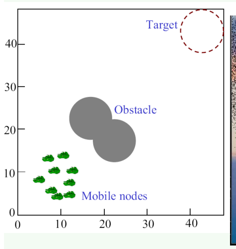
Connectivity in Networked Systems: Consensus Problems and Small World Graphs

Pedram Hovareshti and John S. Baras

Introduction and Motivation



Distributed algorithms

- Group of agents with simple/complex abilities
- Agents sense their local neighborhood
- Communicate with neighbors and process the information
  - Perform a local action
  - Emergence of a global behavior.

Effectiveness of these algorithms depends on:

- The speed of convergence
- Robustness to agent/connection failures
- Energy/ communication efficiency

Graph theoretic abstraction of network

- Group topology affects group performance critically
- Graphs as structural abstractions of neighborhoods/connectivity
- Agents' knowledge of connectivity effects their dynamics
- Structural properties of graphs characterized by relevant matrices

Important graph-related matrices

- Graph Laplacian: L=D-A
- Natural Random walk matrix $P=D^{-1}A$
- Spectrum provides important structural information

Objective

Design problem: Find graph topologies with favorable tradeoff between performance improvement (benefit) of collaborative behaviors vs. costs of collaboration

Performance measures

- The speed of convergence of many distributed algorithms is determined by Second Largest Eigenvalue (SLEM) of P
- The number of spanning trees of a graph is a measure of robustness to losses in many applications

Problem 1 Characterization of Small World networks as efficient topologies (Asymptotic)

Problem 2 Optimal performance enhancement by adding few links

Problem statement and Analysis

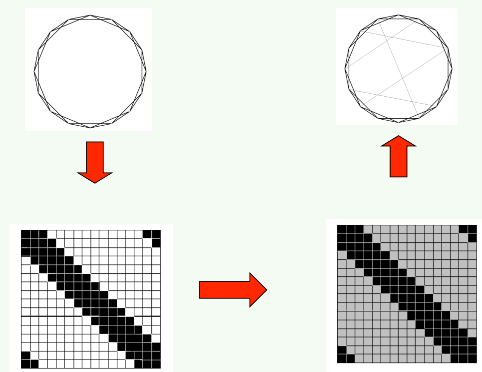
Problem 1 Characterization of Small World graphs

Φ-model: Adding small number of new edges into a regular lattice G_0=C(n,k), (n=number of nodes, 2k=number of initial neighbors of each node)

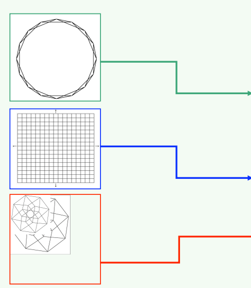
Capture performance measure of G, as property of F(G) → Start from base structure G_0, F_0 →

Perturb zero elements of F_0 by $\varepsilon = Kn^{-\alpha}, \rightarrow F_\varepsilon$ → Analyze $\Delta(F_\varepsilon)$ for large n as a function of α as α varies →

→ Interpret the result as structural perturbation



Results for spectral gap gain



Initial S.G.	SW onset	SW
$O(n^{-2})$	$\varepsilon = O(n^{-3})$	$\varepsilon = O(n^{-2})$
$O(n^{-1})$	$\varepsilon = O(n^{-2})$	$\varepsilon = O(\frac{1}{n\log n})$
$O\left((\log n)^{-1}\right)$	$\varepsilon = O(\frac{1}{n\log n})$	$\varepsilon = O(\frac{1}{n\log\log n})$

Problem 2: Robust network design

Given a base graph topology, add k edges that result in maximum possible number of spanning trees

Matrix-tree theorem

$$\tau(G) = \frac{1}{n}\prod_{j=2}^{n}\lambda_j(L) = \frac{1}{n}\det\left(L + \frac{1}{n}J\right)$$

Optimization problem

Let f_l denote the incidence vector for edge l

$$\text{Maximize} \quad \tau\left(L_0 + \sum_{i=1}^{m} x_i f_i f_i^T\right) \quad \text{or equivalently}$$

$$\log\ \det\ \left(L_0 + \frac{1}{n}J + \sum_{i=1}^{m} x_i f_i f_i^T\right)$$

$$\text{Subject to:}$$

$$1^T x = k$$

$$x \in \{0,1\}^m$$

We relax the problem to find heuristics for design of optimal topologies

Result:

- The optimal graph is determined as a compromise between symmetrizing the graph and minimizing a notion of distance, *effective resistance distance*.
- Effective resistance distance between two nodes i and j: If we consider the graph as a resistive network with unit resistance on the edges, this is the effective resistance between i and j when a unit potential difference is applied between the two node
- The small world effect holds for spanning trees. Asymptotically, union of uniformly generated random spanning trees leads to construction of *expander graphs*.

References

[1] Baras and Hovareshti, Effects of topology in networked systems: stochastic methods and small worlds, CDC08.

[2] Baras and Hovareshti, Efficient and robust communication topologies for distributed decision making in networked systems, CDC09, *Submitted*

[3] Ghosh and Boyd, Growing well-connected graphs, CDC06.

[4] Goyal, Rademacher, and Vempala, Expanders via random spanning trees, SODA2009.

[5] http://www.massimocristaldi.com/