An Explicit Optimal Scheme for Distributed Lossy Compression

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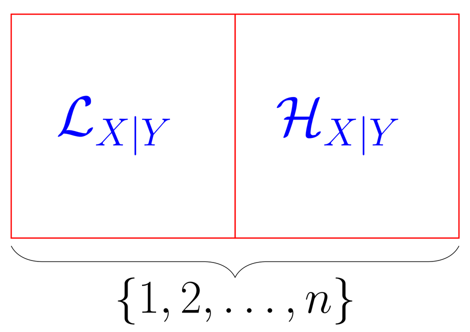
Abstract

We show that polar codes can be used to achieve the rate-distortion functions in the problem of hierarchical lossy compression also known as the successive refinement problem.

Introduction

Polar coding was introduced by Arikan in the seminal paper [Arıkan '09].

Let $n = 2^m, G_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes m}$. Arıkan showed that given a binary-input channel W, there is a sequence of linear codes, whose generator matrices are appropriately chosen from the rows of G_n, achieving the symmetric capacity of W. Later it was proved that polar codes work equally well for source coding [Arıkan '10].



- $X, Y \sim P_{X,Y}$
- Let $U^n = X^n G_n$. $U_{\mathcal{L}_{X|Y}}$ can be determined by previous bits together with Y^n, while $U_{\mathcal{H}_{X|Y}}$ is uniformly random.
- For channel transmission, put information in $U_{\mathcal{L}_{X|Y}}$; for source coding, only need to record $U_{\mathcal{H}_{X|Y}}$.
- $$\lim_{n\to\infty} \frac{1}{n} |\mathcal{H}_{X|Y}| = H(X|Y).$$
  Achieve optimal rates for both channel and source coding.

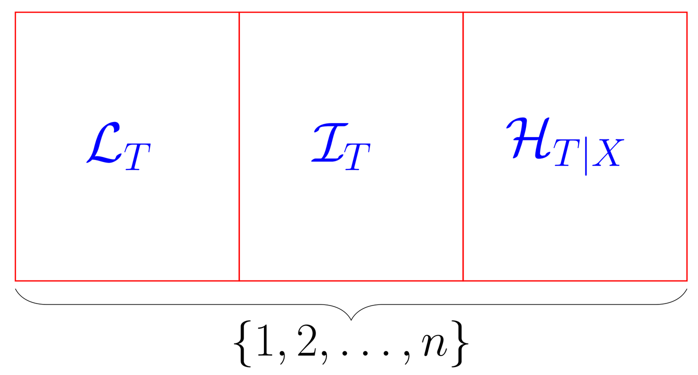
Polar Codes for Lossy Source Coding

Rate-Distortion:

- Source $X \sim P_X$ over a finite alphabet $\mathcal{X}$
- Distortion function $d : \mathcal{X} \times \{0, 1\} \to [0, \infty)$
- Rate distortion function $R(D) = \min_{P_{T|X}} I(X;T)$, where $P_{T|X}$ is such that $E_{XT}(d(X,T)) \le D$.

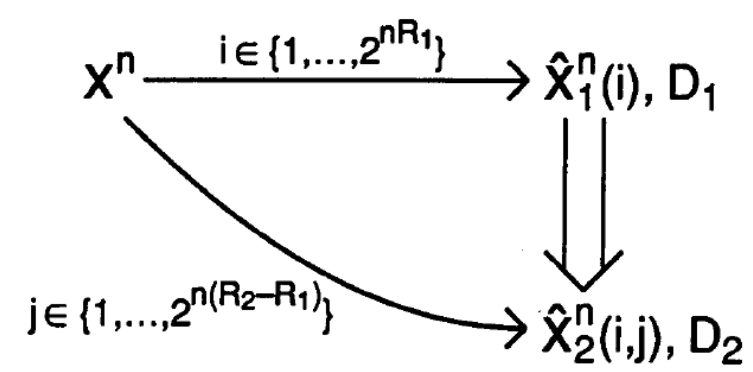
Polar Coding Scheme [Honda and Yamamoto '13]

- Objective: to approximate the distribution $P_{T^n X^n}$.
- Let $U^n = T^n G_n$. $P_{T^n X^n}$ induces a joint distribution $P_{U^n X^n}$. It is equivalent to approximating $P_{U^n X^n}$.
- Divide U^n into three parts.



- For $i \in \mathcal{L}_T$, U_i can be determined by U^{i-1}.
- For $i \in \mathcal{H}_{T|X}$, U_i is independent of U^{i-1} and X^n.
- For $i \in \mathcal{I}_T$, U_i is independent of U^{i-1} but can be determined by U^{i-1} and X^n.
- Set $U_{\mathcal{H}_{T|X}}$ to be uniformly random independently of X^n. Make it known to both the encoder and the decoder.
- The encoder has access to X^n, thus it knows the value of $U_{\mathcal{I}_T}$ and $U_{\mathcal{L}_T}$.
- Transmit $U_{\mathcal{I}_T}$ to the decoder.
  $$\lim_{n\to\infty} \frac{1}{n} |\mathcal{I}_T| = I(X;T) = R(D).$$
- The decoder knows both $U_{\mathcal{H}_{T|X}}$ and $U_{\mathcal{I}_T}$.
- Thus it can recover $U_{\mathcal{L}_T}$ by looking at the previous bits.

Successive Refinement of Information



The source X is said to be successively refinable with distortions D_1 and D_2, $D_2 \le D_1$, if the pair of rate values $(R(D_1), R(D_2))$ is achievable.

Theorem 1 (Koshelev '80, Equitz and Cover '91)**.** *Let X be a source and let T, W be two binary random variables. The source is successively refinable with distortions D_1 and D_2 ($D_2 \le D_1$) if and only if there exists a conditional distribution $P_{TW|X}$ with*

$$E_{XT} d(X,T) \le D_1, \quad E_{XW} d(X,W) \le D_2,$$

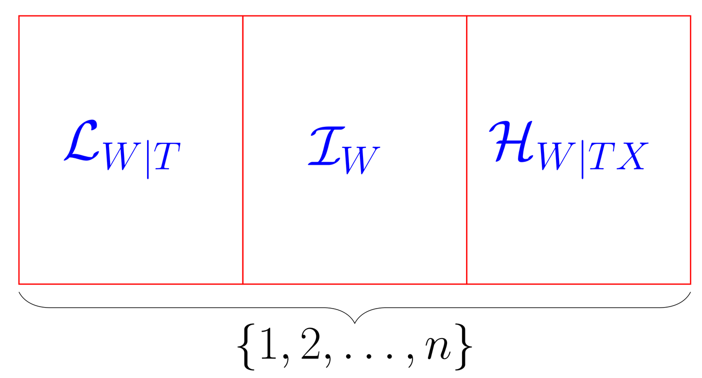
$$I(X;T) = R(D_1), \quad I(X;W) = R(D_2), \tag{1}$$

and such that X, W, T satisfy the Markov condition

$$X \to W \to T. \tag{2}$$

Polar Coding Scheme for Successive Refinement

- Objective: to approximate the distribution $P_{W^n T^n X^n}$.
- Let $U^n = T^n G_n$ and $V^n = W^n G_n$. $P_{W^n T^n X^n}$ induces the joint distribution $P_{V^n U^n X^n}$. It is equivalent to approximate $P_{V^n U^n X^n}$.
- We have shown how to approximate $P_{U^n X^n}$. That is the construction of the coarse layer coding scheme.
- Now we only need to approximate $P_{V^n | U^n X^n}$. Divide V^n into three parts.



- The refinement layer can reconstruct T^n using the encoded sequence at the coarse layer. No need to record $V_{\mathcal{L}_{W|T}}$.
- $V_{\mathcal{H}_{W|TX}}$ is approximately independent of previous bits and $T^n X^n$. No need to record either. Set $V_{\mathcal{H}_{W|TX}}$ to be uniformly random and make these bits known to both encoder and decoder.
- The encoder only needs to record $V_{\mathcal{I}_W}$. Due to the Markov condition,
  $$\frac{|\mathcal{I}_W|}{n} \to R(D_2) - R(D_1).$$

References

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