Binary bridge selection problem

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Selecting between two bridges

The problem: Given two bridges, denoted A and B, of **equal** length, how should ants $n = 1, 2, \ldots$ select one of the bridges so that **eventually all** of them select the **same** bridge.

The algorithm: We encode the branch selection for the $(n+1)^{rst}$ ant by the $\{A, B\}$-valued rv S_{n+1} such that

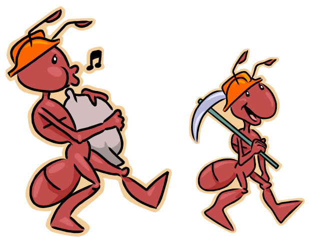
$$S_{n+1} = A \quad \text{if and only if} \quad U_{n+1} \le p_n$$

where

- $\{U_n,\ n = 1, 2, \ldots\}$ is a sequence of i.i.d. rvs **uniformly** distributed on the interval $[0, 1]$,
- p_n is the probability that the $(n+1)^{rst}$ ant selects the branch A; it is given by

$$p_n := \frac{(K + A_n)^\nu}{(K + A_n)^\nu + (K + B_n)^\nu}$$

with $K \ge 0$ and $\nu > 0$ – Here A_n (resp. B_n) is the number of ants among the first n ants which select bridge A (resp. B)



Two-dimensional recursion

$$\begin{array}{l} A_{n+1} = A_n + \mathbf{1}[U_{n+1} \le p_n] \\ B_{n+1} = B_n + \mathbf{1}[U_{n+1} > p_n] \end{array}, \quad n = 1, 2, \ldots$$

The initial condition (A_1, B_1) is **independent** of the driving sequence $\{U_n,\ n = 1, 2, \ldots\}$, and satisfies

$$A_1 + B_1 = 1.$$

E.g., $(A_1, B_1) = (1, 0)$ or $(A_1, B_1) = (0, 1)$.

Consequently,

$$A_n + B_n = n, \quad n = 1, 2, \ldots$$

Thus, we need only consider the evolution of the $\mathbb{R}_+$-valued rvs $\{A_n,\ n = 1, 2, \ldots\}$ which is given through the **one**-dimensional recursion

$$A_{n+1} = A_n + \mathbf{1}[U_{n+1} \le p_n], \quad n = 1, 2, \ldots$$

with

$$p_n = \frac{(K + A_n)^\nu}{(K + A_n)^\nu + (K + n - A_n)^\nu},$$

where the $[0, 1]$-valued rv A_1 is **independent** of the driving sequence $\{U_n,\ n = 1, 2, \ldots\}$.

Main results

Theorem 1 *With $0 < \nu < 1$, it holds that*

$$\lim_{n\to\infty} \frac{A_n}{n} = \lim_{n\to\infty} \frac{B_n}{n} = \frac{1}{2} \quad a.s.$$

whence

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1 \quad a.s.$$

No reinforcement

Theorem 2 *With $\nu = 1$, the sequence of rvs $\{\frac{A_n}{n},\ n = 1, 2, \ldots\}$ converges a.s. to an $[0, 1]$-valued rv $a^\star$; its distribution depends on the initial condition A_1.*

Theorem 3 *With $1 < \nu$, it holds that*

$$\lim_{n\to\infty} \max\left(\frac{A_n}{n}, \frac{B_n}{n}\right) = 1 \quad a.s.$$

with

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{A_n}{n} = 1\right] = \mathbb{P}\left[\lim_{n\to\infty} \frac{B_n}{n} = 1\right] = \frac{1}{2}$$

An equivalent stochastic approximation

Change of variable

$$a_n := \frac{A_n}{n}, \quad n = 1, 2, \ldots$$

so that

$$0 \le a_n \le 1, \quad n = 1, 2, \ldots$$

The original dynamics can now be rewritten as

$$\begin{aligned} a_{n+1} &= \frac{A_n}{n+1} + \frac{1}{n+1}\mathbf{1}[U_{n+1} \le p_n] \\ &= a_n + \frac{1}{n+1}\left(\mathbf{1}[U_{n+1} \le p_n] - a_n\right), \quad n = 1, 2, \ldots \end{aligned}$$

with the $[0, 1]$-valued rv a_1 **independent** of the i.i.d. driving sequence $\{U_n,\ n = 1, 2, \ldots\}$.

Key observation:
This one-dimensional stochastic recursion is
a stochastic approximation of the Robbins-Monro type (of a non-standard type)

Note that

$$p_n = P_\nu\left(a_n, \frac{K}{n}\right)$$

with

$$P_\nu(a, c) := \frac{(a + c)^\nu}{(a + c)^\nu + (1 - a + c)^\nu}, \quad a \in [0, 1],\ c \ge 0.$$

Ant algorithm = Positive feedback

but

Stochastic algorithm = Negative feedback!

A preparatory result and its consequences

Define the $[0, \frac{1}{4}]$-valued rvs $\{V_n,\ n = 1, 2, \ldots\}$ by

$$V_n := \left|a_n - \frac{1}{2}\right|^2, \quad n = 1, 2, \ldots$$

Proposition 1 *Under the summability condition*

$$\sum_{n=1}^{\infty} \frac{1}{n+1} |(2a_n - 1)(p_n - a_n)| < \infty \quad a.s., \tag{1}$$

there exists an $[0, \frac{1}{4}]$-valued rv V such that

$$\lim_{n\to\infty} V_n = V \quad a.s.$$

Corollary 1 *Assume $\nu \neq 1$. Under the assumption (1), we have*

$$\mathrm{Acc}(a_n,\ n = 1, 2, \ldots) \subseteq \left\{0, 1, \frac{1}{2}\right\} \quad a.s.$$

where $\mathrm{Acc}(a_n,\ n = 1, 2, \ldots)$ denotes the set of accumulation points of the sequence $\{a_n,\ n = 1, 2, \ldots\}$, and the limiting rv V appearing in Theorem 1 is therefore an $\{0, \frac{1}{4}\}$-valued rv.

The convergence (1) yields

$$\lim_{n\to\infty} \frac{n}{n+1} |(2a_n - 1)(p_n - a_n)| = 0 \quad a.s.$$

or equivalently,

$$\lim_{n\to\infty} (2a_n - 1)(p_n - a_n) = 0 \quad a.s.$$

Therefore,

$$(2\alpha - 1)(P_\nu(\alpha, 0) - \alpha) = 0, \quad \alpha \in \mathrm{Acc}(a_n,\ n = 1, 2, \ldots)$$

with

$$P_\nu(\alpha, 0) - \alpha = \frac{\alpha^\nu}{\alpha^\nu + (1 - \alpha)^\nu} - \alpha = 0 \quad \textbf{iff} \quad \alpha = 0 \text{ or } 1$$

so that the equation

$$(2\alpha - 1)(P_\nu(\alpha, 0) - \alpha) = 0$$

has only **three** possible solutions, namely $\alpha = 0, \frac{1}{2}, 1$

Proposition 2 *When $\nu \neq 1$, the sequence of rvs $\{a_n, n = 1, 2, \ldots\}$ converges a.s. to an $\{0, \frac{1}{2}, 1\}$-valued rv $a^\star$*

Establishing the summability condition (1)

We do so by showing instead that

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \frac{1}{n+1} |(2a_n - 1)(p_n - a_n)|\right] < \infty$$

Basic ingredients

- **Martingale** methods – Take the expectation of V_{n+1} and use the martingale property for $\{M_n,\ n = 1, 2, \ldots\}$ with $\mathbb{E}[M_{n+1}] = \mathbb{E}[M_1] = 1, \quad n = 1, 2, \ldots$
- **Boundedness** of $\{V_n,\ n = 1, 2, \ldots\}$ with $0 \le V_n \le 1, \quad n = 1, 2, \ldots$
- Properties of $P_\nu(a, c)$ – **Concavity/convexity** properties on $[0, \frac{1}{2}]$ are determined by the value of ν, namely $\nu < 1$ vs. $1 < \nu$.