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Consensus problem under stochastic communication topologies

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PROBLEM FORMULATION

Definition. A *Discrete Markovian Random Graph* is a stochastic process taking values in a set of graphs $\mathcal{G}=\{G_1,\ldots,G_s\}$ which evolves in time according to a underlying finite-state discrete time Markov chain $M(k)$.

Consider a group of n agents which exchange locally information according to a communication protocol. The communication topology is assumed modeled by a *Markovian Random Graph* $G_{M(k)}$. The evolution of the agents is governed by

$$X(k+1)=F_{M(k)}X(k),\quad X(0)=X_0$$

where $X(k)$ represents the n-dimensional state vector of the agents and, $F_{M(k)}$ is a stochastic matrix determined by the communication protocol and which corresponds to the current topology determined by $G_{M(k)}$.

Definition (*Almost sure consensus*). Vector $X(k)$ converges almost surely to consensus if it asymptotically reaches a vector $\alpha\mathbf{1}$ in the almost sure sense. If the state vector converges to $av(X_0)\mathbf{1}$ we say that $X(k)$ converges almost surely to *average consensus*.

Problem. Given a discrete Markovian random graph $G_{M(k)}$ together with the state updating rule $F_{M(k)}$, we derive necessary and sufficient conditions such that the state vector $X(k)$ converges almost surely to consensus for any initial state X_0.

Example of local communication protocols (local state updating rules):

A1) $F_i=I-\varepsilon L_i$

A2) $F_i=(I+D_i)^{-1}(I+A_i)$

where L_i, D_i and A_i are the Laplacian, degree and adjacency matrices respectively of graph G_i.

MOTIVATION

- Presence of communication topologies that varies randomly in time due to link failures, packet drops, appearance or disappearance of nodes, obstacles that interfere with the communication, etc.

- Topology switching may depend on the previous state of the communication topology (systems with memory)

- Applications in
  - Flocking Theory
  - Rendezvous in Space
  - Distributed Sensor Fusion in Sensor Networks
  - Distributed Formation Control

RESULTS

Theorem (*almost sure convergence to consensus*). Under protocol A1 or A2, the state vector $X(k)$ converges *almost surely to consensus* for any initial state X_0 if and only if each of the sets of graphs corresponding to the positive recurrent closed sets of the Markov chain admits a spanning tree. In particular if protocol A1 is used and all graphs in $\mathcal{G}$ are either undirected or directed but *balanced*, then the state vector converges almost surely to *average consensus*.

Bound on the ***rate of convergence*** to consensus in the mean square sense:

$$E\left[\|\varepsilon(k)\|^2\right]\leq\alpha|\lambda_2|^k\|\varepsilon(0)\|^2$$

where $\varepsilon(k)=X(k)-\arg\min_{z\in span\{1\}}\|X(k)-z\|$ and λ_2 is the second largest eigenvalue in absolute value of the matrix

$$\Lambda=\begin{bmatrix}p_{11}F_1\otimes F_1 & p_{21}F_2\otimes F_2 & \ldots & p_{s1}F_s\otimes F_s\\ p_{21}F_1\otimes F_1 & p_{22}F_2\otimes F_2 & \ldots & p_{s2}F_s\otimes F_s\\ \vdots & \vdots & \ddots & \vdots\\ p_{s1}F_1\otimes F_1 & p_{s2}F_2\otimes F_2 & \cdots & p_{ss}F_s\otimes F_s\end{bmatrix}$$

APPROACH

Show for the state vector that

$$E\left[X(k)X(k)^T\right]\longrightarrow\beta\mathbf{1}\mathbf{1}^T$$

for some scalar β, which implies that the second moment of the error vector

$$\varepsilon(k)=\left(I-\frac{\mathbf{1}\mathbf{1}^T}{\mathbf{1}^T\mathbf{1}}\right)X(k)$$

converges to zero exponentially, which in turns shows that

$$\|\varepsilon(k)\|^2\xrightarrow{a.s.}0$$

Key property:

$$\lim_{k\to\infty}\Lambda^k=\begin{bmatrix}p_{11}^\infty\mathbf{1}c_{11}^T & p_{21}^\infty\mathbf{1}c_{12}^T & \ldots & p_{s1}^\infty\mathbf{1}c_{1s}^T\\ p_{12}^\infty\mathbf{1}c_{21}^T & p_{22}^\infty\mathbf{1}c_{22}^T & \ldots & p_{s2}^\infty\mathbf{1}c_{2s}^T\\ \vdots & \vdots & \ddots & \vdots\\ p_{s1}^\infty\mathbf{1}c_{s1}^T & p_{s2}^\infty\mathbf{1}c_{s2}^T & \cdots & p_{ss}^\infty\mathbf{1}c_{ss}^T\end{bmatrix}^k$$

where $p_{ij}^\infty=\mathrm{Prob}(M(k)=j\,|\,M(0)=i)$ for large values of k and c_{ij} are vectors of positive entries which sum up to one.

References

Y. Hatano and M. Mesbahi, "Agreement over Random Networks",*IEEE Trans. Autom. Control*, vol. 50, no. 11, pp.1867–1872, 2005.

M. Porfiri and D.J. Stilwell, "Consensus seeking over RandomDirected Weighted Graphs", \textit{IEEE Trans. Autom. Control},vol. 52, no. 9, Sept. 2007.

Tahbaz Salehi and A. Jadbabaie, "Necessary and SufficientConditions for Consensus over random networks", *IEEETransactions on Automatic Control*, accepted, August 2007.