Data-Fitting using Optimal Control Methods

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Problem Statement

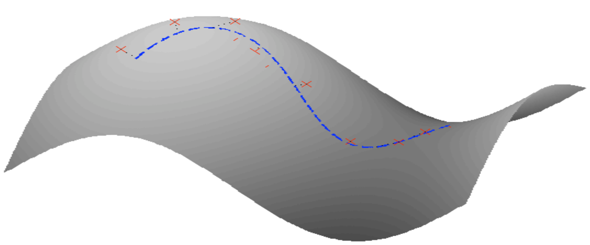
Given noisy, time-indexed observations of the trajectory of a controlled dynamical system on a Riemannian manifold M, we wish to use an optimal-control approach to recover a plausible estimate of the control signal applied (from which we can reconstruct the trajectory, as in the figure below) under the assumption that the physical system is itself implementing a control which is optimal in some sense.

Let $\{q_i\}_{i=0}^N \subset M$ denote the observations and $\{t_i\}_{i=0}^N$ the associated (increasing) time indices, also called "sampling instants." Then our problem is to minimize a cost functional of the form

$$J = \sum_{i=0}^{N} d^2(x(t_i), q_i) + \lambda \int_{t_0}^{t_N} \left(\|u(t)\|^2 + g(x(t))\right) dt,$$

where $d(\cdot,\cdot)$ is the geodesic metric on M, $g(\cdot)$ incorporates state costs, and $\lambda > 0$ is a parameter which reflects our trust in the observations. $u(\cdot)$ is the control and $x(\cdot)$ is the associated trajectory satisfying dynamical constraints

$$\dot{x}(t) = f(t, x(t), u(t)).$$



The Linear Quadratic Case

For $M = \mathbf{R}^n$, we minimize

$$\begin{aligned} J &= \sum_{i=0}^{N} (x(t_i) - q_i)' Q_i (x(t_i) - q_i) \\ &\quad + \lambda \int_{t_0}^{t_N} (x'(t)L(t)x(t) + u'(t)u(t))\, dt, \end{aligned}$$

subject to linear (possibly time-varying) dynamics

$$\dot{x} = Ax + Bu.$$

This problem is solved by recognizing the similarity to the fixed-endpoint problem. We can use path independence lemmas to cancel the quadratic terms $x'(t_i)Q_i x(t_i)$ and linear terms $2x'(t_i)Q_i p_i$ in J if we allow for discontinuities in the control signal. This gives the optimal control

$$u^* = -B'(Kx + \frac{1}{2}\eta),$$

where, for each interval (t_i, t_{i+1}),

$$\begin{aligned} \dot{K} &= -A'K - KA - L + KBB'K, \\ \dot{\eta} &= -(A' - KBB')\eta, \end{aligned}$$

Both K and η are recovered by integrating backwards, where the final conditions at each sampling instant are specified by

$$\begin{aligned} K(t_N^-) &= \lambda^{-1} Q_N, \\ K(t_i^-) - K(t_i^+) &= \lambda^{-1} Q_i, \quad (1 \le i \le N-1) \\ \eta(t_N^-) &= -2\lambda^{-1} Q_N q_N, \\ \eta(t_i^-) - \eta(t_i^+) &= -2\lambda^{-1} Q_i q_i. \quad (1 \le i \le N-1) \end{aligned}$$

Finally, the optimal initial condition satisfies

$$2(Q_0 + \lambda K(t_0))x(t_0) = 2Q_0 q_0 - \lambda \eta(t_0),$$

A Co-state Perspective

If we define $p = Kx + \frac{1}{2}\eta$, then $u = -B'p$, and further, we can show $\dot{p} = -A'p - Lx$. That is, the optimal control is a trajectory of the canonical system

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -L & -A' \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

The discontinuity conditions at the sampling instants become

$$\begin{aligned} Q_0 x(t_0) + \lambda p(t_0) &= Q_0 q_0, \\ Q_i x(t_i) + \lambda(p(t_i^+) - p(t_i^-)) &= Q_i q_i, \ (1 \le i \le N-1) \\ Q_0 x(t_N) - \lambda p(t_N) &= Q_N q_N. \end{aligned}$$

Future Research Directions

Given stereoscopic video data of the trajectory of a bat in pursuit of prey, the problem of recovering the curvatures in the natural Frenet frame was studied [1]. Since the natural Frenet frame lies in $SO(3)$, this problem fits naturally into the present context if we can extend our results to systems defined on Lie groups.

The solution for the linear quadratic problem given above is of the same form as the description of the minimum length curves near data points on Riemannian manifolds given in [2]. Their results coincide with ours when $\dot{x} = u$ and $L = 0$. That is, when the dynamics are trivial and we have control over every component of state, the optimal trajectories are straight lines in $\mathbf{R}^n$. Our work will extend these results to a more general control context, for example, when the number of controls is less than the dimension of the manifold.

[1] Reddy, P.V., *et. al*, Pursuit Laws in Bat-Insect Encounters. Poster session presented at: NSF-NIH Collaborative Research in Computational Neuroscience Program; June 2007.

[2] Machado, Luís, *et. al*, Riemannian means as solutions of variational problems, *LMS J. Comput. Math*, Vol. 9, pp. 86-103, 2006.