Hamiltonians for Collectives

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From Biology – Allelomimesis – to Control

Allelomimetic behavior – activities in which performance of a behavior increases probability of that behavior being performed by other nearby animals (Wikipedia)

Examples – switching activity, initiation of movement, vigilance bouts, schooling, flocking

Function – there are benefits to social animals in behaving in a similar manner to others within their group; synchrony at a fine scale helps in achieving greater cohesion

Allelomimesis is **autocatalytic** – positive feedback in the form of others copying me copying …

Can we construct optimal control problems for collectives with solutions interpretable as allelomimesis?

Can symmetry-reduction techniques be used in finding solutions?

Are there useful hidden symmetries?

Hamilton 1805-1865

Pontryagin 1908-1988

Lie 1842-1899

Noether 1882-1935

References and the Path Ahead

EWJ and PSK (2011), Optimal Natural Frames, *Comm. Info. and Syst.*, **11**(1):17-34.

EWJ and PSK (2014), Optimality, Reduction and Collective Motion, *Proc. R. Soc. A*, DOI: 10.1098/rspa.2014.0606, online 1 April 2015

Proccacini et. al. (2011), Propagating waves in starling, *Sturnus vulgaris*, flock under predation, *Animal Behaviour*, **82**, 759-765.

Understand Information Transfer in flocks

G. Beauchamp, *Social Predation* (2014)

FIGURE 3.10 A murmuration of European starlings. Waves of turning in tight flocks of starlings propagate as waves, which are thought to deter predation from birds of prey. *Photo credit: Muffin.*

Models, Cost Functionals, Collective Optimal Control & Hamiltonian from Maximum Principle

$$\underset{\xi_1,\xi_2,\cdots,\xi_N}{\text{Minimize}} \quad \mathcal{L} = \int_0^T L(\xi_1(t),...,\xi_N(t))dt \tag{1}$$

subject to: Controlled Dynamics: $\dot{g}_k = g_k\xi_k, g_k \in G$

Fixed Endpoints: $g_k(0) = g_{k0}, \quad g_k(T) = g_{kT}, \quad k = 1,...,N$

- Lagrangian: $L(\xi_1,\ldots,\xi_N) = \frac{1}{2}\left(\sum_{k=1}^{N}|\xi_k|^2 + \chi\sum_{k=1}^{N}\sum_{j=1}^{N}a_{kj}|\xi_k-\xi_j|^2\right)$
- $\chi > 0$ is a constant
- Trace norm: $|\xi|^2 = \mathrm{tr}\left(\xi^T\xi\right), \ \xi \in \mathfrak{g}$
- Inner product: $\langle\xi,\eta\rangle = \mathrm{tr}\left(\xi^T\eta\right), \ \xi,\eta \in \mathfrak{g}$
- Moreover, $\sum_{k=1}^{N}\sum_{j=1}^{N}a_{kj}|\xi_k-\xi_j|^2 = 2\sum_{k=1}^{N}\sum_{j=1}^{N}b_{kj}\langle\xi_j,\xi_k\rangle$ (via symmetry of A)

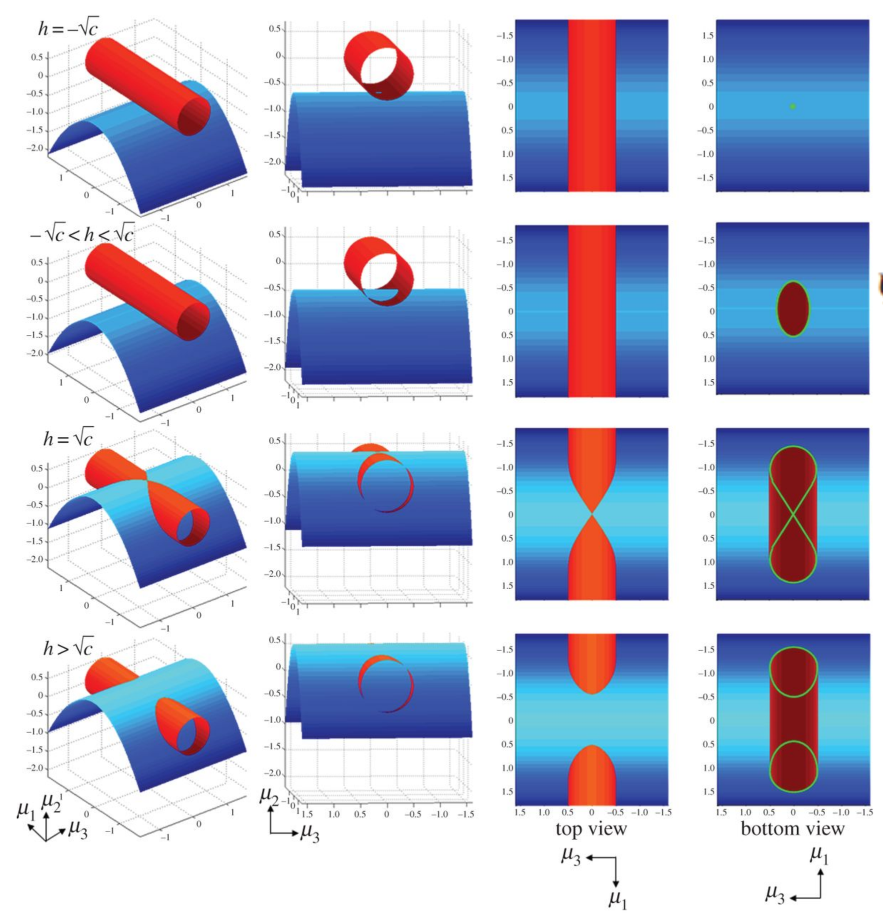
graph adjacency matrix

graph Laplacian matrix

$$H = \sum_{k=1}^{N}\mu_{kq} + \frac{1}{2}[\tilde{\mu}_1^{\mathrm{T}} \ \cdots \ \tilde{\mu}_N^{\mathrm{T}}]\Psi\begin{bmatrix}\tilde{\mu}_1\\ \vdots \\ \tilde{\mu}_N\end{bmatrix} - \frac{1}{2}N,$$

Special case of SE(2) – planar self-steering particles

$$\xi_k = X_2 + u_kX_1, \quad k = 1,\ldots,N,$$



The $\chi \to \infty$ limit

$$\Psi_\infty = \lim_{\chi\to\infty}(\mathbb{I}_N + 2\chi P\tilde{B}P^{\mathrm{T}})^{-1} = P\left[\lim_{\chi\to\infty}(\mathbb{I}_N + 2\chi\tilde{B})^{-1}\right]P^{\mathrm{T}}$$

$$= P\,\mathrm{diag}(1,0,\ldots,0)P^{\mathrm{T}} = \left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)\left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)^{\mathrm{T}},$$

$$\ddot{\alpha}_1 - \frac{h_\alpha}{2}\alpha_1 + \frac{1}{4}\alpha_1^3 = 0.$$

$$u_k = \frac{1}{N}\sum_{j=1}^{N}\mu_{j1} = \alpha_1, \quad k = 1,\ldots,N,$$

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