{P_N A|_N t} P_N b, x \rangle = 0$ for all $t \geq 0$, then it follows from (3) that $\langle P_N e^{At} P_M b, x \rangle = \langle e^{At} b, x \rangle = 0$ for $t \geq 0$, and so $x = 0$ by the definition of M . This proves ii).

We can now use this Theorem and Theorem 1 of sec. 2.5. to produce a canonical balanced realization starting from a given balanced realization. First we show that if the regular realization in Theorem 1 of sec. 2.5 is canonical, then the balanced one is canonical too. The balanced realization $[A, b, c]$ is given by $b = (\lambda I - A)^{-1} b_1$, $c(x) = \langle c_1, (\lambda I - A)x \rangle$ with $\lambda > 1$ and in $\rho(A)$. Suppose $x \in \mathcal{D}_0(A)$ and $c[e^{At} x] = 0$ for $t \geq 0$. Then $\langle c_1, e^{At} (\lambda I - A)x \rangle = 0$ for $t \geq 0$ and hence $(\lambda I - A)x = 0$. This implies $x = 0$ since $\lambda \in \rho(A)$. Suppose now $\langle e^{At} b, x \rangle = 0$ for $t \geq 0$. Then $\langle e^{At} b_1, (\lambda I - A^*)^{-1} x \rangle = 0$ and so $(\lambda I - A^*)^{-1} x = 0$. This implies $x = 0$ since $\lambda \in \rho(A^*)$. Therefore $[A, b, c]$ is canonical.

Hence we have the following:

Corollary 1.1: Given the balanced realization $[A, b, c]$ of T , we construct the associated regular one $[A, b_1, c_1]$ (using Theorem 1 of sec. 2.5) which we reduce according to Theorem 1 of this section to a canonical regular realization say $[F, g_1, h_1]$. Finally we construct (using Theorem 1 of 2.5) the associated balanced realization $[F, g, h]$. Then $[F, g, h]$ is a canonical balanced realization of T .

Based on the above we can restrict our study to canonical regular realizations.

For the rest of this section we restrict to the study of canonical realizations for the class of weighting patterns described in Theorem 3 of sec. 2.6. Extensions of our results to include the larger class

described in Cor. 3.1 of 2.6 are easy.

To proceed we need some facts from the theory of $H^2(\Pi^+)$ functions. For details we refer to Hoffman [7], Duren [54]. A function ϕ is inner if it is analytic in Π^+ with $|\phi(s)| \leq 1$ for $s \in \Pi^+$, while $|\phi(i\omega)| = 1$ a.e. A function $H \in H^2(\Pi^+)$ is outer if its boundary value on the imaginary axis, which is an element of $H^2(\mathbb{I})$, is a cyclic vector for the semigroup of multiplication by $e^{-i\omega t}$ on $H^2(\mathbb{I})$ (see definitions in sec. 3.1). Equivalently H is outer if its inverse Laplace transform h , has the property that its right translations form a dense set in $L_2(0, \infty)$. A subspace of $L_2(0, \infty)$ which is invariant under right translations gets mapped by the Fourier transform to a subspace of $H^2(\mathbb{I})$ which is invariant under multiplication by $e^{-i\omega t}$, $t \geq 0$. Following Lax [55], we call such a subspace of $H^2(\mathbb{I})$ a right translation invariant subspace. The orthogonal complement of a right translation invariant subspace is called a left translation invariant subspace and is invariant under multiplication by $e^{i\omega t}$, $t \geq 0$, followed by projection on $H^2(\mathbb{I})$. The inverse Fourier transform of a left translation invariant subspace is a subspace of $L_2(0, \infty)$ which is invariant under left translations followed by restriction to $(0, \infty)$. Clearly we have corresponding facts for such subspaces in $H^2(\Pi^+)$.

For clarity we use the variable z for complex numbers in \mathbb{D} , while the variable s is used for complex numbers in Π^+ . The map

$$z \xrightarrow{\psi} s = \frac{1+z}{1-z} \quad (4)$$

maps \mathbb{D} onto Π^+ . It is well known (see [7], [6]) that the map \mathcal{V} defined by

$$[\mathcal{V}F](z) = g(z) = \frac{2}{1-z} F\left(\frac{1+z}{1-z}\right) \quad (5)$$

is a unitary map from $H^2(\Pi^+)$ onto $H^2(\mathbb{D})$. \mathcal{V} restricted to boundary values is a unitary map of $H^2(\mathbb{I})$ onto $H^2(\mathbb{T})$. The inverse of \mathcal{V} is given by

$$[\mathcal{V}^{-1}g](s) = F(s) = \frac{1}{s+1} g\left(\frac{s-1}{s+1}\right) \quad (6)$$

We will need a few facts from the functional calculus for contractions developed in [11] (see the end of sec. 3.4) as applied to semigroups of contractions. Let as usual \mathcal{H} denote a Hilbert space and e^{At} a C_0 -semigroup of contractions on \mathcal{H} , with generator A . The operator $B = (A+I)(A-I)^{-1}$ is a contraction called the cogenerator of e^{At} (see [11] p. 141). The relation between the semigroup and its cogenerator is the following:

$$\left. \begin{aligned} e^{At} &= e_t(B), \quad t \geq 0 \\ B &= \lim_{t \rightarrow 0^+} \phi_t(e^{At}) \end{aligned} \right\} \quad (7)$$

where $e_t(\lambda) = \exp\left(t \frac{\lambda+1}{\lambda-1}\right)$, $t \geq 0$, $\phi_t(\lambda) = \frac{\lambda-1+t}{\lambda-1-t}$. Using these relations it is easy to show that a vector $b \in \mathcal{H}$ is a cyclic vector for B if and only if it is a cyclic vector for e^{At} (see [32] where a similar result is proved in the controllability context). Moreover B^* is the cogenerator of the adjoint semigroup $(e^{At})^*$ and the relations described above are valid if we replace B with B^* and e^{At} by $(e^{At})^*$ (see [11] p. 143).

The semigroup of contractions 'multiplication by $e^{-i\omega t}$ ', on $H^2(\mathbb{I})$

has as its cogenerator B, the operator 'multiplication by $\frac{i\omega-1}{i\omega+1}$ '. Then for $g \in H^2(\mathbb{D})$ we have

$$[\mathcal{V}B\mathcal{V}^{-1}g](z) = zg(z) = [Ug](z) \quad (8)$$

Hence B is unitarily equivalent to the forward shift on $H^2(\mathbb{T})$ (denoted as usual by U). The adjoint semigroup is 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})$ ', and its cogenerator is B^* and hence it is unitarily equivalent to the backward shift on $H^2(\mathbb{T})$. Clearly $F \in H^2(\mathbb{I})$ is a cyclic vector for B (resp. B^*) on $H^2(\mathbb{I})$ if and only if $\mathcal{V}F$ is a cyclic vector for the forward (resp. backward shift) on $H^2(\mathbb{T})$. Thus we have proved the following:

Lemma 1: A function $F \in H^2(\Pi^+)$ is outer if and only if $\mathcal{V}F \in H^2(\mathbb{D})$ is outer. A function $F \in H^2(\mathbb{I})$ is a cyclic vector for the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})$ ' if and only if $\mathcal{V}F \in H^2(\mathbb{T})$ is a cyclic vector for the backward shift on $H^2(\mathbb{T})$.

Suppose R is a right translation invariant subspace in $H^2(\Pi^+)$. Then using the second of (7) we see that $BR \subset R$. Hence $U\mathcal{V}R = \mathcal{V}BR \subset \mathcal{V}R$. So \mathcal{V} sends right translation invariant subspaces into subspaces of $H^2(\mathbb{D})$ invariant under multiplication by z (see [6]). By Beurling's theorem $\mathcal{V}R = \phi H^2(\mathbb{D})$ where ϕ is inner. Hence $R = \mathcal{V}^{-1}\phi H^2(\mathbb{D}) = \phi\left(\frac{s-1}{s+1}\right)H^2(\Pi^+) = \Phi H^2(\Pi^+)$. So we have the following:

Theorem 2 ([55]): Every closed subspace R of $H^2(\Pi^+)$ invariant under 'multiplication by e^{-st} , $t \geq 0$, is of the form $\Phi H^2(\Pi^+)$ where Φ is inner. Φ is unique modulo a constant of modulus one. If ϕ is the inner

function associated to $\mathcal{V}R$ by Beurling's theorem then $\phi(s) = \phi\left(\frac{s-1}{s+1}\right)$.

Definition: An inner function ϕ is normalized whenever the corresponding ϕ on the disk is normalized.

A Blaschke product is a function of the form

$$B(s) = \left(\frac{s-1}{s+1}\right)^k \prod_n \frac{|1-\beta_n^2|}{1-\beta_n^2} \cdot \frac{s-\beta_n}{s+\bar{\beta}_n}$$

where β_1, β_2, \dots are complex numbers in Π^+ different from 1 and such that $\sum_n \frac{\operatorname{Re}(\beta_n)}{1+|\beta_n|^2} < \infty$.

A singular function has the form

$$S(s) = e^{-\rho s} \exp\left(-\int_{-\infty}^{\infty} \frac{s\omega+i}{\omega+is} d\mu(\omega)\right)$$

where μ is a finite singular positive measure on the imaginary axis and $\rho > 0$.

Any element $F \in H^2(\Pi^+)$ has a factorization $F = \phi \cdot H$ where ϕ is inner and H is outer. The factors are unique up to a constant of modulus one. Moreover it follows from the above discussion and Lemma 1, that if $\mathcal{V}F = \phi \cdot h$ is the factorization of $\mathcal{V}F \in H^2(\mathbb{D})$ given by Beurling's theorem, then $\phi(s) = \phi\left(\frac{s-1}{s+1}\right)$ and $H(s) = [\mathcal{V}^{-1}h](s)$. Every inner function has a factorization $\phi = cBS$ where c is a constant of modulus 1, B a Blaschke product and S a singular function. Moreover if $\phi\left(\frac{1+z}{1-z}\right) = cb(z)\xi(z)$ is the factorization of the inner function $\phi\left(\frac{1+z}{1-z}\right) = \phi(z)$ on the disk (see sec. 3.3) we have $S(s) = \xi\left(\frac{s-1}{s+1}\right)$ and $B(s) = b\left(\frac{s-1}{s+1}\right)$.

Now using Lemma 1 we can obtain many properties of cyclic or non-cyclic vectors for the semigroup 'multiplication by $e^{i\omega t}$ followed by

projection on $H^2(\mathbb{I})'$, from properties of cyclic or noncyclic vectors for the backward shift on $H^2(\mathbb{T})$ (see [5]).

For the rest of this section when we say cyclic or noncyclic we will mean with respect to the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$, or equivalently with respect to the semigroup of left translations restricted on $L_2(\theta, \infty)$.

Definition: Let $F \in H^2(\mathbb{I})$, e^{At} denote the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$. The left translation invariant subspace generated by F is the closure of the linear span of the vectors $e^{At}F$, $t \geq 0$, in $H^2(\mathbb{I})$.

Theorem 3: An element F of $H^2(\mathbb{I})$ is a noncyclic vector if and only if there exist an inner function ϕ and a function H in $H^2(\Pi^+)$ such that $F(i\omega) = \overline{H(i\omega)}\phi(i\omega)$ a.e. on \mathbb{I} . Moreover if we choose ϕ to be normalized and relatively prime to the inner factor of H this factorization is unique. In this case the left translation invariant subspace generated by F is equal to $(\phi H^2(\mathbb{I}))^\perp$.

Proof: By Lemma 1, F is noncyclic if and only if $\mathcal{V}F$ is noncyclic for the backward shift on $H^2(\mathbb{T})$. The latter can happen if and only if

$$[\mathcal{V}F](e^{i\theta}) = e^{-i\theta} \overline{g(e^{i\theta})} \phi(e^{i\theta}) \text{ a.e. on } \mathbb{T} \quad (9)$$

where $g \in H^2(\mathbb{D})$ and ϕ is inner on the disk, by Theorem 1 of section 3.4.

Now (9) is equivalent to

$$\frac{2}{1-e^{i\theta}} F\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) = e^{-i\theta} \overline{g(e^{i\theta})} \phi(e^{i\theta})$$

or

$$\begin{aligned} F\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) &= \frac{1}{2} (e^{-i\theta}-1)\overline{g(e^{i\theta})}\phi(e^{i\theta}) = \\ &= \frac{1}{2} \overline{(e^{i\theta}-1)g(e^{i\theta})}\phi(e^{i\theta}) = \overline{h(e^{i\theta})}\phi(e^{i\theta}) \end{aligned} \quad (10)$$

$$\text{where } h(e^{i\theta}) = \frac{1}{2} (e^{i\theta}-1)g(e^{i\theta}) \quad (11)$$

Let $H(s) = h\left(\frac{s-1}{s+1}\right)$. Since $g \in H^2(\mathbb{D})$ it follows by (6) and (11) that $H \in H^2(\Pi^+)$. If we let $\Phi(s) = \phi\left(\frac{s-1}{s+1}\right)$, then Φ is inner on the half-plane. Hence (10) reads

$$F(i\omega) = \overline{H(i\omega)}\Phi(i\omega) \quad (12)$$

a.e. on the imaginary axis.

From (11) it follows that if ϕ is prime to the inner factor of g , it is also prime to the inner factor of h . Now if ϕ is normalized and prime to the inner factor of g the factorization in (9) is unique (by Theorem 1 of sec. 3.4). Hence if Φ is normalized and prime to the inner factor of H , then the factorization in (12) is unique. Let M be the left translation invariant subspace generated by F . By Theorem 2, $M^\perp = \Psi H^2(\mathbb{I})$ for some inner function Ψ . Hence $M = (\Psi H^2(\mathbb{I}))^\perp$. So to complete the proof we need to show $\Psi = \Phi$. By (12) $F \perp \Phi H^2(\mathbb{I})$ and since $H^2(\mathbb{I})$ is invariant under 'multiplication by $e^{-i\omega t}$ ', $t \geq 0$, we obviously have $M \subset (\Phi H^2(\mathbb{I}))^\perp$. Hence Ψ divides Φ . Now since $F \perp \Psi H^2(\mathbb{I})$ we must have $F(i\omega) = \overline{G(i\omega)}\Psi(i\omega)$ a.e. on \mathbb{I} , where $G \in H^2(\Pi^+)$. Or equivalently $F(i\omega) = \overline{G'(i\omega)}\Psi'(i\omega)$ where Ψ' is prime to the inner factor of G' , and of course Ψ' divides Ψ . By uniqueness of (12) in this case we get $\Psi' = \Phi$. Hence Φ divides Ψ . So $\Psi = \Phi$ and this completes the proof.

Definition: The inner function uniquely associated to every non-cyclic vector $F \in H^2(\Pi^+)$ by Theorem 3 is called the associated inner function of F .

Remark 1: It follows from Lemma 1 and the fact that any $H^2(\mathbb{D})$ function with isolated branch points on \mathbb{T} is a cyclic vector for the backward shift on $H^2(\mathbb{D})$, that any $F \in H^2(\Pi^+)$ with isolated branch points on the imaginary axis is a cyclic vector.

We are ready now to proceed with the study of canonical realizations for the class of weighting patterns described in Theorem 3. of sec. 2.6. For the rest of this section we will restrict our study to transfer functions in this class, unless explicitly stated otherwise.

Let us recall that a transfer function \tilde{T} in this class, belongs to $H^2(\Pi^+)$ and has a factorization $\tilde{T}(i\omega) = \overline{F_c(i\omega)} F_o(i\omega)$ a.e. on the imaginary axis where F_c, F_o belong to $H^2(\Pi^+)$. By eliminating if necessary any common inner factors between F_c, F_o (recall that if ϕ is inner $\overline{\phi}\phi = 1$ on \mathbb{I}) we can assume that the inner factors of F_c and F_o are relatively prime.

For any such transfer function we have the 'right translation realization' which is constructed by considering as Hilbert space \mathcal{H} the space $H^2(\mathbb{I})$, the semigroup 'multiplication by $e^{-i\omega t}$ ', as e^{At} , F_c as b and F_o as c . Using inverse Fourier transform this realization is described via the semigroup of right translations on $L_2(0, \infty)$ (see Theorem 3 of 2.6). Our plan is to apply Theorem 1 to this realization, obtain a canonical one, and then discuss certain properties of the latter. Recall that according to the definitions given in sec. 3.1 and 3.2 a regular realization $[A, b, c]$ is canonical if and only if b is a cyclic vector of the semigroup e^{At} and c is a cyclic vector of the semigroup e^{A^*t} . The reducibility of

the 'right translation realization' depends on the character of the two factors F_c, F_o and hence on the properties of \tilde{T} itself.

We start with the following preliminary result.

Lemma 2: Suppose $\tilde{T} \in H^2(\Pi^+)$ and has a factorization $\tilde{T}(i\omega) = \overline{F_c(i\omega)} F_o(i\omega)$ a.e. on the imaginary axis, where $F_c, F_o \in H^2(\Pi^+)$ and have no common inner factor. Then \tilde{T} is noncyclic if and only if F_o is.

Proof: Suppose F_o is noncyclic. Then by Theorem 3, $F_o(i\omega) = \overline{H_o(i\omega)} \phi_o(i\omega)$ a.e. on \mathbb{I} , where $H_o \in H^2(\Pi^+)$ and ϕ_o is inner. Hence $\tilde{T} = \overline{F_c(i\omega)H_o(i\omega)} \phi_o(i\omega)$ a.e. on \mathbb{I} . Since $\tilde{T} \in H^2(\Pi^+)$ we have that $F_c H_o \in H^2(\Pi^+)$ and therefore by Theorem 3, \tilde{T} is noncyclic.

Suppose now that \tilde{T} is noncyclic. Then by (5) we have

$$[\mathcal{V}\tilde{T}](e^{i\theta}) = \frac{2}{1-e^{i\theta}} \overline{F_c \left(\frac{1+e^{i\theta}}{1-e^{i\theta}} \right)} F_o \left(\frac{1+e^{i\theta}}{1-e^{i\theta}} \right)$$

$$\text{or } [\mathcal{V}\tilde{T}](e^{i\theta}) = \frac{1}{2} \frac{e^{i\theta}}{1-e^{i\theta}} \overline{F_c \left(\frac{1+e^{i\theta}}{1-e^{i\theta}} \right)} (e^{i\theta}-1) [\mathcal{V}F_o](e^{i\theta})$$

Hence

$$[\mathcal{V}\tilde{T}](e^{i\theta}) = \frac{1}{2} e^{i\theta} \overline{[\mathcal{V}F_c](e^{i\theta})} (e^{i\theta}-1) [\mathcal{V}F_o](e^{i\theta}) \quad (13)$$

Let now

$$\left. \begin{aligned} \mathcal{V}\tilde{T} &= g \\ \frac{1}{2} e^{i\theta} [\mathcal{V}F_c](e^{i\theta}) &= f(e^{i\theta}) \\ (e^{i\theta}-1) [\mathcal{V}F_o](e^{i\theta}) &= h(e^{i\theta}) \end{aligned} \right\} \quad (14)$$

Then $g \in H^2(\mathbb{D})$ and by Lemma 1 is a noncyclic vector for the backward shift; $f \in H^2(\mathbb{D})$ and $f(0) = 0$; $h \in H^2(\mathbb{D})$. By Theorem 1 of sec. 3.4

there exists an element $g_1 \in H^2(\mathbb{D})$ with $g(0) = 0$ and an inner function ϕ (on the disk) such that $g = \bar{g}_1 \phi$ a.e. on \mathbb{T} . Therefore

$$g = \bar{f}h = \bar{g}_1 \phi \quad (15)$$

Now $\phi f \in H^2(\mathbb{D})$. Let U denote, as usual, the forward shift and U^* the backward shift on $H^2(\mathbb{D})$. Then

$$\begin{aligned} \langle \phi f, U^{*n}h \rangle &= \langle U^n \phi f, h \rangle = \\ &= \int_{\mathbb{T}} \overline{z^n \phi f} \, h d\mu(\theta) = \int_{\mathbb{T}} \overline{z^n \phi} \bar{g}_1 \phi d\mu(\theta) = \\ &= \int_{\mathbb{T}} \overline{z^n} \bar{g}_1 d\mu(\theta) = \langle e^{in\theta}, \bar{g}_1 \rangle = 0; \text{ for } n=0,1,2,\dots \end{aligned}$$

because $\bar{g}_1 \perp H^2(\mathbb{D})$. Since $\phi f \perp U^{*n}h$, for $n=0,1,2,\dots$ it follows that h is noncyclic. But $v \in H^2(\mathbb{T})$ is a noncyclic vector for the backward shift if and only if $e^{i\theta}v$ is one (see [5] Theorem 2.2.8). We conclude therefore from the third of (14) that $\forall F_0$ is noncyclic. Then by Lemma 1 F_0 is noncyclic and this completes the proof.

Since the properties of the 'right translation realization' depend heavily on \tilde{T} being cyclic or noncyclic, we study these two cases separately, in the sequel.

i) The Noncyclic Case

Since \tilde{T} is noncyclic it follows from Lemma 2 that F_0 is noncyclic. However F_c can be outer or not.

We study first transfer functions, for which F_c is outer. Then the 'right translation realization' is controllable. Hence applying Theorem 1 we get a canonical realization with state space $N =$ left translation invariant subspace generated by F_0 , in $H^2(\mathbb{I})$, with $P_N e^{-i\omega t} |_N$

as e^{At} , with F_0 as c and $P_N F_c$ as b . Moreover by Theorem 3 since F_0 is noncyclic we have that $N = (\phi_0 H^2(\mathbb{I}))^\perp$, where ϕ_0 is the associated inner function of F_0 .

We give a simple proof of the following theorem, which connects the spectrum of the infinitesimal generator of the semigroup $P_N e^{-i\omega t}|_N$ (where $N = (\phi_0 H^2(\mathbb{I}))^\perp$) with the inner function ϕ_0 . This is done mainly to render the exposition self-contained and to describe this relation precisely in our setting.

Theorem 4 ([39] p. 70): Let N be a left translation invariant subspace of $H^2(\mathbb{I})$, i.e. $N = (\phi H^2(\mathbb{I}))^\perp$ for some inner function ϕ . Consider the semigroup multiplication by $e^{-i\omega t}$, restricted on N (i.e. $P_N e^{-i\omega t}|_N$). The spectrum of its infinitesimal generator is the set \mathfrak{S}_ϕ which consists of

- i) all $\mu \in \mathbb{C}$ with $\text{Re } \mu < 0$, such that $\phi(-\mu) = 0$
- ii) all $\mu \in \mathbb{C}$ with $\text{Re } \mu = 0$ such that ϕ cannot be continued

analytically across the imaginary axis at $-\mu$.

Proof: We are going to use Theorem 2 of sec. 3.4 and the properties of the unitary map \mathcal{V} defined in (5) and (6). Since \mathcal{V} sends right translation invariant subspaces into spaces invariant under the forward shift (which is denoted by U) and is unitary, it sends left translation invariant subspaces into spaces invariant under the backward shift (which is denoted by U^*). So we have the commuting diagram

$$\begin{array}{ccc}
 H^2(\mathbb{I}^+) & \xrightarrow{\mathcal{V}} & H^2(\mathbb{D}) \\
 \downarrow P_N & & \downarrow P_M \\
 N = (\phi H^2(\mathbb{I}^+))^\perp & \xrightarrow{\mathcal{V}} & M = (\phi H^2(\mathbb{D}))^\perp
 \end{array}$$

where $\Phi(s) = \phi\left(\frac{s-1}{s+1}\right)$ by Theorem 2. Let e^{At} denote the semigroup 'multiplication by $e^{-i\omega t}$ ', and B its cogenerator (see the definitions immediately after equation (6) in this section). Let B_1 be the cogenerator of $P_N e^{At}|_N$ and A_1 its generator. Then since $B_1 = \lim_{t \rightarrow 0^+} \phi_t(P_N e^{At}|_N)$ by (7) and because $P_N e^{At} P_N = P_N e^{At}$ by (2) we conclude that

$$B_1 = P_N B|_N$$

Now $\mathcal{V} B_1 \mathcal{V}^{-1} = \mathcal{V} (P_N B|_N) \mathcal{V}^{-1} = \mathcal{V} P_N \mathcal{V}^{-1} \mathcal{V} (B|_N) \mathcal{V}^{-1} = \mathcal{V} P_N \mathcal{V}^{-1} U|_M = P_M U|_M$

since by (8) $\mathcal{V} B \mathcal{V}^{-1} = U$ and the diagram above commutes. Hence $\sigma(B_1)$ is the set s_ϕ in Theorem 2 of sec. 3.4. Now $A_1 = (B_1 + I)(B_1 - I)^{-1}$. Suppose μ, λ are two complex numbers having the following relation $\mu = \frac{\lambda+1}{\lambda-1}$.

Then

$$\mu I - A_1 = \frac{\lambda+1}{\lambda-1} I - (B_1 + I)(B_1 - I)^{-1} = 2(\lambda-1)^{-1} (B_1 - \lambda I)(B_1 - I)^{-1}$$

When $\lambda \in \rho(B_1)$ the operator

$$R(\mu, A) = \frac{1}{2} (\lambda-1) (B_1 - I) (B_1 - \lambda I)^{-1}$$

is an everywhere defined bounded operator and

$$(\mu I - A_1) R(\mu, A_1) = R(\mu, A_1) (\mu I - A_1) = I \quad .$$

Hence $\mu \in \rho(A_1)$.

Since $B_1 = (A_1 + I)(A_1 - I)^{-1}$ using a dual argument we see that if $\mu \in \rho(A_1)$ then $\lambda \in \rho(B_1)$.

The conclusion is that $\sigma(A_1)$ is the image of $\sigma(B_1)$ under the map $\lambda \mapsto \mu = \frac{\lambda+1}{\lambda-1}$. Combining this with the description of the set s_ϕ given in Theorem 2 of sec. 3.4 and using the relation $\Phi(s) = \phi\left(\frac{s-1}{s+1}\right)$ we obtain the result.

Via Fourier transform this theorem also gives the spectrum of the infinitesimal generator of the restriction of the right translation semigroup on a subspace of $L_2(0, \infty)$ which is invariant under left translations.

Combining this result with the previous discussion we arrive at the following theorem.

Theorem 5: Let \tilde{T} be a transfer function which belongs to $H^2(\Pi^+)$, is noncyclic and has a factorization $\tilde{T} = \overline{F_c} F_o$ a.e. on the imaginary axis with F_c, F_o in $H^2(\Pi^+)$ and F_c outer. Then \tilde{T} has a canonical realization with the spectrum of the infinitesimal generator being exactly \mathfrak{S}_{ϕ_o} , where ϕ_o is the associated inner function of F_o . This realization is constructed by taking, as state space the subspace of $H^2(\mathbb{I})$, $N = (\phi_o H^2(\mathbb{I}))^\perp$, as c the function F_o , as b the projection of F_c on N and as semigroup the restriction of the semigroup 'multiplication by $e^{-i\omega t}$, on N (i.e. $P_N e^{-i\omega t} |_{N}$).

Suppose now that \tilde{T} has a meromorphic continuation across the imaginary axis in Π^- (for the definition of a function meromorphic on a half-plane see [19] p. 260). Now

$$\tilde{T}(i\omega) = \overline{F_c(i\omega)} F_o(i\omega) = \overline{F_c(i\omega) H_o(i\omega)} \phi_o(i\omega) \quad (16)$$

where H_o, ϕ_o are the factors of F_o according to Theorem 3. Now since T is real valued we have

$$\tilde{T}(i\omega) = \overline{\tilde{T}(-i\omega)} = \frac{F_c(-i\omega) H_o(-i\omega)}{\phi_o(-i\omega)} \quad (17)$$

The right hand side of (17) is the boundary value of the function

$$G(s) = \frac{F_c(-s)H_0(-s)}{\phi_0(-s)},$$
 which is meromorphic in Π^- . Since F_c does not have any inner factor and ϕ_0 is relatively prime to the inner factor of H_0 we get that G is analytic in Π^- except at points $\mu \in \Pi^-$ where $\phi_0(-\mu) = 0$. Since \tilde{T} has a meromorphic continuation in Π^- we must have that $\tilde{T}(s) = G(s)$ in Π^- . Hence the singularities of \tilde{T} in Π^- are the points μ with $\phi_0(-\mu) = 0$. On the other hand \tilde{T} has an analytic continuation in Π^- , through a point $i\omega$ of the imaginary axis, if and only if K has one through $i\omega$ in the opposite direction in Π^+ , where $K(s) = \overline{\tilde{T}(-\bar{s})} = \tilde{T}(-s)$ (the last equality because T is real valued). But K has an analytic continuation through $i\omega$ in Π^+ if and only if \tilde{T} has an analytic continuation through $-i\omega$ in Π^- . Therefore \tilde{T} has an analytic continuation through $i\omega$ in Π^- , if and only if it has one through $-i\omega$. We have however from (16) that $\tilde{T}(i\omega) = \overline{F_c(i\omega)H_0(i\omega)}\phi_0(i\omega)$ where ϕ_0 and the inner factor of $F_cH_0 \in H^2(\Pi^+)$ are relatively prime. Hence by a theorem in [39] p. 66, \tilde{T} has an analytic continuation in Π^- through $-i\omega$, if and only if ϕ_0 has one through $-i\omega$.

The set of points on the imaginary axis through which ϕ_0 cannot be continued analytically coincides with the union of the following two sets: α) the support of the singular measure of the singular part of ϕ_0 and β) the accumulation points of the sequence of zeros of ϕ_0 (i.e. the sequence of zeros of its Blaschke product part) (see [7]).

Comparing $\sigma(\tilde{T})$ as described above with the set \mathfrak{S}_{ϕ_0} of Theorem 4, we see that for the canonical realization given by Theorem 5 the "spectral inclusion property" becomes again equality

$$\sigma(\tilde{T}) = \sigma(A)$$

where A is the restriction of the operator 'multiplication by $-i\omega$ ' on $(\phi_0 H^2(\mathbb{I}))^\perp$, or restriction of $-\frac{\partial}{\partial z}$ on a left translation invariant subspace of $L_2(0, \infty)$.

This motivates the following definition.

Definition: A canonical realization $[A, b, c]$ (balanced or regular) of a weighting pattern T is called S-minimal (S from spectrum) if and only if $\sigma(\tilde{T}) = \sigma(A)$, multiplicities counted whenever possible.

We have thus proved the following

Corollary 5.1: Any transfer function which satisfies the conditions of Theorem 5 above and is meromorphic in the left half plane, has an S-minimal realization with infinitesimal generator having connected resolvent set. The construction of this realization is given by Theorem 5 above.

We discuss briefly an application of these results to a particularly interesting class of weighting patterns which belong to the class we are considering here. Recall (see Remark 2 of sec. 2.6) that if T belongs to $L_2(0, \infty)$, is locally absolutely continuous, its derivative belongs to $L_2(0, \infty)$ also and $T(0) = 0$, then T belongs to the class we are studying. In this case we have

$$F_c(i\omega) = \frac{1}{1+i\omega}, \quad F_o(i\omega) = (1-i\omega)\tilde{T}(i\omega)$$

Now $[\mathcal{V}F_c](z) = \frac{2}{1-z} \frac{1}{1+\frac{1+z}{1-z}} = 1$ using (5). Hence, by Lemma 1, since 1 is

outer on the disk, F_c is outer. Whenever \tilde{T} is noncyclic all the conditions of Theorem 5 are satisfied and hence it can be applied. If in addition \tilde{T} is meromorphic in the left half-plane Corollary 5.1 applies. We thus have

Corollary 5.2: Let T be a weighting pattern which belongs to $L_2(0, \infty)$, is locally absolutely continuous, its derivative belongs to $L_2(0, \infty)$ and $T(0) = 0$. If \tilde{T} is noncyclic and meromorphic in the left half-plane, T has an S-minimal realization with infinitesimal generator with connected resolvent set. The construction is given in Theorem 5.

Remark 2: The meromorphic assumption is satisfied by many systems governed by several forms of the wave equation (e.g. Schrödinger Equation, Maxwell's Equations) (see [39] Ch. VI and appendix 4, [46]).

We proceed now to the study of transfer functions for which F_c is not outer (we continue to assume that \tilde{T} is noncyclic). Then the 'right translation realization' is neither controllable nor observable. Hence following Theorem 1 we obtain first a realization with state space $M = \{\text{closure of the linear span of the vectors } e^{-i\omega t} F_c, t \geq 0\}$, in $H^2(\mathbb{I})$, with $e^{-i\omega t}|_M$ as e^{At} , $P_M F_o$ as c and F_c as b . By Theorem 2, $M = \phi_c H^2(\mathbb{I})$ where ϕ_c is the normalized inner factor of F_c . Next we reduce the latter realization to obtain a canonical one. This has as state space N , the closure of the linear span of the vectors $P_M e^{i\omega t} P_M F_o, t \geq 0$, in M , $P_M F_o$ as c and $P_N F_c$ as b . The semigroup is $P_N e^{-i\omega t}|_N$. Since M is invariant under $e^{-i\omega t}$, $P_M e^{i\omega t} P_M F_o = P_M e^{i\omega t} F_o$ for $t \geq 0$. Since F_o is noncyclic, the left translation invariant subspace generated by F_o is $(\phi_o H^2(\mathbb{I}))^\perp$ where ϕ_o is the assoc. inner function of F_o , by Th. 3. The situation here looks similar to the previous case. However the existence of the projection P_M complicates the discussion of the spectral properties of this realization. We were not able to conclude whether or not this realization can be S-minimal.

We conclude this section by

ii) The Cyclic Case

Since \tilde{T} is cyclic, it follows from Lemma 2 that F_c is cyclic also.

This class is very interesting because it contains transfer functions with branch points, such as those usually appearing in systems governed by partial differential equations. Remarks similar to the ones given in sec. 3.5 are valid here as well.

F_c can be outer or not. We consider first the case when F_c is outer. Then the 'right translation realization' is canonical. The spectrum of the infinitesimal generator in this realization is however the whole closed left half-plane and hence generally it is far from being equal to $\sigma(\tilde{T})$. So again (see sec. 3.5) canonical by no means imply S-minimal.

In this class belong also weighting patterns which satisfy the conditions of Theorem 1 of sec. 2.6 and have Laplace transforms with branch points. By the "spectral inclusion property" all the points on branch cuts of \tilde{T} are included in the spectrum of any infinitesimal generator with connected resolvent set which realizes T . Hence there is no unique "minimal" spectrum, due to the nonuniqueness of the branch cuts.

We consider now the case when F_c is not outer. Then the 'right translation realization' is observable. Reducing this realization by Theorem 1 we obtain a canonical realization with state space $M = \{\text{closure of the linear span of } e^{-i\omega t} F_c, t \geq 0\}$, in $H^2(\mathbb{I})$, with $P_M e^{-i\omega t}|_M$ as e^{At} , with $P_M F_c$ as c and F_c as b . Obviously M is a right translation invariant subspace and hence by Theorem 2, $M = \phi_c H^2(\mathbb{I})$ where ϕ_c is the normalized inner factor of F_c . Since M is obviously invariant under 'multiplication by $e^{-i\omega t}$,

we see that the following diagram commutes

$$\begin{array}{ccc}
 H^2(\mathbb{I}) & \xleftarrow{\mathcal{W}^{-1}} & M \\
 \downarrow \begin{array}{l} \text{mult. by} \\ e^{-i\omega t} \end{array} & & \downarrow \begin{array}{l} \text{mult. by} \\ e^{-i\omega t} \end{array} \\
 H^2(\mathbb{I}) & \xrightarrow{\mathcal{W}} & M
 \end{array}$$

where \mathcal{W} is the unitary operator given by $\mathcal{W}H = \Phi_c H$. Hence the spectrum of the infinitesimal generator of the semigroup in this realization is the whole closed left half-plane. So essentially the 'right translation realization' restricted to a right translation invariant subspace, where it is canonical, is generally far from being S-minimal.

Therefore independently of F_c being outer or not, whenever \tilde{T} is cyclic the 'right translation realization' does not reduce to an S-minimal realization. Only when \tilde{T} is noncyclic, we can reduce the 'right translation realization' to obtain an S-minimal one.

It may be possible to construct S-minimal realizations for \tilde{T} cyclic, by other means however (compare with section 3.5).

Remark 3: We do not have a complete picture for the relations between canonical (resp. S-minimal) realizations of the same weighting pattern T , for the class considered in this section. It is apparent however that there is no analogue of the state space isomorphism theorem of the finite dimensional theory. Counterexamples similar to the ones presented for the bounded case (see sec. 3.4, 3.5), can be constructed easily. To obtain a state space isomorphism theorem we need more assumptions. W. Helton [19] introduced the notions of exact controllability and observability and

obtained a state space isomorphism theorem for bounded realizations. This result can be extended in a straightforward manner to regular and balanced realizations.

CHAPTER IV
CONCLUSIONS AND SUGGESTIONS
FOR FURTHER RESEARCH

We summarize briefly the major results of this thesis.

i) Classification of the transfer functions which admit bounded, balanced and regular realizations.

ii) Existence and study of canonical realizations with spectral properties closely reflecting the properties of the singularities of the transfer functions.

iii) Existence of universal dynamical models for the class of systems considered.

It is of interest to investigate further the class of irreducible transfer functions (bounded realizations) and the cyclic case of the last section (regular realizations), and in particular to classify these classes according to the possibility of existence or not of S-minimal realizations. A systematic way of obtaining S-minimal realizations is another subject which needs investigation. The study of the connections between the realizability theory presented here (and especially of the concept of an S-minimal realization) with certain modeling methods in engineering (waveguide systems [46]) will hopefully provide a better understanding of these methods.

The study of scattering theory by Lax-Phillips using translation representations of the semigroups involved in this problem, and the results obtained via this method, give sufficient evidence that application of the

universal dynamical models in other physical systems will prove to be successful as well.

Finally since in many interesting physical problems the semigroup of the state transition operators is either analytic (systems governed by parabolic partial differential equations) or compact (delay systems) it is of importance to characterize the transfer functions which admit realizations with such properties. On the other hand the question of existence of simple universal models for these systems has its own theoretical and practical interest.

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