Reconstruction from Magnitudes of Redundant Representations

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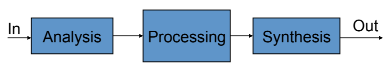
Problem Formulation

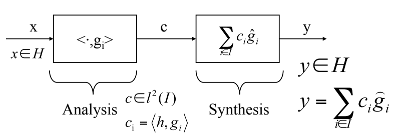
Typical signal processing pipeline:



Features:

- ❑ Relative low complexity O(Nlog(N))
- ❑ On-line version is possible

The Analysis/Synthesis Components:



Example: Short-Time Fourier Transform

$$c_{k,f}=\langle x,g_{k,f}\rangle \qquad g_{k,f}(t)=e^{2\pi i f(t-kb)}g(t-kb)$$

$$I=\{(k,f);k\in Z,0\le f\le F-1\}=Z\times Z_F$$

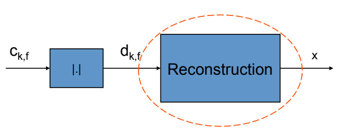
$$H=l^2(Z)$$

Problem:
Given the Short-Time Fourier Amplitudes (STFA):

$$d_{k,f}=\left|\langle x,g_{k,f}\rangle\right|=\left|\sum_{t=kb}^{kb+M-1}x(t)\overline{g(t-kb)}e^{-2\pi i f t/F}\right|$$

we want an *efficient* reconstruction algorithm:

- ❑ Reduced computational complexity
- ❑ On-line ("on-the-fly") processing



Where is this problem important:

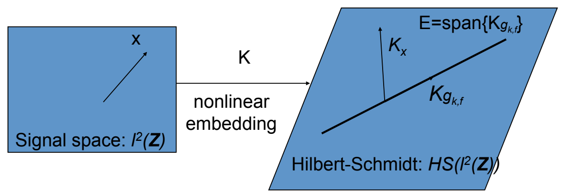
- Speech enhancement
- Speech separation
- Neural speech production
- X-Ray Crystallography
- Optical Communication

Solution/Approach

First observation [BBCE'09]:

$$d_{k,f}^2=\left|\langle x,g_{k,f}\rangle\right|^2=tr\left\{K_x K_{g_{k,f}}^{\ *}\right\}=\langle K_x,K_{g_{k,f}}\rangle_{HS}$$

$$K_x(y)=\langle y,x\rangle x \quad , \quad K_{g_{k,f}}(y)=\langle y,g_{k,f}\rangle g_{k,f}$$



Assume $\{K_{g_{k,f}}\}$ form a frame for its span, *E*. Then the projection P_E can be written as:

$$P_E=\sum_{k,f}\langle\cdot,K_{g_{k,f}}\rangle_{HS}Q_{k,f}$$

where $\{Q_{k,f}\}$ is the canonical dual of $\{K_{g_{k,f}}\}$.

Frame operator

$$X\mapsto S(X)=\sum_{k,f}\langle X,K_{g_{k,f}}\rangle_{HS}K_{g_{k,f}}$$

$$Q_{k,f}=S^{-1}(K_{g_{k,f}})$$

Second Observation: Since

$$g_{k,f}=M^fT^kg$$

where $M:h\mapsto Mh(t)=e^{2\pi i t/F}h(t)\ ,\ T:h\mapsto Th(t)=h(t-b)$

it follows:

$$K_{g_{k,f}}=\Lambda^f\Theta^kK_g \qquad Q_{k,f}=\Lambda^f\Theta^kQ_{0,0}$$

where $\Lambda:X\mapsto\Lambda X=MXM^*\ ,\ \Theta:X\mapsto\Theta X=TXT^*$

Explicitly

$$\left(Q_{k,f}\right)_{t_1,t_2}=e^{2\pi i f(t_1-t_2)/F}\left(Q_{0,0}\right)_{t_1-kb,t_2-kb}$$

Set: $H(\omega,m)=\sum_{k\in Z}\sum_{f\in Z_F}e^{2\pi i\left(k\omega+\frac{mf}{F}\right)}\left|\langle g,g_{k,f}\rangle\right|^2$

Main Result:
Theorem Assume $\{g_{k,f}\}_{(k,f)\in Z\times Z_F}$ is a frame for $l^2(\mathbf{Z})$.
Then $\{K_{g_{k,f}};(k,f)\in Z\times Z_F\}$

1. is a frame for its span in $HS(l^2(\mathbf{Z}))$ iff for each $m\in Z_F$, $H(\omega,m)$ either vanishes identically in ω, or it is never zero;
2. is a Riesz basis for its span in $HS(l^2(\mathbf{Z}))$ iff for each $m\in Z_F$ and ω, $H(\omega,m)$ is never zero.

Under these hypotheses:

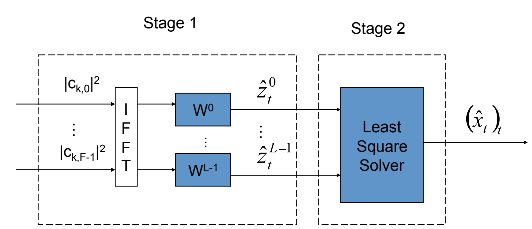
$$\left(Q_{0,0}\right)_{t,t-\tau}=\frac{1}{F}\int_0^1\frac{e^{2\pi i t\omega}}{\sum_p\overline{g(p)g(p-\tau)}e^{-2\pi i p\omega}}d\omega$$

Final step: Signal reconstruction from Q_x by solving:

$$\min_x\left[\left|\left|x\right|^2-\left(K_x\right)_{t,t}\right|+w_1\left|x\hat{x}_{t-1}-\left(K_x\right)_{t,t-1}\right|^2+\cdots+w_J\left|x\hat{x}_{t-J}-\left(K_x\right)_{t,t-J}\right|^2\right]$$

for x_t, assuming we already estimated x_s for *s<t,*

Reconstruction scheme:



$$W^\tau(z^{-1})=\sum_n\left(Q_{0,0}\right)_{n,n-\tau}z^{-n}$$

Example: