A Step Beyond the State of the Art
Robust Model Predictive Control
Synthesis Methods

by

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(based on recent collaborative research with

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- Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks, IEEE-TAC Regular Paper, (Published)
  (Raković and Barić),

- Parameterized Tube MPC, IEEE-TAC Regular Paper, (Accepted)
  (Raković, Kouvaritakis, Cannon, Panos and Findeisen),

- Fully Parameterized Tube MPC, IFAC 2011, (Published)
  (Raković, Kouvaritakis, Cannon, Panos and Findeisen),

- Fully Parameterized Tube MPC, IJRNC D. W. Clarke's Special Issue Paper, (Accepted)
  (Raković, Kouvaritakis, Cannon and Panos),

§0 – Outlook

Outlook

- Setting & Objectives §1
- Earlier Robust Model Predictive Control Methods §2
- Fully Parameterized Tube Optimal & Model Predictive Control §3
- Comparative Remarks & Illustrative Examples §4
- Concluding Remarks §5

§1 – Setting & Objectives

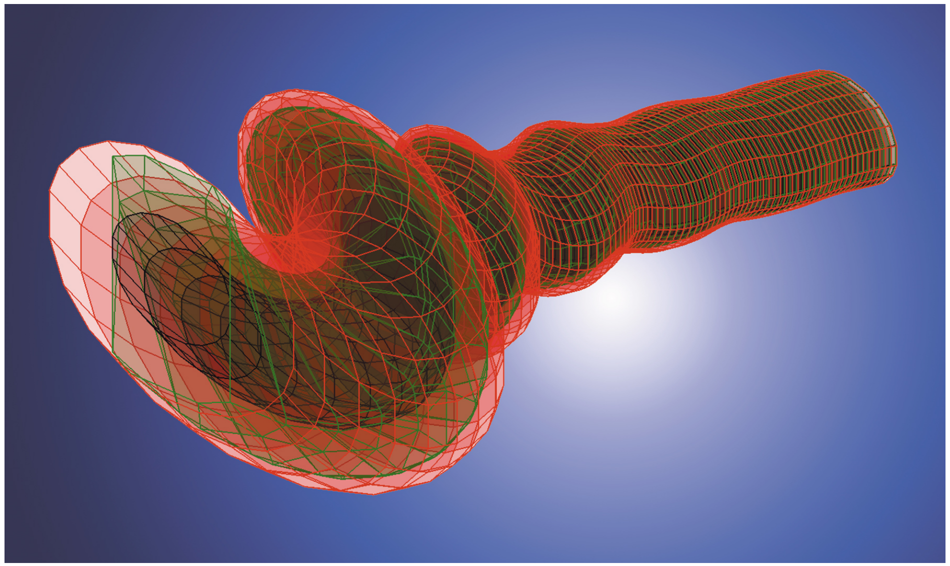
- System Description
- Problem Description
- Synthesis Objectives

System Description – Min–Max Linear–PCP Case

- Linear discrete time system $x^+ = Ax + Bu + w$,
- Variables $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^n$ and $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$,
- Constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $w \in \mathbb{W}$,
- Sets $\mathbb{X} \in \mathrm{PolyPC}(\mathbb{R}^n)$, $\mathbb{U} \in \mathrm{PolyPC}(\mathbb{R}^m)$ and $\mathbb{W} \in \mathrm{PolyC}(\mathbb{R}^n)$,
- Matrix pair (A, B) stabilizable,
- Information is variable x so feedback rules $u(x) \ : \ \mathbb{X} \to \mathbb{U}$.

Brief Problem Description – Illustration

- Given an integer $N \in \mathbb{N}_+$ and $x \in \mathbb{X}$, select (if possible):
  - State Tube $\mathbf{X}_N := \{X_k\}_{k \in \mathbb{N}_{[0:N]}}$,
  - Control Tube $\mathbf{U}_{N-1} := \{U_k\}_{k \in \mathbb{N}_{[0:N-1]}}$, and
  - Control Policy $\Pi_{N-1} := \{\pi_k(\cdot)\}_{k \in \mathbb{N}_{[0:N-1]}}$ such that



$$
\begin{aligned}
& x \in X_0, \ \forall k \in \mathbb{N}_{N-1}, \\
& X_k \subseteq \mathbb{X}, \\
& U_k \subseteq \mathbb{U}, \\
& \forall y \in X_k, \ Ay + B\pi_k(y) \oplus \mathbb{W} \subseteq X_{k+1}, \\
& \forall y \in X_k, \ \pi_k(y) \in U_k, \\
& X_N \subseteq \mathbb{X}_f \subseteq \mathbb{X},
\end{aligned}
$$

which optimize $V_N(\mathbf{X}_N, \mathbf{U}_{N-1}) := \sum_{k \in \mathbb{N}_{N-1}} L(X_k, U_k) + V_F(X_N)$.

Synthesis Objectives & Key Ingredients

- Equally Important Objectives
  - Robust Constraint Satisfaction,
  - Robust Stability (Boundedness and Attractiveness),
  - Computational Practicability,
  - Optimized (Meaningful) Performance.
- Key Ingredients
  - Fully Parameterized Tubes,
  - Induced, More General, Non–Linear Control Policy,
  - Repetitive Online Implementation.

§2 – Earlier Robust Model Predictive Control Methods

- Open–Loop Min–Max MPC $\times$, $\times$, $\times$,
- Feedback Min–Max MPC $\checkmark$, $\checkmark$, $\times$,
- Dynamic Programming Based Robust MPC $\checkmark$, $\times$, $\times$,
- Tube MPC $\checkmark$, $\times$, $\checkmark$,
- Disturbance Affine Feedback RMPC $\checkmark$, $\checkmark$, $\checkmark$.

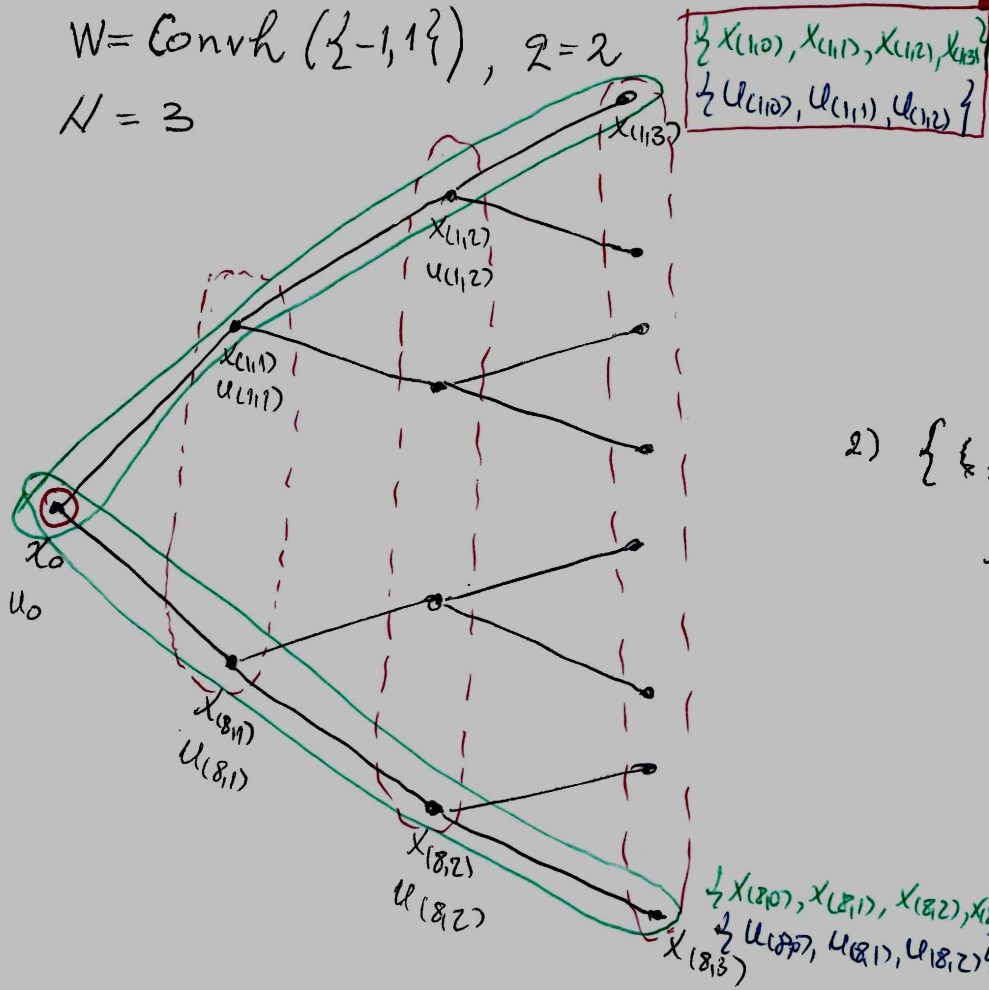
Feedback Min–Max OC and MPC – Preview

Feedback Min-Max Optimal Control

$x_{k+1} = x_k + u_k + w_k$

$W = \mathrm{Convh}(\{-1,1\})$, $q = 2$

$N = 3$



1) $\boxed{\underline{w}_{(1,2)} = \{1,1,1\}}$

$\underline{w}_{(2,2)} = \{1,1,-1\}$

$\underline{w}_{(3,2)} = \{1,-1,1\}$

$\underline{w}_{(4,2)} = \{1,-1,-1\}$

$\underline{w}_{(5,2)} = \{-1,1,1\}$

$\underline{w}_{(6,2)} = \{-1,1,-1\}$

$\underline{w}_{(7,2)} = \{-1,-1,1\}$

$\underline{w}_{(8,2)} = \{-1,-1,-1\}$

$N_d = q^N = 2^3 = 8$

2) $\{\{\underline{x}_{(i,3)}\},\{\underline{u}_{(i,2)}\}\}$, $i = 1,2,\dots,8$

$x_{(i,k+1)} = x_{(i,k)} + u_{(i,k)} + w_{(i,k)}$

$k = 0,1,2$

$x_{(1,1)} = x_{(1,0)} + u_{(1,0)} + w_{(1,0)}$
$= x_0 + u_0 + 1$

$x_{(1,2)} = x_{(1,1)} + u_{(1,1)} + w_{(1,1)}$
$= x_0 + u_0 + u_{(1,1)} + 1 + 1$

$x_{(1,3)} = x_{(1,2)} + u_{(1,2)} + w_{(1,2)}$
$= x_0 + u_0 + u_{(1,1)} + u_{(1,2)} + 1 + 1 + 1$

- The set $\mathbf{w}_{N_d}$ of extreme disturbance sequences $\mathbf{w}_{(i,N-1)} := \{w_{(i,k)}\}_{k\in\mathbb{N}_{N-1}}$, with $w_{(i,k)} \in \mathrm{Vertices}(\mathbb{W})$,
- A set $\mathbf{u}_{N_d}$ of extreme control sequences $\mathbf{u}_{(i,N-1)} := \{u_{(i,k)}\}_{k\in\mathbb{N}_{N-1}}$,
- A set $\mathbf{x}_{N_d}$ of extreme state sequences $\mathbf{x}_{(i,N)} := \{x_{(i,k)}\}_{k\in\mathbb{N}_{N}}$,
- A sensible decision making process for selecting $\mathbf{u}_{N_d} := \{\mathbf{u}_{(i,N-1)} \ : \ i \in \mathbb{N}_{[1:N_d]}\}$, and $\mathbf{x}_{N_d} := \{\mathbf{x}_{(i,N)} \ : \ i \in \mathbb{N}_{[1:N_d]}\}$. (here $N_d := q^N$, and $q := \mathrm{Cardinality}(\mathrm{Vertices}(\mathbb{W}))$.)

Feedback Min–Max OC – Decision Making Process

- Given $N \in \mathbb{N}_+$ and $x \in \mathbb{X}$, select (if possible) sets of extreme:
  - State Sequences $\mathbf{x}_{N_d} = \{\mathbf{x}_{(i,N)} \ : \ i \in \mathbb{N}_{[1:N_d]}\}$ and
  - Control Sequences $\mathbf{u}_{N_d} = \{\mathbf{u}_{(i,N-1)} \ : \ i \in \mathbb{N}_{[1:N_d]}\}$ such that

$$\begin{aligned}
&\forall i \in \mathbb{N}_{[1:N_d]},\ \forall k \in \mathbb{N}_{N-1},\\
&x_{(i,k+1)} = Ax_{(i,k)} + Bu_{(i,k)} + w_{(i,k)}, \text{ with } x_{(i,0)} = x,\\
&x_{(i,k)} \in \mathbb{X},\ u_{(i,k)} \in \mathbb{U}, \text{ and } x_{(i,N)} \in \mathbb{X}_f,\\
&\forall (i_1, i_2) \in \mathbb{N}_{[1:N_d]} \times \mathbb{N}_{[1:N_d]},\ \forall k \in \mathbb{N}_{N-1},\\
&x_{(i_1,k)} = x_{(i_2,k)} \Rightarrow u_{(i_1,k)} = u_{(i_2,k)}
\end{aligned}$$

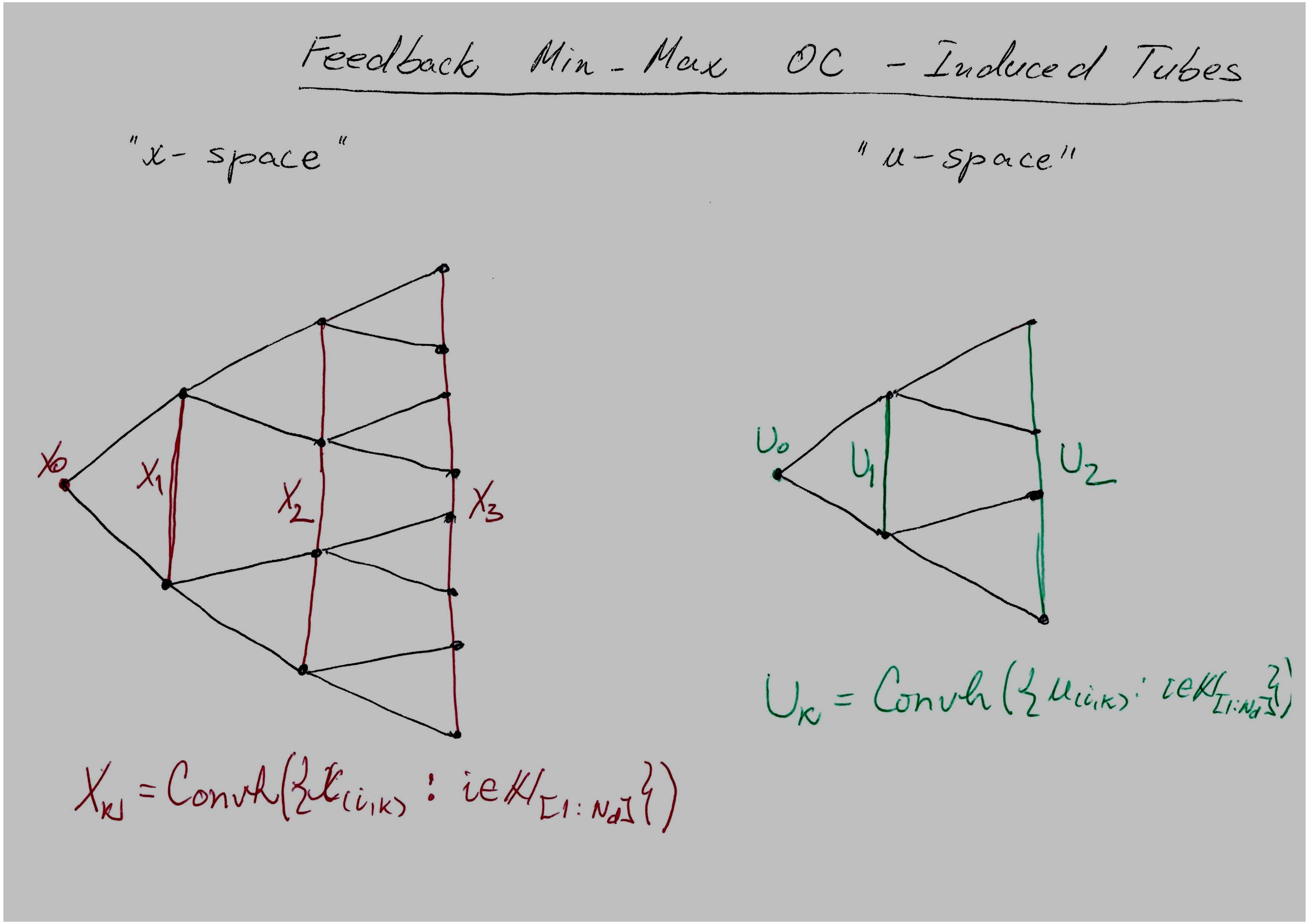
which minimize

$$\begin{aligned}
&V_N(\mathbf{x}_{N_d}, \mathbf{u}_{N_d}) := \max_i \{V_{(i,N)}(\mathbf{x}_{N_d}, \mathbf{u}_{N_d}) \ : \ i \in \mathbb{N}_{[1:N_d]}\}, \text{ where}\\
&V_{(i,N)}(\mathbf{x}_{N_d}, \mathbf{u}_{N_d}) := \sum_{k \in \mathbb{N}_{N-1}} \ell(x_{(i,k)}, u_{(i,k)}) + V_f(x_{(i,N)}).
\end{aligned}$$

Feedback Min–Max MPC – Summarized

- Repetitive Online Application of Feedback Min–Max OC,
- Dimension of Decision Variable Proportional to $N_d = q^N$,
- Number of Constraints Proportional to $N_d = q^N$,
- Computation Exceedingly Demanding and Impracticable.

Feedback Min–Max OC and MPC – Summarized



Feedback Min–Max OC – Important Remarks

- Feedback Min–Max OC Utilizes:
  - State Tubes $\mathbf{X}_N := \{X_k\}_{k \in \mathbb{N}_N}$, with $X_k := \mathrm{Convh}(\{x_{(i,k)} \ : \ i \in \mathbb{N}_{[1:N_d]}\})$, and
  - Control Tubes $\mathbf{U}_{N-1} := \{U_k\}_{k \in \mathbb{N}_{N-1}}$, with $U_k := \mathrm{Convh}(\{u_{(i,k)} \ : \ i \in \mathbb{N}_{[1:N_d]}\})$.
  - Induced Control Policy $\mathbf{\Pi}_{N-1} := \{\pi_k(\cdot, X_k, U_k)\}_{k \in \mathbb{N}_{N-1}}$, with $\pi_k(\cdot, X_k, U_k) \ : \ X_k \to U_k$.
- Feedback Min–Max OC Indicates Weakness of Open Loop Min–Max OC:
  - Additional Constraints $\forall i \in \mathbb{N}_{[1:N_d]}, \ \forall k \in \mathbb{N}_{N-1}, \ u_{(i,k)} = u_k$.

Disturbance Affine Feedback ROC and RMPC – Preview

Disturbance Affine Feedback Robust OC - Illustration

$$(z_0, v_0) \longrightarrow (z_1, v_1) \longrightarrow (z_2, v_2) \longrightarrow \ldots \longrightarrow (z_{N-1}, v_{N-1}) \longrightarrow z_N$$

$$(T_{(1,0)} w_0, M_{(1,0)} w_0) \longrightarrow (T_{(2,0)} w_0, M_{(2,0)} w_0) \longrightarrow \ldots \longrightarrow (T_{(N-1,0)} w_0, M_{(N-1,0)} w_0) \longrightarrow T_{(N,0)} w_0$$

$$(T_{(2,1)} w_1, M_{(2,1)} w_1) \longrightarrow \ldots \longrightarrow (T_{(N-1,1)} w_0, M_{(N-1,1)} w_0) \longrightarrow T_{(N,1)} w_1$$

Novel Interpretation

$$(T_{(N-1,N-2)} w_{(N-2)}, M_{(N-1,N-2)} w_{N-2}) \longrightarrow T_{(N,N-2)} w_{N-2}$$

$$\longrightarrow T_{(N,N-1)} w_{N-1}$$

$$\boxed{z_{k+1} = A z_k + B v_k}$$

$u_0 = v_0$	$u_1 = v_1 + M_{(1,0)} w_0$	$\ldots$	$u_k = v_k + \sum_{j=0}^{k-1} M_{(k,j)} w_j$
$x_0 = z_0$	$x_1 = z_1 + T_{(1,0)} w_0$		$x_k = z_k + \sum_{j=0}^{k-1} T_{(k,j)} w_j$

$$M_{(k,j)} \in \mathbb{R}^{m \times n}, \quad T_{(k,j)} \in \mathbb{R}^{n \times n}, \quad T_{(k+1,j)} = A T_{(k,j)} + B M_{(k,j)}$$

$$T_{(k+1,k)} = I$$

Disturbance Affine Feedback (DAF) ROC – Basic Idea

- Control Parameterization $u_0 = v_0,\ u_k = v_k + \sum_{j=0}^{k-1} M_{(k,j)} w_j$, with $M_{(k,j)} \in \mathbb{R}^{m \times n}$,
- State Parameterization $x = x_0 = z_0,\ x_k = z_k + \sum_{j=0}^{k-1} T_{(k,j)} w_j$, with $T_{(k,j)} \in \mathbb{R}^{n \times n}$,
- A set $\mathbf{M}_{N-1}$ of control matrices $\{M_{(k,j)}\ :\ j \in \mathbb{N}_{k-1},\ k \in \mathbb{N}_{[1:N-1]}\}$,
- A nominal control sequence $\mathbf{v}_{N-1} := \{v_k\}_{k \in \mathbb{N}_{N-1}}$,
- A set $\mathbf{T}_N$ of state matrices $\{T_{(k,j)}\ :\ j \in \mathbb{N}_{k-1},\ k \in \mathbb{N}_{[1:N]}\}$,
- A nominal state sequence $\mathbf{z}_N := \{z_k\}_{k \in \mathbb{N}_N}$,
- A sensible decision making process for selecting $\mathbf{M}_{N-1}$, $\mathbf{v}_{N-1}$, $\mathbf{T}_N$, and $\mathbf{z}_N$.

DAF ROC – Decision Making Process

- Given $N \in \mathbb{N}_+$ and $x \in \mathbb{X}$, select (if possible) sets of:
  - State and Control Matrices $\mathbf{T}_N$ and $\mathbf{M}_{N-1}$ and
  - Nominal State and Control Sequences $\mathbf{z}_N$ and $\mathbf{v}_{N-1}$ such that

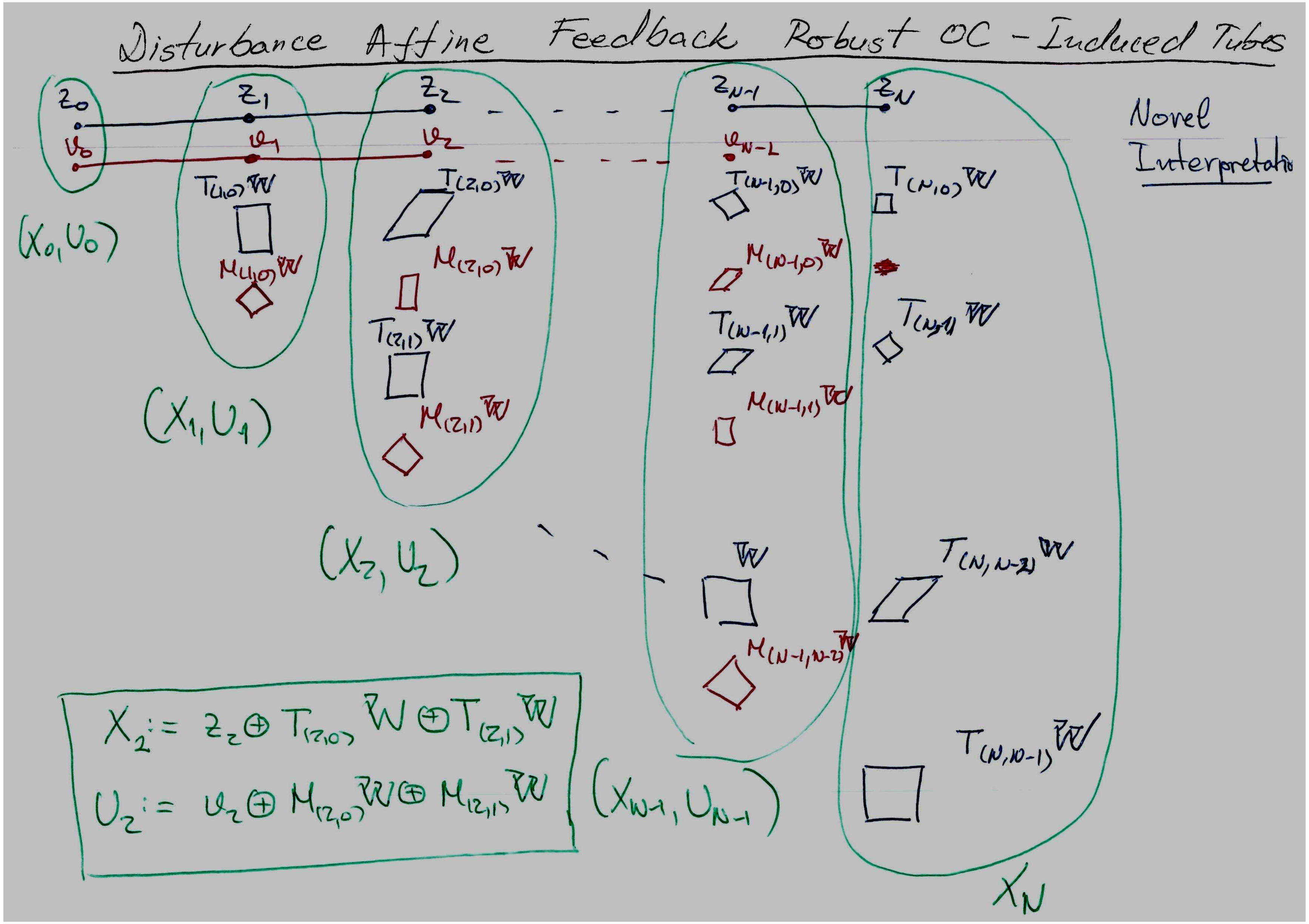
$$\boxed{\begin{array}{l}
\forall k \in \mathbb{N}_{N-1}, z_{k+1} = Az_k + Bv_k, \text{ with } z_0 = x \in \mathbb{X},\ v_0 = u_0 \in \mathbb{U}, \\
\forall k \in \mathbb{N}_{[1:N-1]}, \\
z_k \oplus \bigoplus_{j=0}^{k-1} T_{(k,j)}\mathbb{W} \subseteq \mathbb{X},\ v_k \oplus \bigoplus_{j=0}^{k-1} M_{(k,j)}\mathbb{W} \subseteq \mathbb{U}, \text{ and, } z_N \oplus \bigoplus_{j=0}^{N-1} T_{(N,j)}\mathbb{W} \subseteq \mathbb{X}_f, \\
\forall j \in \mathbb{N}_{k-1}, \\
T_{(k+1,j)} = AT_{(k,j)} + BM_{(k,j)} \text{ with } T_{(k+1,k)} = I.
\end{array}}$$

which minimize a sensible cost

$$\boxed{V_N(\mathbf{x}_N, \mathbf{u}_{N-1}, \mathbf{T}_N, \mathbf{M}_{N-1}) := \sum_{k \in \mathbb{N}_{N-1}} \ell(z_k, v_k, T_k, M_k) + V_f(z_N, T_N).}$$

Disturbance Affine Feedback RMPC – Summarized

- Repetitive Online Application of Disturbance Affine Feedback ROC,
- Dimension of Decision Variable Proportional to hN^2,
- Number of Constraints Proportional to hN^2,
- Computation Practicable.



Disturbance Affine Feedback Robust OC - Induced Tubes
Novel Interpretation
z_0
v_0
(X_0, U_0)
z_1
v_1
$T_{(1,0)}\mathbb{W}$
$M_{(1,0)}\mathbb{W}$
(X_1, U_1)
z_2
v_2
$T_{(2,0)}\mathbb{W}$
$M_{(2,0)}\mathbb{W}$
$T_{(2,1)}\mathbb{W}$
$M_{(2,1)}\mathbb{W}$
(X_2, U_2)
z_{N-1}
v_{N-1}
$T_{(N-1,0)}\mathbb{W}$
$M_{(N-1,0)}\mathbb{W}$
$T_{(N-1,1)}\mathbb{W}$
$M_{(N-1,1)}\mathbb{W}$
$\mathbb{W}$
$M_{(N-1,N-2)}\mathbb{W}$
(X_{N-1}, U_{N-1})
z_N
$T_{(N,0)}\mathbb{W}$
$T_{(N,1)}\mathbb{W}$
$T_{(N,N-2)}\mathbb{W}$
$T_{(N,N-1)}\mathbb{W}$
X_N
$X_2 := z_2 \oplus T_{(2,0)}\mathbb{W} \oplus T_{(2,1)}\mathbb{W}$
$U_2 := v_2 \oplus M_{(2,0)}\mathbb{W} \oplus M_{(2,1)}\mathbb{W}$

Disturbance Affine Feedback ROC – Important Remarks

- Disturbance Affine Feedback ROC Utilizes:
  - State Tubes $\mathbf{X}_N := \{X_k\}_{k \in \mathbb{N}_N}$, with $X_k := z_k \oplus \bigoplus_{j=0}^{k-1} T_{(k,j)} \mathbb{W}$, and
  - Control Tubes $\mathbf{U}_{N-1} := \{U_k\}_{k \in \mathbb{N}_{N-1}}$, with $U_k := v_k \oplus \bigoplus_{j=0}^{k-1} M_{(k,j)} \mathbb{W}$.
  - Disturbance Affine Control Policy $\mathbf{\Pi}_{N-1} := \{\pi_k(\cdot, X_k, U_k)\}_{k \in \mathbb{N}_{N-1}}$.
- Disturbance Affine Feedback ROC Indicates Weakness of Open Loop Min–Max OC:
  - Additional Constraints $M_{(k,j)} = 0$ and $T_{(k,j)} = A^{k-1}$ (Problems for Unstable A).

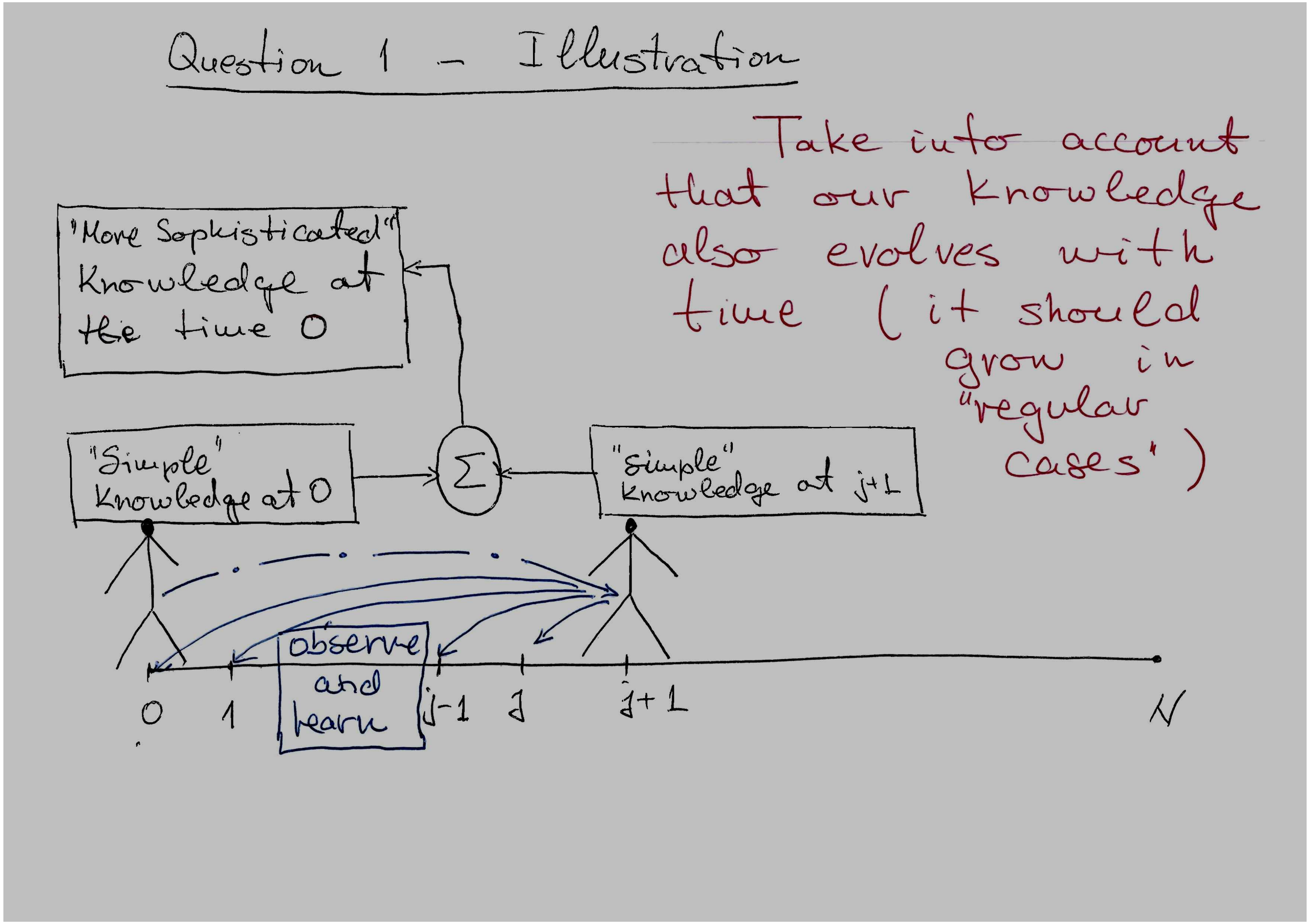
§3 – Fully Parameterized Tube Optimal & Model Predictive Control

- Prediction Structure
- Constraint Handling
- Sensible Cost
- FPT Optimal & Model Predictive Control
- System Theoretic Properties

FPT Prediction Structure – Question 1

- What if:
  - $x_{(j,0)} = x \in \mathbb{R}^n$ was known,
  - w–player acted only once at $j \in \mathbb{N}_{N-1}$, and
  - $j \in \mathbb{N}_{N-1}$ (at which $w_j \in \mathbb{W}$ would happen) was also known?

Question 1 – Illustration



FPT Prediction Structure – Answer to Question 1

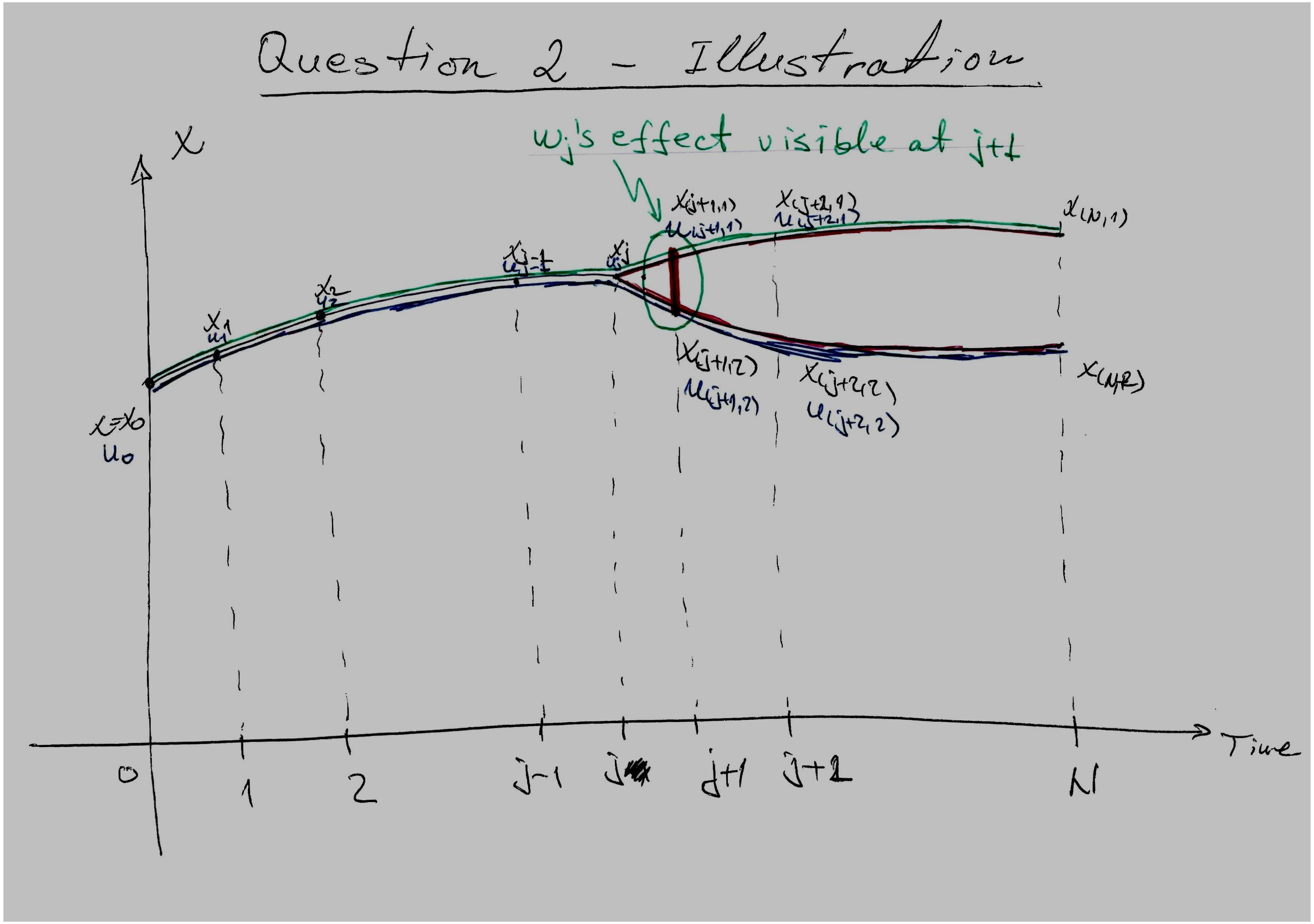
- Question was: What if:
  - $x_{(j,0)} = x \in \mathbb{R}^n$ was known,
  - w–player acted only once at $j \in \mathbb{N}_{N-1}$, and
  - $j \in \mathbb{N}_{N-1}$ (at which $w_j \in \mathbb{W}$ would happen) was also known?
- An answer could be:

$$\forall k \in \mathbb{N}_{N-1},\ x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)} + \delta_{(j,k)} w_k \text{ with}$$
$$\delta_{(j,j)} = 1 \text{ for } j = k \text{ and } \delta_{(j,k)} = 0 \text{ otherwise.}$$

  - Use a Simple Sequence $\mathbf{u}_{(j,N-1)}(\cdot) := \{u_{(j,k)}(\cdot)\}_{k \in \mathbb{N}_{N-1}}$!
  - $\{u_{(j,k)}(\cdot)\}_{k \in \mathbb{N}_j}$ function of $x_{(j,0)}$,
  - $\{u_{(j,k)}(\cdot)\}_{k \in \mathbb{N}_{[j+1:N-1]}}$ function of $x_{(j,j+1)}$!

- What if:
  - $x_{(j,0)} = x \in \mathbb{R}^n$ was known,
  - w–player acted only once at $j \in \mathbb{N}_{N-1}$,
  - $j \in \mathbb{N}_{N-1}$ (at which $w_j \in \mathbb{W}$ would happen) was known, and,
  - $\mathbb{W} = \mathrm{Convh}(\{\tilde{w}_i \ : \ i \in N_{[1:q]}\})$ and points $\tilde{w}_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:q]}$ were known?

Question 2 – Illustration



FPT Prediction Structure – Answer to Question 2

- Question was: What if:
  - $x_{(j,0)} = x \in \mathbb{R}^n$ was known,
  - w–player acted only once at $j \in \mathbb{N}_{N-1}$,
  - $j \in \mathbb{N}_{N-1}$ (at which $w_j \in \mathbb{W}$ would happen) was known, and,
  - $\mathbb{W} = \text{Convh}(\{\tilde{w}_i \ : \ i \in N_{[1:q]}\})$ and points $\tilde{w}_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:q]}$ were known?

- An answer could be:

$\forall i \in \mathbb{N}_{[1:q]}, \ \forall k \in \mathbb{N}_{N-1}, \ x_{(i,j,k+1)} = Ax_{(i,j,k)} + Bu_{(i,j,k)} + \delta_{(j,k)}\tilde{w}_i$ with $\delta_{(j,j)} = 1$ for $j = k$ and $\delta_{(j,k)} = 0$ otherwise.

  - Use q Control Sequences $\mathbf{u}_{(i,j,N-1)} := \{u_{(i,j,k)}\}_{k \in \mathbb{N}_{N-1}}, \ i \in \mathbb{N}_{[1:q]}$!
  - Each $\{u_{(i,j,k)}\}_{k \in \mathbb{N}_j}$ function of $x_{(j,0)}$!

FPT Prediction Structure – Question 3

- Can we Make Use of q Control Sequences $\mathbf{u}_{(i,j,N-1)} := \{u_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}},\ i \in \mathbb{N}_{[1:q]}$?

FPT Prediction Structure – Answer to Question 3

- Question was: Can we Make Use of q Control Sequences $\mathbf{u}_{(i,j,N-1)} := \{u_{(i,j,k)}\}_{k \in \mathbb{N}_{N-1}},\ i \in \mathbb{N}_{[1:q]}$?
- An answer could be: YES

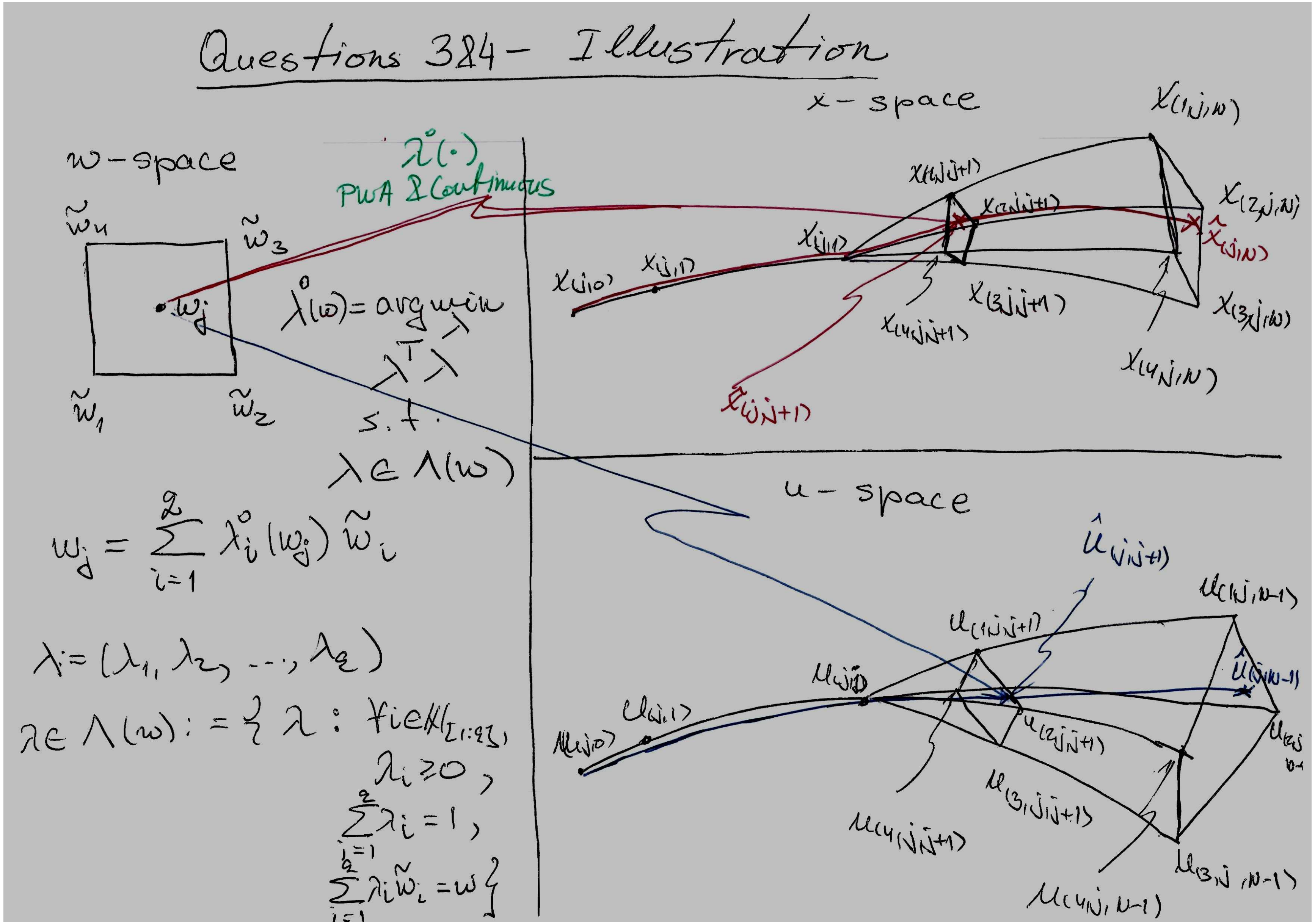
$$\boxed{\begin{array}{l} \forall k \in \mathbb{N}_j, \\ x_{(1,j,k)} = x_{(2,j,k)} = \ldots = x_{(q,j,k)} = x_{(j,k)} \text{ and} \\ u_{(1,j,k)} = u_{(2,j,k)} = \ldots = u_{(q,j,k)} = u_{(j,k)}. \end{array}}$$

- ♦ Ensure Causality,
- ♦ Employ Linearity and Convexity!

FPT Prediction Structure – Question 4

- How to Make Use of q Control Sequences $\mathbf{u}_{(i,j,N-1)} := \{u_{(i,j,k)}\}_{k \in \mathbb{N}_{N-1}},\ i \in \mathbb{N}_{[1:q]}$?

Questions 3 and 4 – Illustration



FPT Prediction Structure – Answer to Question 4

- Question was: How to Make Use of q Control Sequences $\mathbf{u}_{(i,j,N-1)} := \{u_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}, \; i \in \mathbb{N}_{[1:q]}$?
- An answer could be: Easy because

$$x_{(i,j,k+1)} = Ax_{(i,j,k)} + Bu_{(i,j,k)} + \delta_{(j,k)}\tilde{w}_i \Rightarrow$$

$$\lambda_i x_{(i,j,k+1)} = A\lambda_i x_{(i,j,k)} + B\lambda_i u_{(i,j,k)} + \delta_{(j,k)}\lambda_i\tilde{w}_i \Rightarrow$$

$$\sum_{i=1}^{q}\lambda_i x_{(i,j,k+1)} = A\sum_{i=1}^{q}\lambda_i x_{(i,j,k)} + B\sum_{i=1}^{q}\lambda_i u_{(i,j,k)} + \delta_{(j,k)}\sum_{i=1}^{q}\lambda_i\tilde{w}_i$$

$$\forall\lambda\in\Lambda := \{\lambda\in\mathbb{R}^q_+ \; : \; \sum_{i=1}^{q}\lambda_i = 1\}$$

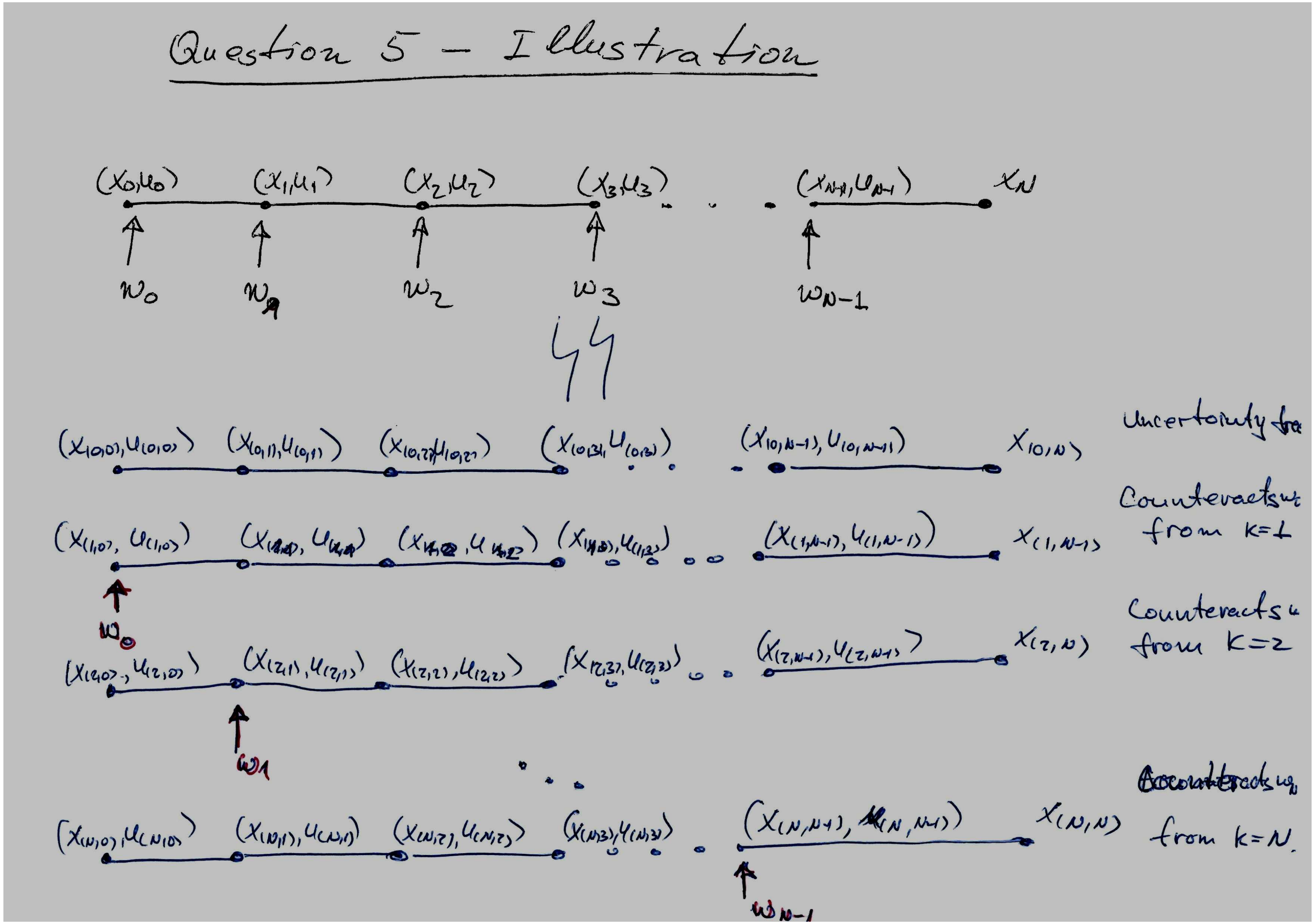
$x_{(j,k+1)}(\lambda) = Ax_{(j,k)}(\lambda) + Bu_{(j,k)}(\lambda) + \delta_{(j,k)}w_k(\lambda)$ with

$x_{(j,k)}(\lambda) = \sum_{i=1}^{q}\lambda_i x_{(i,j,k)}, \; u_{(j,k)}(\lambda) = \sum_{i=1}^{q}\lambda_i u_{(i,j,k)}$ and $w_k(\lambda) = \sum_{i=1}^{q}\lambda_i\tilde{w}_i$

FPT Prediction Structure – Question 5

- What if:
  - $x \in \mathbb{R}^n$ was known,
  - w–player acted at all $k \in \mathbb{N}_{N-1}$ with $w_k \in \mathbb{W}$, and
  - $\mathbb{W} = \mathrm{Convh}(\{\tilde{w}_i \ : \ i \in N_{[1:q]}\})$ and points $\tilde{w}_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:q]}$ were known?

Question 5 – Illustration



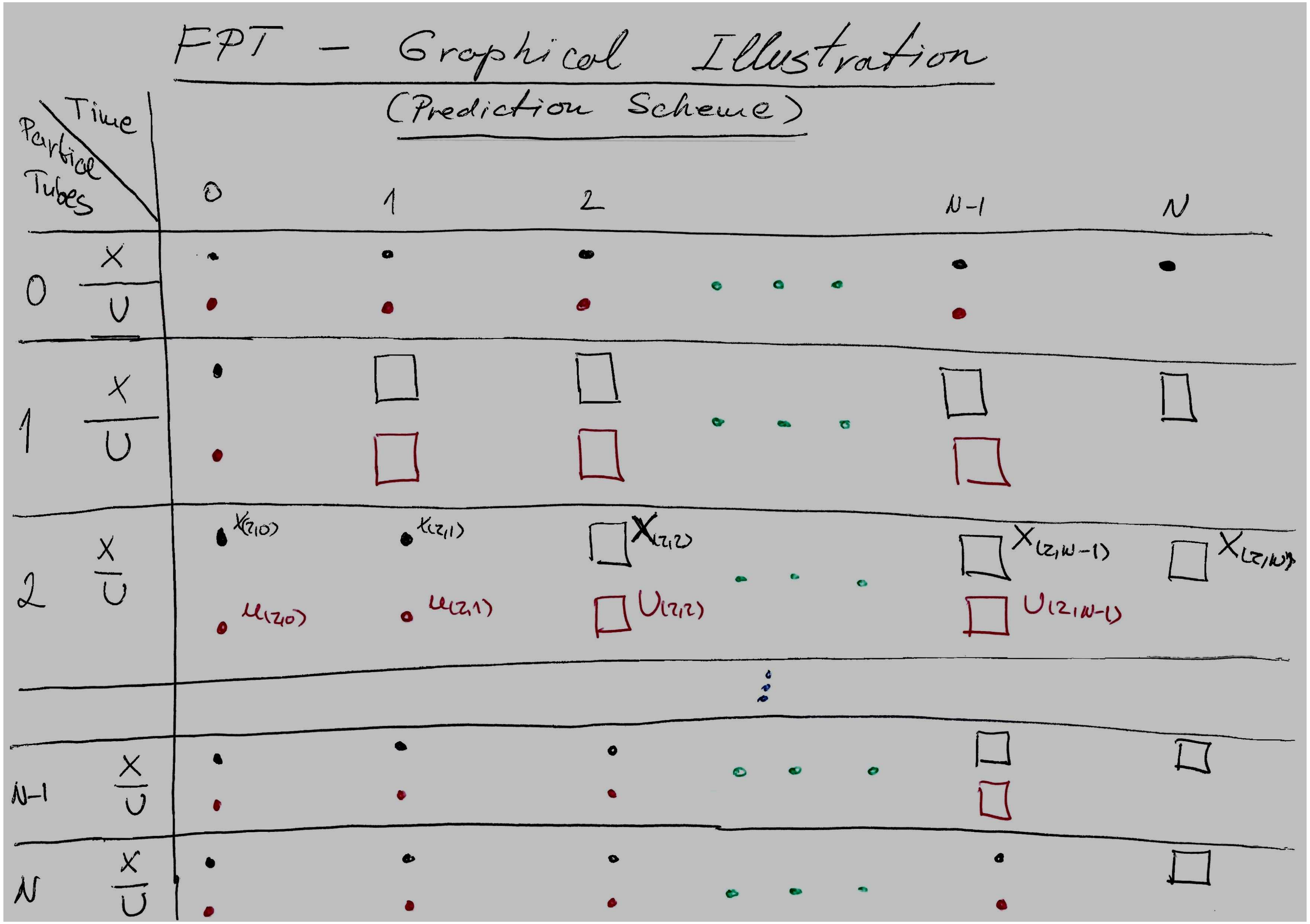
FPT Prediction Structure – Answer to Question 5

- Question was: What if:
  - $x \in \mathbb{R}^n$ was known, and
  - w–player acted at all $k \in \mathbb{N}_{N-1}$ with $w_k \in \mathbb{W}$, and
  - $\mathbb{W} = \mathrm{Convh}(\{\tilde{w}_i \ : \ i \in N_{[1:q]}\})$ and points $\tilde{w}_i \in \mathbb{R}^n, \ i \in \mathbb{N}_{[1:q]}$ were known?

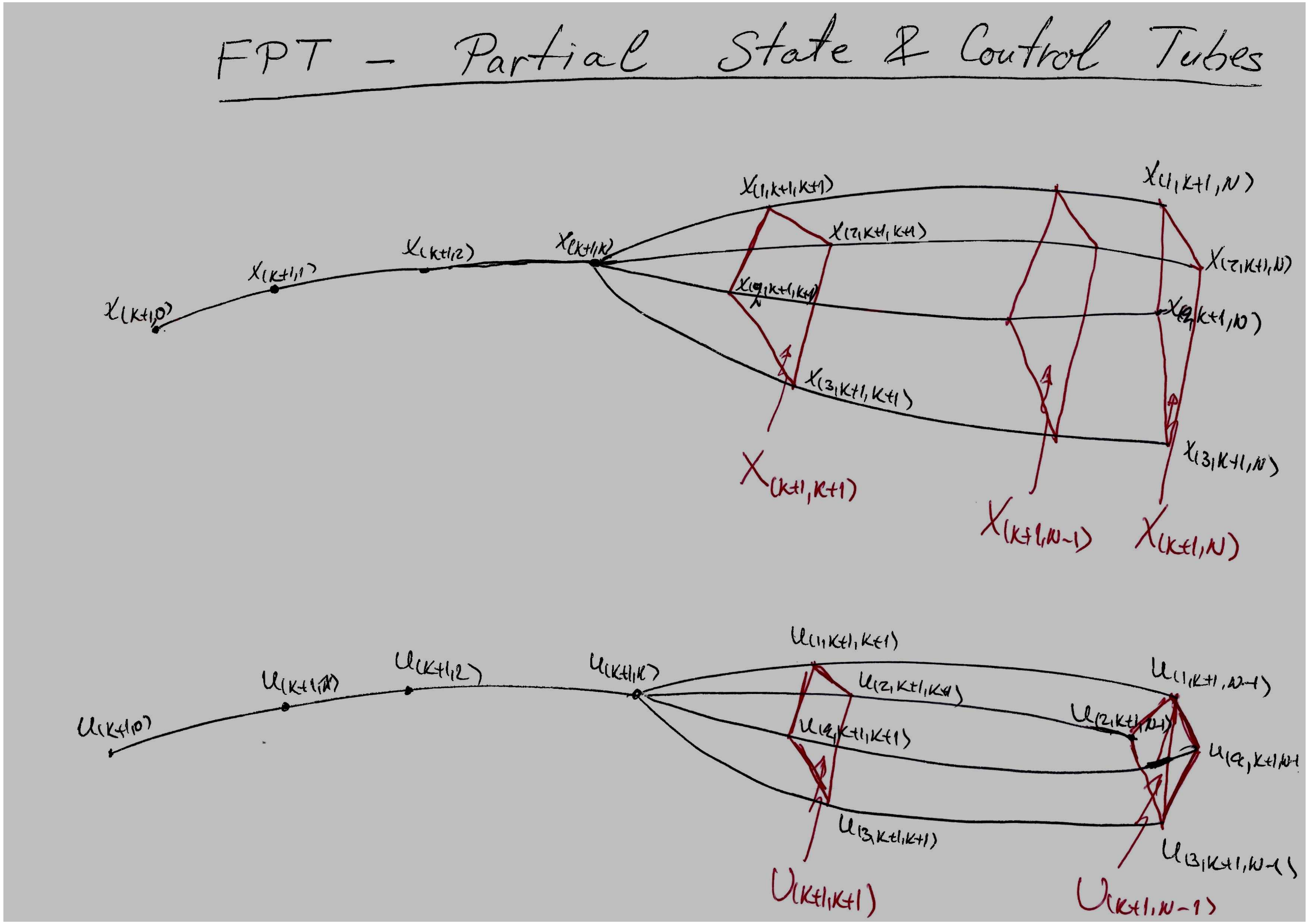
- An answer could be:

$$\boxed{\begin{array}{l} \forall k \in \mathbb{N}_{N-1}, \ x_{k+1} = Ax_k + Bu_k + w_k \text{ with} \\ x_k = \sum_{j=0}^{N} x_{(j,k)}, \ u_k = \sum_{j=0}^{N} u_{(j,k)}. \end{array}}$$

  - Decomposition into $N+1$ State and Control Sequences $\{x_{(j,k)}\}_{k \in \mathbb{N}_N}$ and $\{u_{(j,k)}\}_{k \in \mathbb{N}_{N-1}}$ with $j \in \mathbb{N}_N$
  - Utilization of Answers to Previous Questions!


FPT – Graphical Illustration
(Prediction Scheme)
Time
Particle Tubes
0
1
2
N-1
N
X
U
$X_{(2,0)}$
$X_{(2,1)}$
$X_{(2,2)}$
$X_{(2,N-1)}$
$X_{(2,N)}$
$U_{(2,0)}$
$U_{(2,1)}$
$U_{(2,2)}$
$U_{(2,N-1)}$

FPT – Partial State and Control Tubes



FPT Prediction Structure – Partial Tubes

$\mathbf{X}_{(0,N)}$	$x_{(0,0)}$	$x_{(0,1)}$	$x_{(0,2)}$	$\cdots$	$x_{(0,N-1)}$	$x_{(0,N)}$
$\mathbf{U}_{(0,N-1)}$	$u_{(0,0)}$	$u_{(0,1)}$	$u_{(0,2)}$	$\cdots$	$u_{(0,N-1)}$	
$\mathbf{X}_{(1,N)}$	$x_{(1,0)}$	$X_{(1,1)}$	$X_{(1,2)}$	$\cdots$	$X_{(1,N-1)}$	$X_{(1,N)}$
$\mathbf{U}_{(1,N-1)}$	$u_{(1,0)}$	$U_{(1,1)}$	$U_{(1,2)}$	$\cdots$	$U_{(1,N-1)}$	
$\mathbf{X}_{(2,N)}$	$x_{(2,0)}$	$x_{(2,1)}$	$X_{(2,2)}$	$\cdots$	$X_{(2,N-1)}$	$X_{(2,N)}$
$\mathbf{U}_{(2,N-1)}$	$u_{(2,0)}$	$u_{(2,1)}$	$U_{(2,2)}$	$\cdots$	$U_{(2,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{X}_{(N-1,N)}$	$x_{(N-1,0)}$	$x_{(N-1,1)}$	$x_{(N-1,2)}$	$\cdots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$
$\mathbf{U}_{(N-1,N-1)}$	$u_{(N-1,0)}$	$u_{(N-1,1)}$	$u_{(N-1,2)}$	$\cdots$	$U_{(N-1,N-1)}$	
$\mathbf{X}_{(N,N)}$	$x_{(N,0)}$	$x_{(N,1)}$	$x_{(N,2)}$	$\cdots$	$x_{(N,N-1)}$	$X_{(N,N)}$
$\mathbf{U}_{(N,N-1)}$	$u_{(N,0)}$	$u_{(N,1)}$	$u_{(N,2)}$	$\cdots$	$u_{(N,N-1)}$	

- $X_{(j,k)} := \mathrm{Convh}(\{x_{(i,j,k)} : i \in \mathbb{N}_{[1:q]}\})$,
- $U_{(j,k)} := \mathrm{Convh}(\{u_{(i,j,k)} : i \in \mathbb{N}_{[1:q]}\})$, and
- $\{X_{(j,k)}\}_{k\in\mathbb{N}_N}$ and $\{U_{(j,k)}\}_{k\in\mathbb{N}_{N-1}}$ Deterministic!

FPT Prediction Structure – Partial Tubes & Policy

- Partial State Tubes $\mathbf{X}_{(j,N)}$,
- Partial Control Tubes $\mathbf{U}_{(j,N-1)}$,
- Pairs $\mathbf{X}_{(j,N)}$ and $\mathbf{U}_{(j,N-1)}$ Counteract to Disturbances w_{j-1} with $j \in \mathbb{N}_{[1:N]}$,
- Pair $\mathbf{X}_{(0,N)}$ and $\mathbf{U}_{(0,N-1)}$ Represents Nominal State and Control Sequences ($x_{(0,k+1)} = Ax_{(0,k)} + Bu_{(0,k)}$).
- Partial Policy $\mathbf{\Pi}_{(j,N-1)}$ via $\{u_{(j,k)}(\cdot)\}_{k\in\mathbb{N}_{N-1}}$

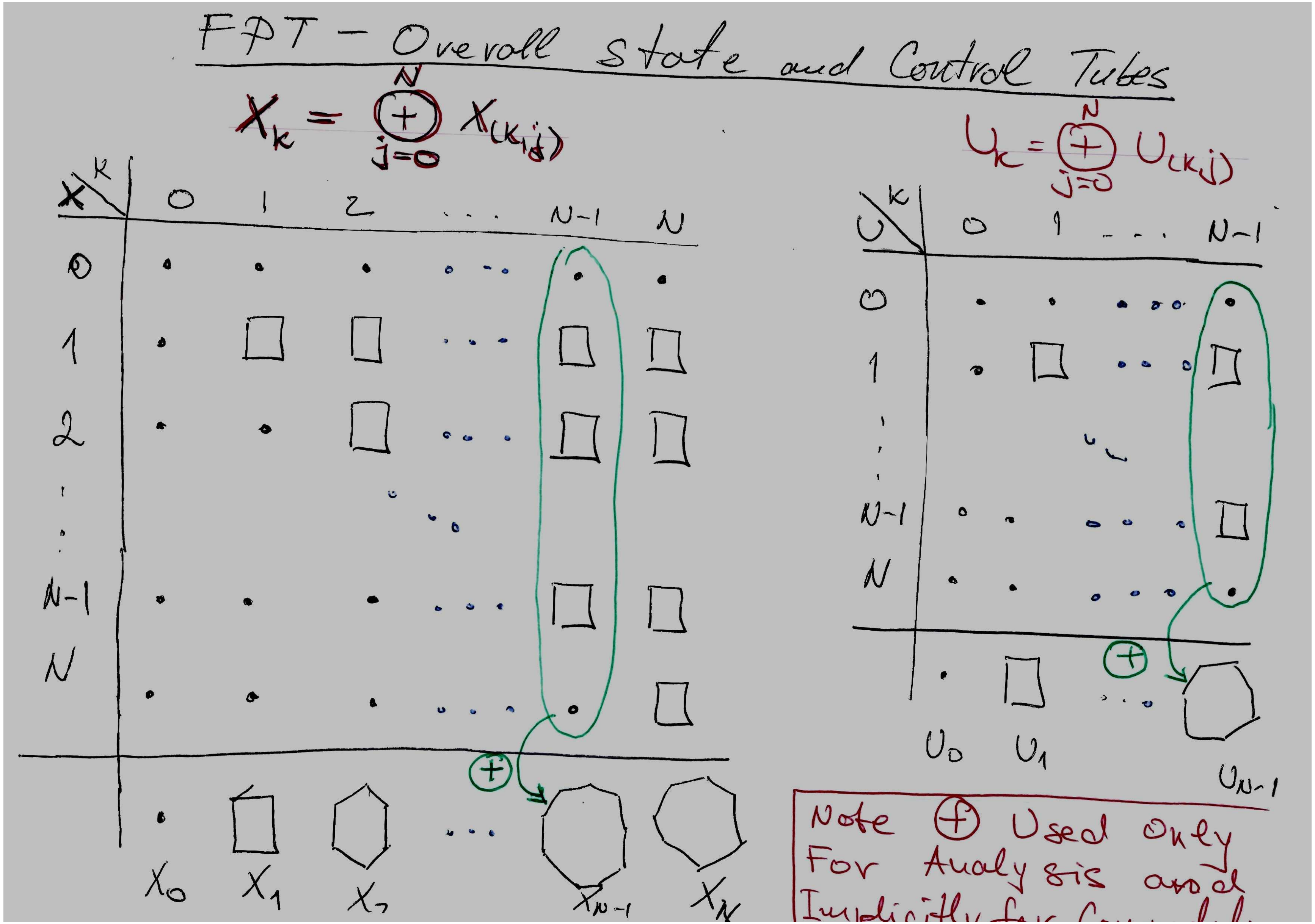
$w_{j-1} \in \mathbb{W} \Rightarrow w_{j-1} = \sum_{i=1}^{q} \lambda_i(w_{j-1})\tilde{w}_i$ for some $\lambda(w_{j-1}) \in \Lambda$

$x_{(j,k+1)}(w_{(j-1)}) = Ax_{(j,k)}(w_{(j-1)}) + Bu_{(j,k)}(w_{(j-1)}) + \delta_{(j-1,k)}w_{(j-1)}$ with

$x_{(j,k)}(w_{(j-1)}) = \sum_{i=1}^{q} \lambda_i(w_{j-1})x_{(i,j,k)} \in X_{(j,k)},\ x_{(j,k)}(\cdot)$ PWA and continuous

$u_{(j,k)}(w_{(j-1)}) = \sum_{i=1}^{q} \lambda_i(w_{j-1})u_{(i,j,k)} \in U_{(j,k)},\ u_{(j,k)}(\cdot)$ PWA and continuous

FPT – Overall State and Control Tubes



FPT Prediction Structure – Overall Tubes

$\mathbf{X}_{(0,N)}$	$x_{(0,0)}$	$x_{(0,1)}$	$\ldots$	$x_{(0,N-1)}$	$x_{(0,N)}$
$\mathbf{U}_{(0,N-1)}$	$u_{(0,0)}$	$u_{(0,1)}$	$\ldots$	$u_{(0,N-1)}$	
$\mathbf{X}_{(1,N)}$	$x_{(1,0)}$	$X_{(1,1)}$	$\ldots$	$X_{(1,N-1)}$	$X_{(1,N)}$
$\mathbf{U}_{(1,N-1)}$	$u_{(1,0)}$	$U_{(1,1)}$	$\ldots$	$U_{(1,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{X}_{(N-1,N)}$	$x_{(N-1,0)}$	$x_{(N-1,1)}$	$\ldots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$
$\mathbf{U}_{(N-1,N-1)}$	$u_{(N-1,0)}$	$u_{(N-1,1)}$	$\ldots$	$U_{(N-1,N-1)}$	
$\mathbf{X}_{(N,N)}$	$x_{(N,0)}$	$x_{(N,1)}$	$\ldots$	$x_{(N,N-1)}$	$X_{(N,N)}$
$\mathbf{U}_{(N,N-1)}$	$u_{(N,0)}$	$u_{(N,1)}$	$\ldots$	$u_{(N,N-1)}$	
$\mathbf{X}_N$	$X_0 = \bigoplus_{j=0}^{N} X_{(j,0)}$	$X_1 = \bigoplus_{j=0}^{N} X_{(j,1)}$	$\ldots$	$X_{N-1} = \bigoplus_{j=0}^{N} X_{(j,N-1)}$	$X_N = \bigoplus_{j=0}^{N} X_{(j,N)}$
$\mathbf{U}_{N-1}$	$U_0 = \bigoplus_{j=0}^{N} U_{(j,0)}$	$U_1 = \bigoplus_{j=0}^{N} U_{(j,1)}$	$\ldots$	$U_{N-1} = \bigoplus_{j=0}^{N} U_{(j,N-1)}$	

- $X_k = \bigoplus_{j=0}^{N} X_{(j,k)}$,
- $U_k = \bigoplus_{j=0}^{N} U_{(j,k)}$, and
- $\{X_k\}_{k \in \mathbb{N}_N}$ and $\{U_k\}_{k \in \mathbb{N}_{N-1}}$ Deterministic!

FPT Prediction Structure – Overall Tubes & Policy

- Overall State Tube $\mathbf{X}_N$,
- Overall Control Tube $\mathbf{U}_{N-1}$,
- Pairs $\mathbf{X}_N$ and $\mathbf{U}_{N-1}$ Counteract to Disturbance Sequences $\mathbf{w}_{N-1} = \{w_{j-1}\}_{j \in \mathbb{N}_{[1:N]}}$,
- Control Policy $\mathbf{\Pi}_{N-1}$ via Partial Policies $\mathbf{\Pi}_{(j,N-1)}$

$$\forall \mathbf{w}_{N-1} \in \mathbb{W}^N,$$

$$x_k(\mathbf{w}_N) = x_{(0,k)} + \sum_{j=1}^{N} x_{(j,k)}(w_{(j-1)}) \;\&\; u_k(\mathbf{w}_N) = u_{(0,k)} + \sum_{j=1}^{N} u_{(j,k)}(w_{(j-1)}) \Rightarrow$$

$$x_{k+1}(\mathbf{w}_N) = Ax_k(\mathbf{w}_N) + Bu_k(\mathbf{w}_N) + w_k = x_{(0,k+1)} + \sum_{j=1}^{N} x_{(j,k+1)}(w_{(j-1)}) \text{ with}$$

$x_k(\mathbf{w}_N) \in X_k, \; x_k(\cdot)$ PWA and continuous

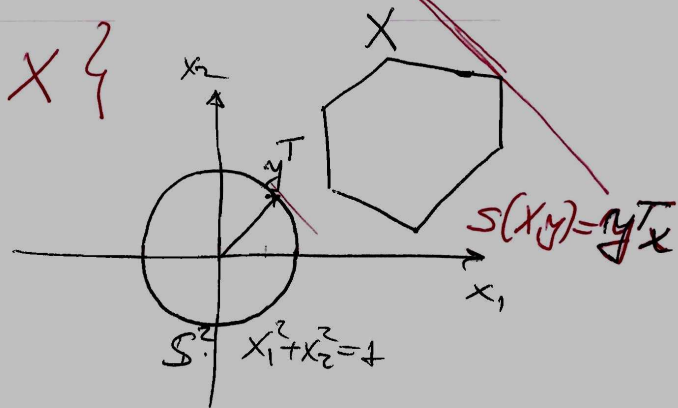
$u_k(\mathbf{w}_N) \in U_k, \; u_k(\cdot)$ PWA and continuous

FPT Constraint Handling – Two Key Questions

- What is $\operatorname{support}(X_k, F)$ for a given $F \in \mathbb{R}^n$?
- Can we find $\operatorname{support}(X_k, F)$ without computing explicitly $X_k = \bigoplus_{j=0}^{N} X_{(j,k)}$ and $X_{(j,k)} := \operatorname{Convh}(\{x_{(i,j,k)} \ : \ i \in \mathbb{N}_{[1:q]}\})$?

Support Function Illustration

$$s(X,y) := \max_{x} \{ y^T x : x \in X \}$$



Key Facts:

i) $s(X_1 \oplus X_2, y) = s(X_1, y) + s(X_2, y)$

ii) if $X := \mathrm{Convh}(\{ x_i : i \in \mathcal{K}_{[1:q]} \})$

$$s(X,y) = \max_{i} \{ y^T x_i : i \in \mathcal{K}_{[1:q]} \}$$

FPT Constraint Handling – Support Function Trick

$$\begin{aligned}
\mathrm{support}(X_k, F) &= \\
&= \mathrm{support}(\bigoplus_{j=0}^{N} X_{(j,k)}, F) \\
&= \sum_{j=0}^{N} \mathrm{support}(X_{(j,k)}, F) \\
&= \sum_{j=0}^{N} f_{(j,k)} \text{ where}
\end{aligned}$$

$f_{(0,k)} = F^T x_{(0,k)}$ and $f_{(j,k)} = \max_i \{F^T x_{(i,j,k)} \ : \ i \in \mathbb{N}_{[1:q]}\}$

FPT Constraint Handling – Support Function Trick

$\mathrm{support}(X_k, F) \leq 1 \Leftrightarrow$

$\exists \{f_{(j,k)} \in \mathbb{R} \; : \; j \in \mathbb{N}_N\}$ such that

$\sum_{j=0}^{N} f_{(j,k)} \leq 1$ with

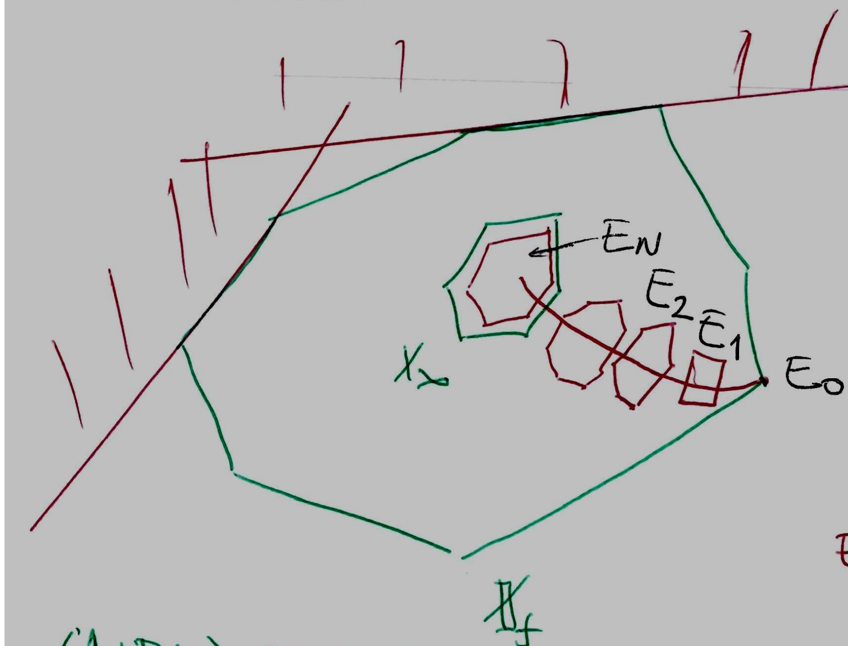
$F^T x_{(0,k)} \leq f_{(0,k)},$ and

$\forall j \in \mathbb{N}_N$ and $\forall i \in \mathbb{N}_{[1:q]},\; F^T x_{(i,j,k)} \leq f_{(j,k)}.$

- State Constraints $\forall k \in \mathbb{N}_{N-1},\; X_k \subseteq \mathbb{X}$,
- Control Constraints $\forall k \in \mathbb{N}_{N-1},\; U_k \subseteq \mathbb{U}$, and
- Terminal Constraints $X_N \subseteq \mathbb{X}_f$, reduce to a tractable set of linear/affine inequalities!

Local Behavior – Illustration

Local Behaviour Illustration



$$E_{k+1} = (A+BK)E_k \oplus W$$

$\mathbb{X}_f$ – Maximal RPI Set for $x^+ = (A+BK)x + w$.

$$E_0 = \{e_0\} \subseteq \mathbb{X}_f \implies$$

$$E_k \subseteq \mathbb{X}_f \subseteq \mathbb{X}$$

$$K E_k \subseteq K\mathbb{X}_f \subseteq \mathbb{U}$$

$$(A+BK)\mathbb{X}_f \oplus W \subseteq \mathbb{X}_f;$$

$$\mathbb{X}_f \subseteq \mathbb{X}$$

$$K\mathbb{X}_f \subseteq \mathbb{U}$$

Operating with $u = Kx$ yields "zero cost"

FPT Sensible Cost – Terminal Constraint Set

- Local Linear Dynamics $x^+ = (A + BK)x + w$,
- Constraints $x \in \mathbb{X}_K := \{x \in \mathbb{X} \ : \ Kx \in \mathbb{U}\}$,
- Terminal Constraint Set
  $\mathbb{X}_f \in \mathrm{PolyPC}(\mathbb{R}^n) \ : \ (A + BK)\mathbb{X}_f \oplus \mathbb{W} \subseteq \mathbb{X}_f \subseteq \mathbb{X}_K$,

FPT Sensible Cost – Local Behavior

$\mathbf{E}_{(0,N)}$	$e_{(0,0)}$	$(A+BK)e_{(0,0)}$	$(A+BK)^2e_{(0,0)}$	$\ldots$	$(A+BK)^{N-1}e_{(0,0)}$	$(A+BK)^{N}e_{(0,0)}$
$\mathbf{KE}_{(0,N-1)}$	$Ke_{(0,0)}$	$K(A+BK)e_{(0,0)}$	$K(A+BK)^2e_{(0,0)}$	$\ldots$	$K(A+BK)^{N-1}e_{(0,0)}$	
$\mathbf{E}_{(1,N)}$	0	$\mathbb{W}$	$(A+BK)\mathbb{W}$	$\ldots$	$(A+BK)^{N-2}\mathbb{W}$	$(A+BK)^{N-1}\mathbb{W}$
$\mathbf{KE}_{(1,N-1)}$	0	$K\mathbb{W}$	$K(A+BK)\mathbb{W}$	$\ldots$	$K(A+BK)^{N-2}\mathbb{W}$	
$\mathbf{E}_{(2,N)}$	0	0	$\mathbb{W}$	$\ldots$	$(A+BK)^{N-2}\mathbb{W}$	$(A+BK)^{N-2}\mathbb{W}$
$\mathbf{KE}_{(2,N-1)}$	0	0	$K\mathbb{W}$	$\ldots$	$K(A+BK)^{N-2}\mathbb{W}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{E}_{(N-1,N)}$	0	0	0	$\ldots$	$\mathbb{W}$	$(A+BK)\mathbb{W}$
$\mathbf{KE}_{(N-1,N-1)}$	0	0	0	$\ldots$	$K\mathbb{W}$	
$\mathbf{E}_{(N,N)}$	0	0	0	$\ldots$	0	$\mathbb{W}$
$\mathbf{KE}_{(N,N-1)}$	0	0	0	$\ldots$	0	

- Key Observation:
- $\forall k \in \mathbb{N}_{N-1},\ E_k := (A+BK)^k e_{(0,0)} \oplus \bigoplus_{j=0}^{k-1} (A+BK)^{k-1-j}\mathbb{W} \subseteq \mathbb{X}_f$, implies:
- $E_{k+1} = (A+BK)E_k \oplus \mathbb{W} \subseteq \mathbb{X}_f$, and
- $KE_k \subseteq K\mathbb{X}_f \subseteq \mathbb{U}$.

FPT Sensible Cost – Equivalent Representation I

- $\{x_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$ are Deterministic:

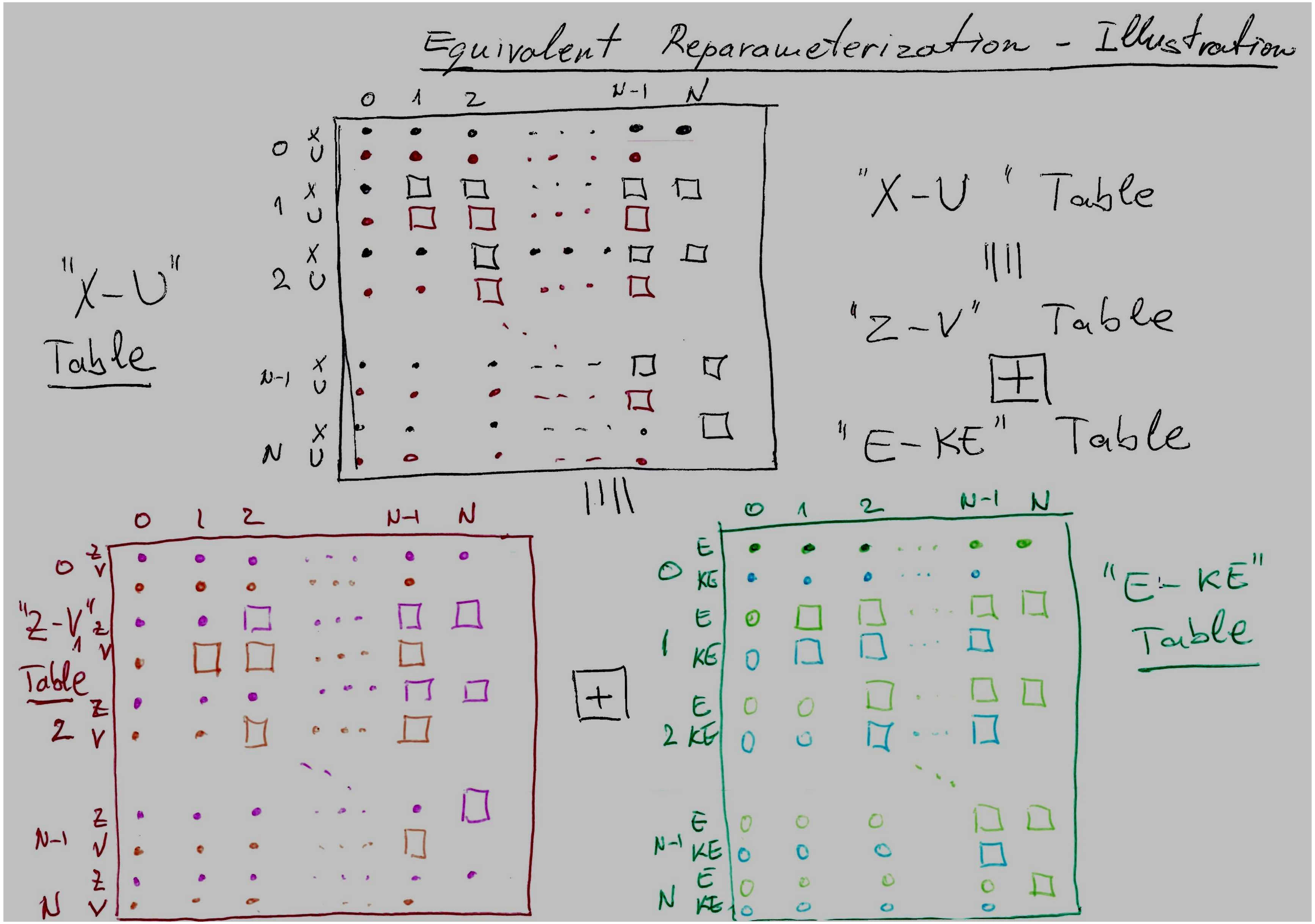
- Equivalent Representation $x_{(0,k)} = z_{(0,k)} + e_{(0,k)}$ and $u_{(0,k)} = v_{(0,k)} + Ke_{(0,k)}$

- Dynamics $z_{(0,k+1)} = Az_{(0,k)} + Bv_{(0,k)}$ and $e_{(0,k+1)} = (A + BK)e_{(0,k)}$.

FPT Sensible Cost – Equivalent Representation II

- $\{x_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ and $\{u_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ are Deterministic:
- Equivalent Representation $x_{(i,j,k)} = z_{(i,j,k)} + e_{(i,j,k)}$ and $u_{(i,j,k)} = v_{(i,j,k)} + Ke_{(i,j,k)}$
- Dynamics $z_{(i,j,k+1)} = Az_{(i,j,k)} + Bv_{(i,j,k)}$ and $e_{(i,j,k+1)} = (A + BK)e_{(i,j,k)} + \delta_{(j-1,k)}\tilde{w}_i$.
- Interesting Facts:
  - Sequences $\{z_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ and $\{v_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$ do not carry uncertainty
  - Deterministic Dynamics $z_{(i,j,k+1)} = Az_{(i,j,k)} + Bv_{(i,j,k)}$

Equivalent Reparameterization – Illustration



FPT Sensible Cost – Equivalent Reparameterization II

$\mathbf{Z}_{(0,N)}$	$z_{(0,0)}$	$z_{(0,1)}$	$z_{(0,2)}$	$\cdots$	$z_{(0,N-1)}$	$z_{(0,N)}$
$\mathbf{V}_{(0,N-1)}$	$v_{(0,0)}$	$v_{(0,1)}$	$v_{(0,2)}$	$\cdots$	$v_{(0,N-1)}$	
$\mathbf{Z}_{(1,N)}$	$z_{(1,0)}$	$z_{(1,1)}$	$Z_{(1,2)}$	$\cdots$	$Z_{(1,N-1)}$	$Z_{(1,N)}$
$\mathbf{V}_{(1,N-1)}$	$v_{(1,0)}$	$V_{(1,1)}$	$V_{(1,2)}$	$\cdots$	$V_{(1,N-1)}$	
$\mathbf{Z}_{(2,N)}$	$z_{(2,0)}$	$z_{(2,1)}$	$z_{(2,2)}$	$\cdots$	$Z_{(2,N-1)}$	$Z_{(2,N)}$
$\mathbf{V}_{(2,N-1)}$	$v_{(2,0)}$	$v_{(2,1)}$	$V_{(2,2)}$	$\cdots$	$V_{(2,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{Z}_{(N-1,N)}$	$z_{(N-1,0)}$	$z_{(N-1,1)}$	$z_{(N-1,2)}$	$\cdots$	$z_{(N-1,N-1)}$	$Z_{(N-1,N)}$
$\mathbf{V}_{(N-1,N-1)}$	$v_{(N-1,0)}$	$v_{(N-1,1)}$	$v_{(N-1,2)}$	$\cdots$	$V_{(N-1,N-1)}$	
$\mathbf{Z}_{(N,N)}$	$z_{(N,0)}$	$z_{(N,1)}$	$z_{(N,2)}$	$\cdots$	$z_{(N,N-1)}$	$z_{(N,N)}$
$\mathbf{V}_{(N,N-1)}$	$v_{(N,0)}$	$v_{(N,1)}$	$v_{(N,2)}$	$\cdots$	$v_{(N,N-1)}$	

- $Z_{(j,k)} := \mathrm{Convh}(\{z_{(i,j,k)} \;:\; i \in \mathbb{N}_{[1:q]}\})$,
- $V_{(j,k)} := \mathrm{Convh}(\{v_{(i,j,k)} \;:\; i \in \mathbb{N}_{[1:q]}\})$, and
- $\{Z_{(j,k)}\}_{k\in\mathbb{N}_N}$ and $\{V_{(j,k)}\}_{k\in\mathbb{N}_{N-1}}$ Completely Deterministic!

FPT Sensible Cost – Sensible Cost Functions

- Decomposition of $X - U$ prediction table into:
  - Uncertainty Free $Z - V$ prediction table, and
  - Uncertainty Absorbing $E - KE$ prediction table.
- Penalize Distance of $Z - V$ prediction table from its target $0 - 0$ table!

FPT Sensible Cost – Generic Cost Functions I

$\mathcal{V}_{(0,N)}$	$\ell(z_{(0,0)}, v_{(0,0)})$	$\ell(z_{(1,0)}, v_{(1,0)})$	$\ell(z_{(2,0)}, v_{(2,0)})$	$\dots$	$\ell(z_{(N-1,0)}, v_{(N-1,0)})$	$V_f(z_{(N,0)})$
$\mathcal{V}_{(1,N)}$	$L(Z_{(1,0)}, V_{(1,0)})$	$L(Z_{(1,1)}, V_{(1,1)})$	$L(Z_{(1,2)}, V_{(1,2)})$	$\dots$	$L(Z_{(1,N-1)}, V_{(1,N-1)})$	$V_F(Z_{(1,N)})$
$\mathcal{V}_{(2,N)}$	$L(Z_{(2,0)}, V_{(2,0)})$	$L(Z_{(2,1)}, V_{(2,1)})$	$L(Z_{(2,2)}, V_{(2,2)})$	$\dots$	$L(Z_{(2,N-1)}, V_{(2,N-1)})$	$V_F(Z_{(2,N)})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathcal{V}_{(N-1,N)}$	$L(Z_{(N-1,0)}, V_{(N-1,0)})$	$L(Z_{(N-1,1)}, V_{(N-1,1)})$	$L(Z_{(N-1,2)}, V_{(N-1,2)})$	$\dots$	$L(Z_{(N-1,N-1)}, V_{(N-1,N-1)})$	$V_F(Z_{(N-1,N)})$
$\mathcal{V}_{(N,N)}$	$L(Z_{(N,0)}, V_{(N,0)})$	$L(Z_{(N,1)}, V_{(N,1)})$	$L(Z_{(N,2)}, V_{(N,2)})$	$\dots$	$L(Z_{(N,N-1)}, V_{(N,N-1)})$	$V_F(Z_{(N,N)})$
$\mathcal{V}_N$	$\sum_{j=0}^{N} L(Z_{(j,0)}, V_{(j,0)})$	$\sum_{j=0}^{N} L(Z_{(j,1)}, V_{(j,1)})$	$\sum_{j=0}^{N} L(Z_{(j,2)}, V_{(j,2)})$	$\dots$	$\sum_{j=0}^{N} L(Z_{(j,N-1)}, V_{(j,N-1)})$	$\sum_{j=0}^{N} V_F(Z_{(j,N)})$

- $L(Z_{(j,k)}, V_{(j,k)}) = \sum_{i=1}^{q} \ell(z_{(i,j,k)}, v_{(i,j,k)})$,
- $V_F(Z_{(j,N)}) = \sum_{i=1}^{q} V_f(z_{(i,j,N)})$,
- $\ell(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $V_f(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$:
  - Convex and Sub–Additive,
  - Satisfy Condition:
    $\forall z \in \mathbb{R}^n,\ V_f((A + BK)z) - V_f(z) \leq -\ell(z, Kz)$
  - Adequately Lower– and Upper–Bounded

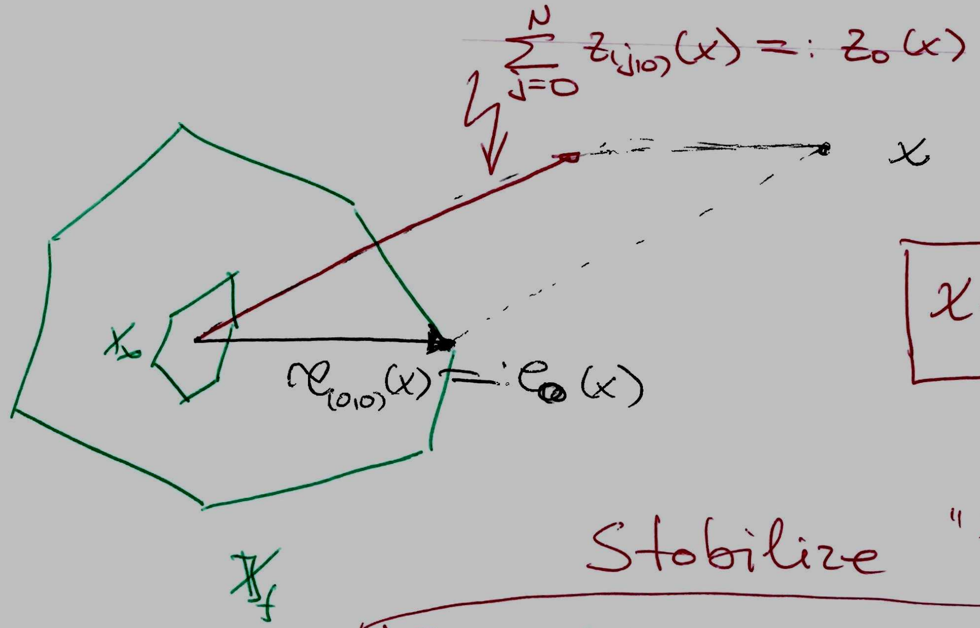
FPT Sensible Cost – Generic Cost Functions II

$\mathcal{V}_{(0,N)}$	$\ell(z_{(0,0)}, v_{(0,0)})$	$\ell(z_{(1,0)}, v_{(1,0)})$	$\ell(z_{(2,0)}, v_{(2,0)})$	$\dots$	$\ell(z_{(N-1,0)}, v_{(N-1,0)})$	$V_f(z_{(N,0)})$
$\mathcal{V}_{(1,N)}$	$L(Z_{(1,0)}, V_{(1,0)})$	$L(Z_{(1,1)}, V_{(1,1)})$	$L(Z_{(1,2)}, V_{(1,2)})$	$\dots$	$L(Z_{(1,N-1)}, V_{(1,N-1)})$	$V_F(Z_{(1,N)})$
$\mathcal{V}_{(2,N)}$	$L(Z_{(2,0)}, V_{(2,0)})$	$L(Z_{(2,1)}, V_{(2,1)})$	$L(Z_{(2,2)}, V_{(2,2)})$	$\dots$	$L(Z_{(2,N-1)}, V_{(2,N-1)})$	$V_F(Z_{(2,N)})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathcal{V}_{(N-1,N)}$	$L(Z_{(N-1,0)}, V_{(N-1,0)})$	$L(Z_{(N-1,1)}, V_{(N-1,1)})$	$L(Z_{(N-1,2)}, V_{(N-1,2)})$	$\dots$	$L(Z_{(N-1,N-1)}, V_{(N-1,N-1)})$	$V_F(Z_{(N-1,N)})$
$\mathcal{V}_{(N,N)}$	$L(Z_{(N,0)}, V_{(N,0)})$	$L(Z_{(N,1)}, V_{(N,1)})$	$L(Z_{(N,2)}, V_{(N,2)})$	$\dots$	$L(Z_{(N,N-1)}, V_{(N,N-1)})$	$V_F(Z_{(N,N)})$
$\mathcal{V}_N$	$\sum_{j=0}^{N} L(Z_{(j,0)}, V_{(j,0)})$	$\sum_{j=0}^{N} L(Z_{(j,1)}, V_{(j,1)})$	$\sum_{j=0}^{N} L(Z_{(j,2)}, V_{(j,2)})$	$\dots$	$\sum_{j=0}^{N} L(Z_{(j,N-1)}, V_{(j,N-1)})$	$\sum_{j=0}^{N} V_F(Z_{(j,N)})$

- $L(Z_{(j,k)}, V_{(j,k)}) = \max_i \{\ell(z_{(i,j,k)}, v_{(i,j,k)}) \; : \; i \in \mathbb{N}_{[1:q]}\}$,
- $V_F(Z_{(j,N)}) = \max_i \{V_f(z_{(i,j,N)}) \; : \; i \in \mathbb{N}_{[1:q]}\}$,
- $\ell(\cdot,\cdot) \; : \; \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $V_f(\cdot) \; : \; \mathbb{R}^n \to \mathbb{R}_+$:
  - Convex,
  - Satisfy Condition:
    $\forall z \in \mathbb{R}^n, \; V_f((A+BK)z) - V_f(z) \leq -\ell(z, Kz)$
  - Adequately Lower– and Upper–Bounded

Sensible Cost – Illustration

Cost Illustration



$$\boxed{x = e_0(x) + z_0(x)}$$

Stabilize "z" System:

$\forall x \in \mathbb{X}_f$, $e_{(0,0)}(x) = 0$ &
$\forall j \in \mathbb{N}_N$, $z_{(j,0)}(x) = 0$
is feasible?

$$\boxed{x - \sum_{j=0}^{N} z_{(j,0)} \in \mathbb{X}_f}$$

"Stabilizing Constraint"

- Sequences $\{z_{(0,k)}\}_{k\in\mathbb{N}_N}$ and $\{v_{(0,k)}\}_{k\in\mathbb{N}_{N-1}}$
- Sequences $\{z_{(i,j,k)}\}_{k\in\mathbb{N}_N}$ and $\{v_{(i,j,k)}\}_{k\in\mathbb{N}_{N-1}}$
- Initial Error State $e_{(0,0)} = x - \sum_{j=0}^{N} z_{(j,0)}$ (Can be Eliminated),
- Dimension of Decision Variable $\mathbf{d}_N$ Proportional to qN^2 !

- Given $N \in \mathbb{N}_+$ and $x \in \mathbb{X}$, select (if possible):
  - Decision Variable $\mathbf{d}_N$ such that

$$\boxed{\begin{array}{l}
\forall k \in \mathbb{N}_{N-1}, z_{(0,k+1)} = Az_{(0,k)} + Bv_{(0,k)}, \\
\forall i \in \mathbb{N}_{[1:q]},\ \forall j \in \mathbb{N}_{[1:N]},\ \forall k \in \mathbb{N}_{N-1},\ z_{(i,j,k+1)} = Az_{(i,j,k)} + Bv_{(i,j,k)}, \\
\text{with } e_{(0,0)} + \sum_{j=0}^{N} z_{(j,0)} = x, \\
\forall k \in \mathbb{N}_{N-1}, \\
X_k = Z_k \boxplus E_k \subseteq \mathbb{X},\ U_k = V_k \boxplus KE_k \subseteq \mathbb{U},\ \text{and } X_N = Z_N \boxplus E_N \subseteq \mathbb{X}_f,\ \text{and,} \\
e_{(0,0)} = x - \sum_{j=0}^{N} z_{(j,0)} \in \mathbb{X}_f
\end{array}}$$

which minimize a cost function

$$\boxed{\mathcal{V}_N(\mathbf{d}_N) := \sum_{j \in \mathbb{N}_N} \mathcal{V}_{(j,N)}(\mathbf{d}_N).}$$

FPT OC – Topological Properties

- The set of $x \in \mathbb{X}$ for which FPT OC is feasible, say $\mathcal{X}_N$, is a PC–polytope in $\mathbb{R}^n$.
- Under Convexity of $\ell(\cdot,\cdot)$ and $V_f(\cdot)$:
  - ♦ $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is continuous and convex, and
  - ♦ $\exists\, \mathbf{d}_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ which is continuous.
- Under “Linearity” of $\ell(\cdot,\cdot)$ and $V_f(\cdot)$:
  - ♦ $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is PWA, convex and continuous, and
  - ♦ $\exists\, \mathbf{d}_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ which is PWA and continuous.

FPT MPC – Summarized

- Repetitive Online Application of FPT OC,
- Dimension of Decision Variable Proportional to qN^2,
- Number of Constraints Proportional to qN^2,
- Computation Practicable,
- More General than Disturbance Affine Feedback RMPC (due PWA structure of employed feedback!).

- FPTMPC
  $\forall x \in \mathcal{X}_N,\ \kappa_N^0(x) = \sum_{j=0}^{N} v_{(j,0)}^0(x) + K(x - \sum_{j=0}^{N} z_{(j,0)}^0(x))$,

- Controlled Uncertain Dynamics
  $\forall x \in \mathcal{X}_N,\ x^+ \in \mathcal{F}(x),\ \mathcal{F}(x) := Ax + B\kappa_N^0(x) \oplus \mathbb{W}$,

- Also $\forall x \in \mathbb{X}_f,\ \kappa_N^0(x) = Kx$ and $\mathcal{F}(x) := (A + BK)x \oplus \mathbb{W}$.

FPT MPC – System Theoretic Properties

- The set $\mathcal{X}_N \subseteq \mathbb{X}$ is an RPI set, i.e. $\forall x \in \mathcal{X}_N, \ \kappa_N^0(x) \in \mathbb{U}$ $\mathcal{F}(x) \subseteq \mathcal{X}_N$.
- The set $\mathbb{X}_f \subseteq \mathcal{X}_N$ is robustly exponentially stable set for $x^+ \in \mathcal{F}(x)$ with the basin of attraction $\mathcal{X}_N$, i.e. any $\{x_k\}_{k\in\mathbb{N}}$ with $\forall k \in \mathbb{N}, \ x_{k-1} \in \mathcal{F}(x_k)$ converges exponentially fast, in stable manner, to $\mathbb{X}_f$, and
- The set $\mathbb{X}_\infty := \bigoplus_{k=0}^{\infty}(A+BK)^k\mathbb{W} \subseteq \mathbb{X}_f$ is the minimal robustly exponentially stable set for $x^+ \in \mathcal{F}(x)$ with the basin of attraction $\mathcal{X}_N$.

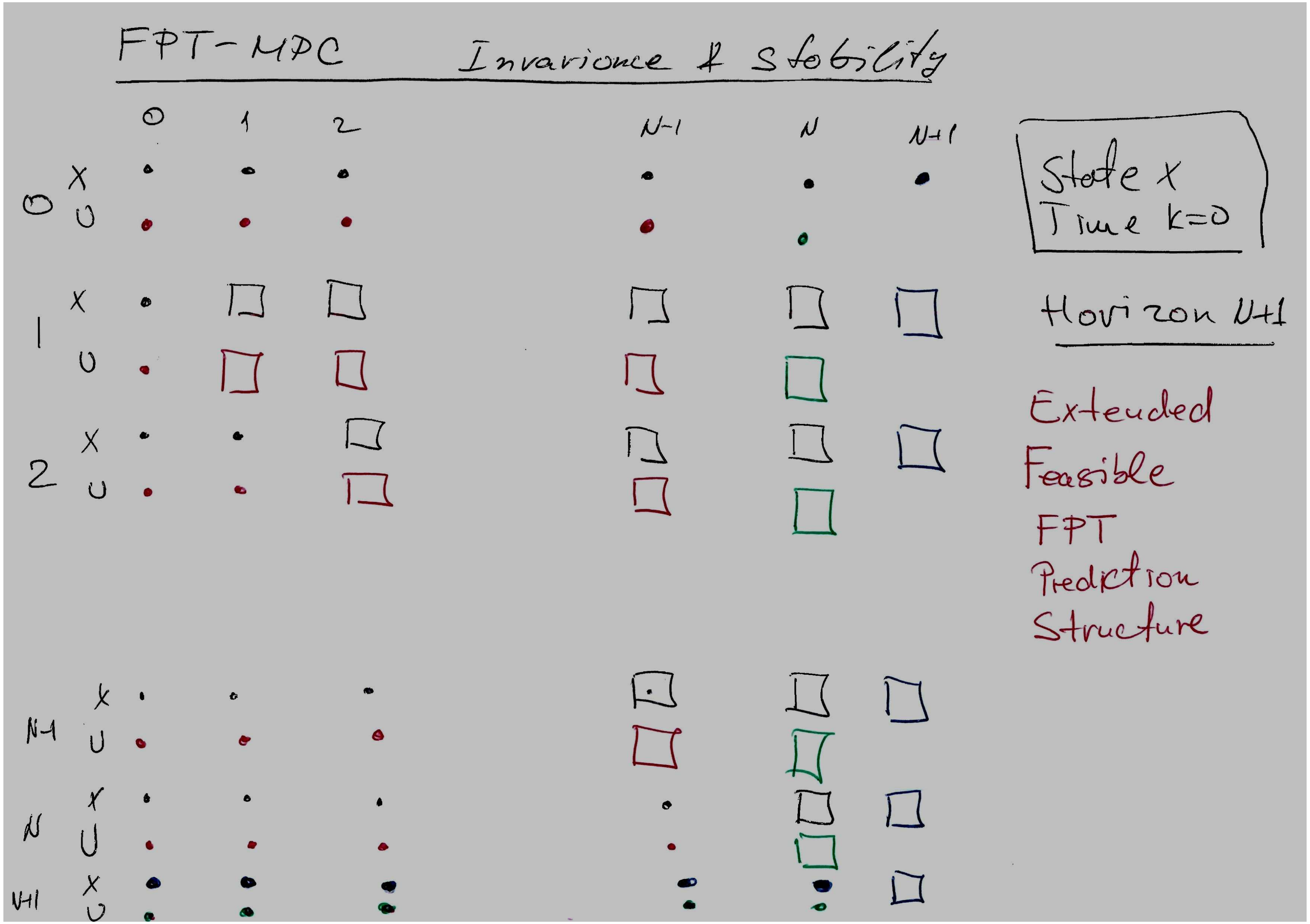
FPT MPC – Invariance and Stability Illustration

FPT MPC – Invariance and Stability Properties

$x_{(0,0)}$	$x_{(0,1)}$	$\cdots$	$x_{(0,N-1)}$	$x_{(0,N)}$
$u_{(0,0)}$	$u_{(0,1)}$	$\cdots$	$u_{(0,N-1)}$	
$x_{(1,0)}$	$X_{(1,1)}$	$\cdots$	$X_{(1,N-1)}$	$X_{(1,N)}$
$u_{(1,0)}$	$U_{(1,1)}$	$\cdots$	$U_{(1,N-1)}$	
	$\vdots$	$\vdots$	$\vdots$	
$x_{(N-1,0)}$	$x_{(N-1,1)}$	$\cdots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$
$u_{(N-1,0)}$	$u_{(N-1,1)}$	$\cdots$	$U_{(N-1,N-1)}$	
$x_{(N,0)}$	$x_{(N,1)}$	$\cdots$	$x_{(N,N-1)}$	$X_{(N,N)}$
$u_{(N,0)}$	$u_{(N,1)}$	$\cdots$	$u_{(N,N-1)}$	

Feasible FPT Prediction Structure at $k = 0$.

FPT MPC – Invariance and Stability Illustration

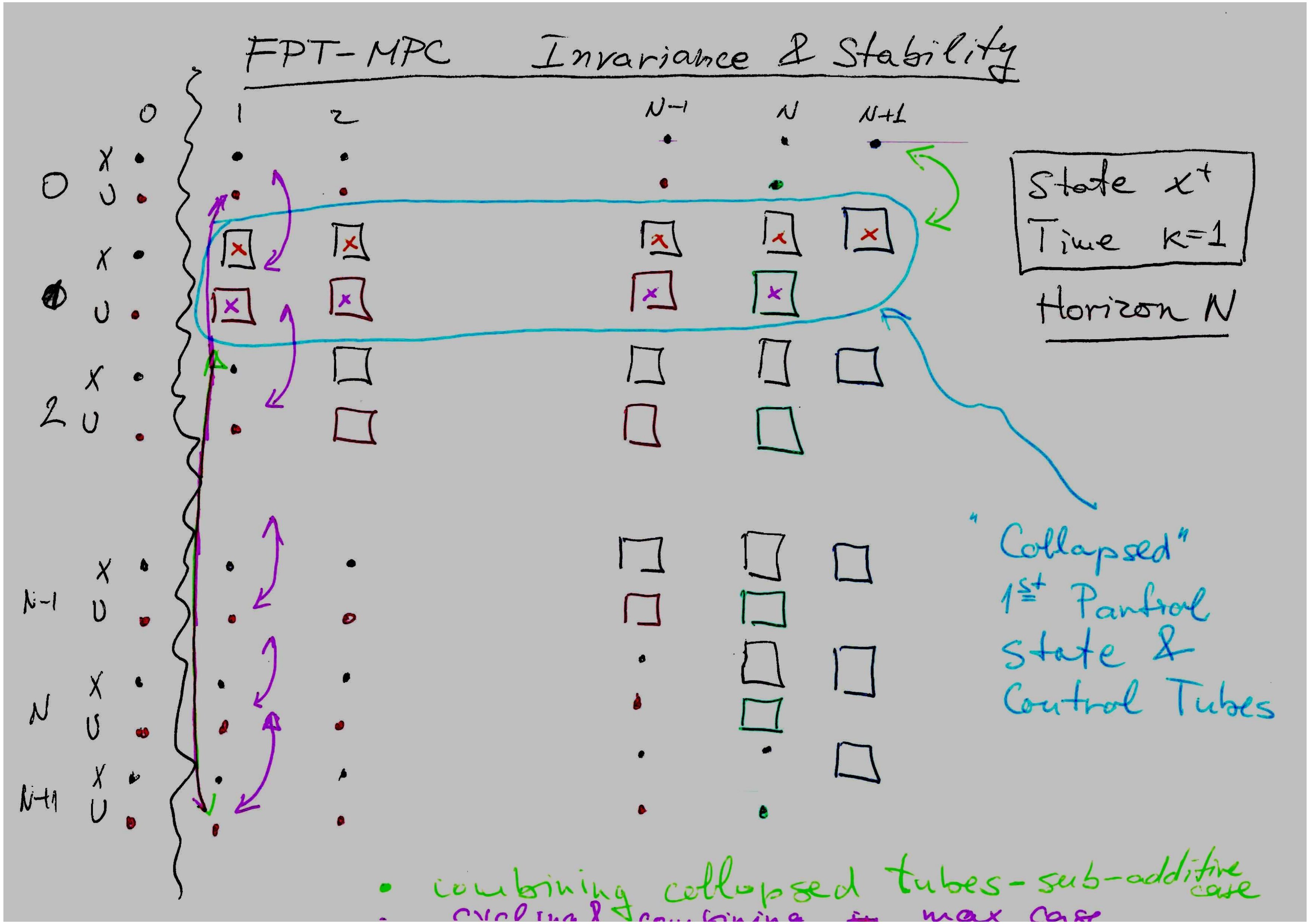


FPT MPC – Invariance and Stability Properties

$x_{(0,0)}$	$x_{(0,1)}$	$\dots$	$x_{(0,N-1)}$	$x_{(0,N)}$	$(A+BK)x_{(0,N)}$
$u_{(0,0)}$	$u_{(0,1)}$	$\dots$	$u_{(0,N-1)}$	$Kx_{(0,N)}$	
$x_{(1,0)}$	$X_{(1,1)}$	$\dots$	$X_{(1,N-1)}$	$X_{(1,N)}$	$(A+BK)X_{(1,N)}$
$u_{(1,0)}$	$U_{(1,1)}$	$\dots$	$U_{(1,N-1)}$	$KX_{(1,N)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{(N-1,0)}$	$x_{(N-1,1)}$	$\dots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$	$(A+BK)X_{(N-1,N)}$
$u_{(N-1,0)}$	$u_{(N-1,1)}$	$\dots$	$U_{(N-1,N-1)}$	$KX_{(N-1,N)}$	
$x_{(N,0)}$	$x_{(N,1)}$	$\dots$	$x_{(N,N-1)}$	$X_{(N,N)}$	$(A+BK)X_{(N,N)}$
$u_{(N,0)}$	$u_{(N,1)}$	$\dots$	$u_{(N,N-1)}$	$KX_{(N,N)}$	
0	0	$\dots$	0	0	$X_{(N+1,N+1)} = \mathbb{W}$
0	0	$\dots$	0		0

Extended Feasible FPT Prediction Structure at $k = 0$.

FPT MPC – Invariance and Stability Illustration



FPT MPC – Invariance and Stability Properties

$x_{(0,0)}$	$x_{(0,1)}$	$\ldots$	$x_{(0,N-1)}$	$x_{(0,N)}$	$(A+BK)x_{(0,N)}$
$u_{(0,0)}$	$u_{(0,1)}$	$\ldots$	$u_{(0,N-1)}$	$Kx_{(0,N)}$	
$x_{(1,0)}$	$\hat{x}_{(1,1)}$	$\ldots$	$\hat{x}_{(1,N-1)}$	$\hat{x}_{(1,N)}$	$(A+BK)\hat{x}_{(1,N)}$
$u_{(1,0)}$	$\hat{u}_{(1,1)}$	$\ldots$	$\hat{u}_{(1,N-1)}$	$K\hat{x}_{(1,N)}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_{(N-1,0)}$	$x_{(N-1,1)}$	$\ldots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$	$(A+BK)X_{(N-1,N)}$
$u_{(N-1,0)}$	$u_{(N-1,1)}$	$\ldots$	$U_{(N-1,N-1)}$	$KX_{(N-1,N)}$	
$x_{(N,0)}$	$x_{(N,1)}$	$\ldots$	$x_{(N,N-1)}$	$X_{(N,N)}$	$(A+BK)X_{(N,N)}$
$u_{(N,0)}$	$u_{(N,1)}$	$\ldots$	$u_{(N,N-1)}$	$KX_{(N,N)}$	
0	0	$\ldots$	0	0	$X_{(N+1,N+1)} = \mathbb{W}$
0	0	$\ldots$	0		0

Collapsed Version of Extended Feasible FPT Prediction Structure at $k = 1$.

$x_{(0,1)} + \hat{x}_{(1,1)}$	$\ldots$	$x_{(0,N-1)} + \hat{x}_{(1,N-1)}$	$x_{(0,N)} + \hat{x}_{(1,N)}$	$(A+BK)x_{(0,N)} + (A+BK)\hat{x}_{(1,N)}$
$u_{(0,1)} + \hat{u}_{(1,1)}$	$\ldots$	$u_{(0,N-1)} + \hat{u}_{(1,N-1)}$	$Kx_{(0,N)} + K\hat{x}_{(1,N)}$	
	$\vdots$	$\vdots$	$\vdots$	
$x_{(N-1,1)}$	$\ldots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$	$(A+BK)X_{(N-1,N)}$
$u_{(N-1,1)}$	$\ldots$	$U_{(N-1,N-1)}$	$KX_{(N-1,N)}$	
$x_{(N,1)}$	$\ldots$	$x_{(N,N-1)}$	$X_{(N,N)}$	$(A+BK)X_{(N,N)}$
$u_{(N,1)}$	$\ldots$	$u_{(N,N-1)}$	$KX_{(N,N)}$	
0	$\ldots$	0	0	$X_{(N+1,N+1)} = \mathbb{W}$
0	$\ldots$	0	0	

Feasible FPT Prediction Structure at $k=1$ for "Sub–Additive" Cost.

FPT MPC – Invariance and Stability Properties

$\hat{x}_{(1,1)}$	$\ldots$	$\hat{x}_{(1,N-1)}$	$\hat{x}_{(1,N)}$	$(A+BK)\hat{x}_{(1,N)}$
$\hat{u}_{(1,1)}$	$\ldots$	$\hat{u}_{(1,N-1)}$	$K\hat{x}_{(1,N)}$	
	$\vdots$	$\vdots$	$\vdots$	
$x_{(N-1,1)}$	$\ldots$	$X_{(N-1,N-1)}$	$X_{(N-1,N)}$	$(A+BK)X_{(N-1,N)}$
$u_{(N-1,1)}$	$\ldots$	$U_{(N-1,N-1)}$	$KX_{(N-1,N)}$	
$x_{(N,1)}$	$\ldots$	$x_{(N,N-1)}$	$X_{(N,N)}$	$(A+BK)X_{(N,N)}$
$u_{(N,1)}$	$\ldots$	$u_{(N,N-1)}$	$KX_{(N,N)}$	
$x_{(0,1)}$	$\ldots$	$x_{(0,N-1)}$	$x_{(0,N)}$	$(A+BK)x_{(0,N)} \oplus X_{(N+1,N+1)}$
$u_{(0,1)}$	$\ldots$	$u_{(0,N-1)}$	$Kx_{(0,N)}$	

Feasible FPT Prediction Structure at $k=1$ for "Max" Cost.

§4 – Comparative Remarks & Illustrative Examples

- Comparative Remarks
- Example 1: Feedback Min–Max MPC vs FPTMPC
- Example 2: Disturbance Affine Feedback RMPC vs FPTMPC
- Example 3: FPTMPC in Action

Comparisons of Main Existing RMPC Methods Based on Computational Practicability:

RMPC vs Facts	CER	CP	PS	CO	# DV	# C
CLMM RMPC	5	$u_k(x_k)$	Nonlinear	YES	$O(q^N)$	$O(q^N)$
RTMPC	4	$u_k(x_k) = Kx_k + v_k(x_0)$	Affine	YES	$O(N)$	$O(N)$
TVA RMPC	3	$u_k(x_k) = K_k x_k + v_k(x_0)$	Affine	NO	$O(qN^2)$	$O(qN^2)$
DA RMPC	2	$u_k(x_k) = \sum_{j=0}^{k-1} M_{(j,k)} w_j + v_k(x_0)$	Affine	YES	$O(qN^2)$	$O(qN^2)$
FPTMPC	1	$u_k(x_k) = \sum_{j=0}^{N} u_{(j,k)}(x_{(j,k)})$	Nonlinear	YES	$O(qN^2)$	$O(qN^2)$

Comparisons of Main Existing RMPC Methods Based on Size of Domain of Attraction:

RMPC vs Facts	SDAR	CP	PS
RTMPC	5	$u_k(x_k) = Kx_k + v_k(x_0)$	Affine
TVA RMPC	$3-4$	$u_k(x_k) = K_k x_k + v_k(x_0)$	Affine
DA RMPC	$3-4$	$u_k(x_k) = \sum_{j=0}^{k-1} M_{(j,k)} w_j + v_k(x_0)$	Affine
FPTMPC	2 or $(1-2)?$	$u_k(x_k) = \sum_{j=0}^{N} u_{(j,k)}(x_{(j,k)})$	Nonlinear
CLMM RMPC	1 or $(1-2)?$	$u_k(x_k)$	Nonlinear

Same Holds for Performance when Same Cost Functions are Used.

Main Existing RMPC Methods and Feasibility–Wise Equivalence to DP:

RMPC vs Facts	FWEDPR	$N \in \mathbb{N}_+$ $n = m = 1$	$N = 1$ $n \in \mathbb{N}_+,\ m \in \mathbb{N}_+$	$N = 2$ $n \in \mathbb{N}_+,\ m \in \mathbb{N}_+$	$N > 2$ $n \in \mathbb{N}_+,\ m \in \mathbb{N}_+$
RTMPC	5	No	YES	NO	NO
TVA RMPC	$3-4$	YES	YES	NO	NO
DA RMPC	$3-4$	YES	YES	NO	NO
FPTMPC	2 or $(1-2)?$	YES	YES	YES	?
CLMM RMPC	1 or $(1-2)?$	YES	YES	YES	YES

Feasibility–Wise Equivalence to DP of PTMPC and FPTMPC Discussed in IEEE–TAC and IJRNC Papers.

FPT MPC – Illustrative Examples

online optimization is feasible then it is guaranteed that the online optimization will remains feasible for all the future time $k \in \mathbb{N}$ and at all possible states x_k encountered in the control process. The previous fact is true as the initial optimization is feasible for all state $x \in \mathcal{X}_N$ and the N-step parameterized tubes controllability set $\mathcal{X}_N$ is robust positively invariant set for the system $x^+ = Ax + B\kappa_N^*(x) + w$ and the constraint set $(\mathbb{X}_{\kappa_N^*}, \mathbb{W})$ as asserted in Theorem 1 (see also Proposition 7). Finally, since $\forall x \in \mathbb{X}_f,\ V_N^0(x) = 0$, and, consequently, $\forall x \in \mathbb{X}_f,\ \kappa_N^*(x) = Kx$ we note that online optimization can be terminated once the state x_k enters the set $\mathbb{X}_f$.

Remark 11 *We also note that the variable $\mathbf{d}_{(N,\mathbf{X})}$ in (7.5), can be eliminated by employing the relationships (3.3b) and (3.4b) which, in turn, reduces the number of decision variables by $\frac{1}{2}qnN(N+1)$. However, the form of constraints we have utilized seems to be both the numerically and structurally preferred option even in the case of the nominal MPC [46].*

7.2 Illustrative Examples

We provide first two examples illustrating advantages of the developed PTOC and PTMPC over the methods proposed in [13, 14] as well as in [15–18].

Illustrative Example 1 *The first illustrative example is a variant of the example used in [13]. The system is one dimensional system given by:*

$$x^+ = x + u + w,$$

with the state and control constraint sets:

$$\mathbb{X} = [-30, 30] = \{x \in \mathbb{R} \ : \ \frac{1}{30}x \le 1,\ -\frac{1}{30}x \le 1\},\ \text{and,}\ \mathbb{U} = [-2, 2] = \{u \in \mathbb{R} \ : \ \frac{1}{2}u \le 1,\ -\frac{1}{2}u \le 1\}.$$

The disturbance set $\mathbb{W}$ is given by:

$$\mathbb{W} = \mathrm{convh}(\{1, -1\})\ \text{so that}\ q = 2,\ \tilde{w}_1 = 1\ \text{and}\ \tilde{w}_2 = -1.$$

The matrices Q and R defining the sets $\mathcal{Q}$ and $\mathcal{R}$ as well as their gauge functions utilized for the cost function $V_N(\cdot)$ are given by:

$$Q = (1,\ -1)^T\ \text{and}\ R = (2,\ -2)^T.$$

The local linear feedback $u = kx$, the terminal constraint set $\mathbb{X}_f$ and the matrix P defining the set $\mathcal{P}$ and its gauge function utilized for the cost function $V_N(\cdot)$ are given by:

$$k = -\frac{1}{2},\ \mathbb{X}_f = [-4, 4] = \{x \in \mathbb{R} \ : \ \frac{1}{4}x \le 1,\ -\frac{1}{4}x \le 1\},\ \text{and,}\ P = (4,\ -4)^T.$$

In this example, any initial condition $x_0 \in \mathbb{X}$ is min–max controllable to a target set within $N = 26$ steps. In particular, for the initial conditions $x_0' = 30$ and $x_0'' = -30$ the min–max controllability to a target set $\mathbb{X}_f$ can be guaranteed if and only if $N = 26$. So, we choose the initial condition $x_0 = 30$ and consider the horizon length

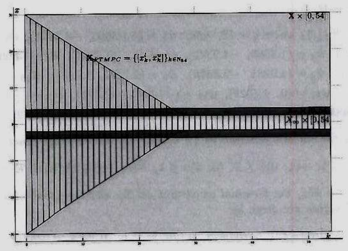


Figure 1: Parameterized Model Predictive Control State Tube.

$N = 26$. In this setting, the methods of [13, 14] require the utilization of the decision variable whose dimension is $2^{26} = 67108864$ and the corresponding optimization requires the utilization of a number of constraints that is linearly

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FPT MPC – Illustrative Examples

proportional to $2^{26} = 67108864$. Clearly, even with the most sophisticated optimization software, this would lead to an almost impossible computational task (realistically speaking, this task is, in fact, an impossible mission). On the contrary, the proposed PTMPC method requires the utilization of the decision variable $\mathbf{d}_N$ whose dimension is $N_{tot} = 4162$ while the total number of equality and inequality constraints for the underlying linear programming problem satisfy $N_{ineq} = 5674$ and $N_{eq} = 728$. This simple scalar example demonstrates clearly computational advantages of the proposed method over those suggested in [13, 14]. The performance of the PTMPC is additionally illustrated in Figure 1 where we show the tube $\mathbf{X}_{PTMPC} := \{[x_k^l, x_k^u]\}_{k \in \mathbb{N}}$ composed from the extreme trajectories generated from $\{x_k^l\}_{k\in\mathbb{N}}$ and $\{x_k^u\}_{k\in\mathbb{N}}$ which satisfy:

$$x_{k+1}^l = x_k^l + \kappa_N^*(x_k^l) + w^*(x_k^l, \kappa_N^*(x_k^l)), \text{ and } x_{k+1}^u = x_k^u + \kappa_N^*(x_k^u) + w^*(x_k^u, \kappa_N^*(x_k^u)),$$

where $x_0^l = -30$, $x_0^u = 30$ and the function $w^*(\cdot,\cdot)$ is the maximizing disturbance function taking the form $w^*(x,u) = 1$ if $x+u \geq 0$, $w^*(x,u) = -1$ if $x+u \leq 0$ and $w^*(x,u)$ can be either 1 or -1 if $x+u=0$. As evident from the Figure, the PTMPC law induces the controlled, uncertain, dynamics with strong system theoretic properties. Clearly, the tube $\mathbf{X}_{PTMPC} := \{[x_k^l, x_k^u]\}_{k\in\mathbb{N}}$ composed from the extreme trajectories converges exponentially fast to the minimal robust positively invariant set $X_\infty = [-2, 2]$ (exhibiting also the exponential convergence to the maximal robust positively invariant set $\mathbb{X}_f = [-4, 4]$) as expected by Theorem 1 and Corollary 3.

Illustrative Example 2 *Our second illustrative example demonstrates the advantages of the developed PTOC and PTMPC over the methods utilizing the so–called afine disturbance feedbacks discussed in [15–18]. At the conceptual level, these methods are subsumed within our method as they can be recovered from our method by invoking an additional constraint on the control tubes, namely, by requiring that:*

$$\forall i \in \mathbb{N}_{[1:q]},\ \forall k \in \mathbb{N}_{[1:N-1]},\ \forall j \in \mathbb{N}_{[k:N-1]},\ \tilde{u}_{(i,j,k)} = M_{(j,k)}\tilde{w}_i$$

for a set of design matrices $M_{(j,k)} \in \mathbb{R}^{m\times n}$ and introducing these relationships as an additional set of constraints in the corresponding optimization for PTOC (where one optimizes over the set of $M_{(j,k)}$ which then induce the set of $\tilde{u}_{(i,j,k)}$). Clearly, whenever the optimization with this additional set of constraints is feasible so is the one without enforcing these additional constraints. Hence, whenever the methods of [15–18] are feasible so is our proposal. We prove that the opposite is not true by providing an example where the methods of [15–18] fail to be feasible for the RMPC synthesis while our PTMPC method allows for a feasible RMPC synthesis.

The system is two dimensional system specified by:

$$x^+ = Ax + Bu + w, \text{ with, } A = \frac{1}{2}\begin{pmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix}, \text{ and, , } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The state and control constraint sets are given by:

$$\begin{aligned} \mathbb{X} := \{x \in \mathbb{R}^2 :\ & (0.0176,\ 0.0655)x \leq 1,\ (-0.0428,\ 0.0428)x \leq 1,\ (-0.0747,\ -0.0200)x \leq 1, \\ & (-0.0176,\ -0.0655) \leq 1,\ (0.0428,\ -0.0428)x \leq 1,\ (0.0747,\ 0.0200)x \leq 1\}, \text{ and,} \\ \mathbb{U} := \{u \in \mathbb{R} :\ & \frac{1}{3}u \leq 1,\ -\frac{1}{3}u \leq 1\}. \end{aligned}$$

The disturbance set $\mathbb{W}$ is given by:

$$\begin{aligned} &\mathbb{W} = \mathrm{convh}(\{\tilde{w}_i \ :\ i \in \mathbb{N}_{[1:q]}\}) \text{ where } q = 18, \text{ and, } \tilde{w}_1 = (-1.8660,\ -4.2321),\ \tilde{w}_2 = (-2.7321,\ -3.7321), \\ &\tilde{w}_3 = (2.7321,\ -3.7321),\ \tilde{w}_4 = (1.8660,\ -4.2321),\ \tilde{w}_5 = (0,\ -4.7321),\ \tilde{w}_6 = (4.5981,\ 0.5000), \\ &\tilde{w}_7 = (4.5981,\ -0.5000),\ \tilde{w}_8 = (4.0981,\ -2.3660),\ \tilde{w}_9 = (2.7321,\ 3.7321),\ \tilde{w}_{10} = (4.0981,\ 2.3660), \\ &\tilde{w}_{11} = (1.8660,\ 4.2321),\ \tilde{w}_{12} = (0,\ 4.7321),\ \tilde{w}_{13} = (-1.8660,\ 4.2321),\ \tilde{w}_{14} = (-4.5981,\ 0.5000), \\ &\tilde{w}_{15} = (-4.5981,\ -0.5000),\ \tilde{w}_{16} = (-4.0981,\ -2.3660),\ \tilde{w}_{17} = (-2.7321,\ 3.7321),\ \tilde{w}_{18} = (-4.0981,\ 2.3660). \end{aligned}$$

The sets $\mathcal{Q}$ and $\mathcal{R}$ utilized for the cost function $V_N(\cdot)$ via the associated gauge functions are given by:

$$\mathcal{Q} := \{x \in \mathbb{R}^2 \ :\ (1,\ 0)x \leq 1,\ -(1,\ 0)x \leq 1,\ (0,\ 1)x \leq 1,\ -(0,\ 1)x \leq 1\}, \text{ and, } \mathcal{R} := \{u \in \mathbb{R}\ :\ u \leq 1,\ -u \leq 1\}.$$

The local linear feedback $u = Kx$, the terminal constraint set $\mathbb{X}_f$ and the set $\mathcal{P}$ utilized in the cost function $V_N(\cdot)$ via the associated gauge function are given by:

$$\begin{aligned} &\mathbb{X}_f = \{x \in \mathbb{R}^2 \ :\ (0.0473,\ 0.1765)x \leq 1,\ -(0.0473,\ 0.1765)x \leq 1,\ (0.1314,\ 0.0352)x \leq 1,\ -(0.1314,\ 0.0352)x \leq 1\}, \\ &K = -(0.3943,\ 0.1057), \text{ and,} \\ &\mathcal{P} := \{x \in \mathbb{R}^2 \ :\ (1.0635,\ 3.9692)x \leq 1,\ -(1.0635,\ 3.9692)x \leq 1,\ (2.9564,\ 0.7922)x \leq 1,\ -(2.9564,\ 0.7922)x \leq 1\}. \end{aligned}$$

Similarly as in the previous example, any initial condition $x_0 \in \mathbb{X}$ is min–max controllable to a target set $\mathbb{X}_f$ within $N = 2$ steps. In particular, for the initial conditions equal to the extreme points of the set $\mathbb{X}$ the min–max

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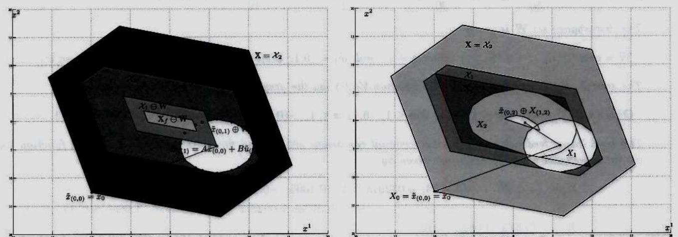
controllability to a target set $\mathbb{X}_f$ can be guaranteed if and only if $N = 2$. The state constraint set $\mathbb{X}$ is in this case equal to 2–step parameterized tubes controllability set $\mathcal{X}_2$ for this example. We consider the initial condition $x_0 = -(10.0127,\ 12.5832)$ and the horizon length $N = 2$. Let $\tilde{x}_{(0,0)} = x_0$. The only feasible control $u \in \mathbb{U}$ for this initial condition permitting for the min–max controllability to a target set $\mathbb{X}_f$ is $u = 3$ so we are forced to set $\tilde{u}_{(0,0)} = 3$ and, in turn, we get $\tilde{x}_{(0,1)} = (5.9455,\ -4.4814)^T$ which is one of the extreme points of the set $\mathcal{X}_1 \ominus \mathbb{W}$. This fixes the state tube cross–section X_1 to $X_1 = \tilde{x}_{(0,1)} \oplus \mathbb{W}$. The initial condition $\tilde{x}_{(0,0)} = x_0$, the state $\tilde{x}_{(0,1)}$ and the state tube cross–section X_1 are shown in Figure 2. The 1– and 2–step parameterized tubes controllability sets $\mathcal{X}_1$, $\mathcal{X}_2$ as well as effective target sets at times 1 and 2, namely the sets $\mathcal{X}_1 \ominus \mathbb{W}$ and $\mathbb{X}_f \ominus \mathbb{W}$ are also show in Figure 2 (a) using the different levels of gray scale shading. In order to implement the proposed PTMPC we need to find the



(a) Geometry of the Example. (b) PTOC State Tube.

Figure 2: Geometry and PTOC State Tubes for Illustrative Example 2.

partial control tube cross sections $U_{(0,1)} = \{\tilde{u}_{(0,1)}\}$ and $U_{(1,1)} = \{\tilde{u}_{(i,1,1)}\ :\ i \in \mathbb{N}_{[1:q]}\}$ satisfying the constraints:

$$\forall i \in \mathbb{N}_{[1:18]},\ \tilde{u}_{(0,1)} + \tilde{u}_{(i,1,1)} \in \mathbb{U},\ \textit{and},$$
$$\forall i \in \mathbb{N}_{[1:18]},\ A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} + A\tilde{w}_i + B\tilde{u}_{(i,1,1)} \in \mathbb{X}_f \ominus \mathbb{W}.$$

This task is possible to accomplish since, by inspection of Figure 2 (b), we see that the state tube cross–section X_1 satisfies $X_1 \subseteq \mathcal{X}_1$ and the set $\mathcal{X}_1$ is the 1–step parameterized tubes controllability set. We solve PTOC for this problem and in Figure 2 (b) we show the corresponding PTOC state tubes $\mathbf{X}_2 = \{X_0 = \{\tilde{x}_{(0,0)}\}, X_1 = \tilde{x}_{(0,1)} \oplus X_{(1,1)}, X_2 = \tilde{x}_{(0,2)} \oplus X_{(1,2)} \oplus X_{(2,2)}\}$ obtained from the corresponding PTOC control tubes $\mathbf{U}_1 = \{U_0 = \{\tilde{u}_{(0,0)}\}, U_1 = \tilde{u}_{(0,1)} \oplus U_{(1,1)}\}$ given by:

$$\tilde{u}_{(0,0)} = 3,\ \tilde{u}_{(0,1)} = -1.9251\ \textit{and}$$
$$U_{(1,1)} = \mathrm{convh}(\{0.5,\ 0.8943,\ -0.6830,\ -0.5774,\ -0.1830,\ -1.0749,\ -1.0749,\ -1.0749,\ -1.0749,$$
$$-1.0749,\ -1.0749,\ -0.5000,\ 0.2887,\ 1.7604,\ 1.866,\ 1.683,\ 0.683,\ 1.366\})\ \textit{so that}$$
$$U_{(1,1)} = [-1.0749, 1.8660],\ \textit{and},\ \tilde{u}_{(0,1)} \oplus U_{(1,1)} = [-3, -0.059] \subseteq \mathbb{U}.$$

In Figure 2 (b), we also depict the set $\tilde{x}_{(0,2)} \oplus X_{(1,2)}$ in order to illustrate that the partial state tube cross–section $X_{(1,2)}$ has been transformed in a non linear fashion by the control rule induced from the partial control tube $U_{(1,1)}$ in order to meet the constraints $X_2 = \tilde{x}_{(0,2)} \oplus X_{(1,2)} \oplus X_{(2,2)} \subseteq \mathbb{X}_f$.

By construction it follows that for the methods of [15–18] to be applicable it is necessary to find $\tilde{u}_{(0,1)}$ and a matrix $M \in \mathbb{R}^{1\times 2}$ satisfying the constraints:

$$\forall i \in \mathbb{N}_{[1:18]},\ \tilde{u}_{(0,1)} + M\tilde{w}_i \in \mathbb{U},\ \textit{and},$$
$$\forall i \in \mathbb{N}_{[1:18]},\ A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} + (A + BM)\tilde{w}_i \in \mathbb{X}_f \ominus \mathbb{W}.$$

However, these constraints can not be satisfies as we have verified numerically and, hence, the methods of [15–18] are not applicable to this problem. From the inspection of the Figure, it is not surprising that we can not find an affine function of states belonging to the state tube cross–section X_1 generating the admissible controls actions for the extreme points of X_1 ensuring that the state tube cross–section $X_2 = A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} \oplus (A + BM)\mathbb{W} \oplus \mathbb{W}$ satisfies $X_2 \subseteq \mathbb{X}_f$ (or equivalently $A\tilde{x}_{(0,1)} + B\tilde{u}_{(0,1)} \oplus (A + BM)\mathbb{W} \subseteq \mathbb{X}_f \ominus \mathbb{W}$) and that the corresponding control tube cross–section $U_1 = \tilde{u}_{(0,1)} \oplus M\mathbb{W} \subseteq \mathbb{U}$. This should come without any surprises as there are 18 extreme points 4 of which lie on the boundary of the 1–step parameterized tubes controllability set $\mathcal{X}_1$ and we are allowed to select

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FPT MPC – Illustrative Examples

only three variables namely $\tilde{u}_{(0,1)}$, m_1 and m_2 (where $M = (m_1,\ m_2)$). To sum up, our example demonstrates that the methods of [15–18] may fail to be applicable when our proposal is applicable.

Illustrative Example 3 *Our third illustrative example is two dimensional system specified by:*

$$x^+ = Ax + Bu + w, \ \textit{with,}\ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \textit{and,}\ ,\ B = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

The state and control constraint sets are given by:

$$\mathbb{X} := \{x \in \mathbb{R}^2 \ :\ (\frac{1}{50},\ 0)x \leq 1,\ (-\frac{1}{20},\ 0)x \leq 1,\ (0,\ \frac{1}{2})x \leq 1,\ (0,\ -\frac{1}{10})x \leq 1\}, \ \textit{and,}\ \mathbb{U} := \{u \in \mathbb{R} \ :\ u \leq 1,\ -u \leq 1\}.$$

The disturbance set $\mathbb{W}$ is given by:

$$\mathbb{W} = \mathrm{convh}(\{\tilde{w}_i \ :\ i \in \mathbb{N}_{[1:q]}\}) \ \textit{where}\ q = 4,\ \textit{and,}\ \tilde{w}_1 = (0.1,\ 0.1),\ \tilde{w}_2 = (0.1,\ -0.1),\ \tilde{w}_3 = -\tilde{w}_1 \ \textit{and}\ \tilde{w}_4 = -\tilde{w}_2.$$

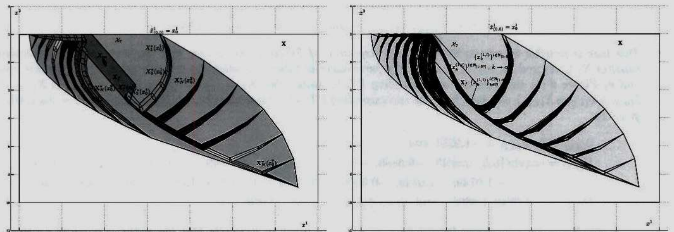
The sets $\mathcal{Q}$ and $\mathcal{R}$ utilized for the cost function $V_N(\cdot)$ via the associated gauge functions are given by:

$$\mathcal{Q} := \{x \in \mathbb{R}^2 \ :\ (1,\ 0)x \leq 1,\ -(1,\ 0)x \leq 1,\ (0,\ 1)x \leq 1,\ -(0,\ 1)x \leq 1\}, \ \textit{and,}\ \mathcal{R} := \{u \in \mathbb{R} \ :\ u \leq 1,\ -u \leq 1\}.$$

The local linear feedback $u = Kx$, the terminal constraint set $\mathbb{X}_f$ and the set $\mathcal{P}$ utilized in the cost function $V_N(\cdot)$ via the associated gauge function are given by:

$$\begin{aligned} \mathbb{X}_f = \{x \in \mathbb{R}^2 \ :\ & -(0.1974,\ 0.0329)x \leq 1,\ (0.1095,\ -0.1277)x \leq 1,\ (0,\ 0.5)x \leq 1, \\ & (0.3750,\ 1.0625)x \leq 1,\ -(0.3750,\ 1.0625)x \leq 1\}, \\ K = -(0.375,\ 1.0625),\ & \textit{and,} \\ \mathcal{P} := \{x \in \mathbb{R}^2 \ :\ & (35.8103,\ 5.9684)x \leq 1,\ -(35.8103,\ 5.9684)x \leq 1,\ (0,\ 90.7194)x \leq 1, \\ & -(0,\ 90.7194)x \leq 1,\ (68.0396,\ 192.7788)x \leq 1,\ -(68.0396,\ 192.7788)x \leq 1\}. \end{aligned}$$

We set the horizon length $N = 7$. In Figure 3 (a), we show the state constraint set $\mathbb{X}$, the 7–step parameterized tubes controllability set $\mathcal{X}_7$, the terminal constraint set $\mathbb{X}_f$ and the minimal robust positively invariant set X_∞. The 7–step



(a) PTOC State Tubes. (b) Sample PTMPC Controlled Trajectories.

Figure 3: PTOC State Tubes and Sample PTMPC Controlled Trajectories from Extreme Point of the Set $\mathcal{X}_7$.

parameterized tubes controllability set $\mathcal{X}_7$ has 17 extreme points and we solve PTOC for a set of initial conditions being equal to the extreme points of $\mathcal{X}_7$. The corresponding PTOC state tubes are also shown in Figure 3 (a). As evident, they satisfy state constraints as they remain with the sets $\mathcal{X}_7$ and the final tube tube cross–sections are all included in the terminal constraint set $\mathbb{X}_f$. In Figure 3 (b), we show a number of trajectories generated by the controlled uncertain dynamics obtained from the implementation of the PTMPC law for a set of random extreme disturbance sequences. As expected in view of results established in Theorem 1 and Corollary 3, all these trajectories converge exponentially fast to the minimal robust positively invariant set X_∞.

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$\boxed{\text{§5 – Concluding Remarks}}$

- Summary
- Historical Remarks

Summary

- Well–posed parameterized tube optimal control problems,
  - Meaningful solution process & 2–dimensional thinking,
  - Repetitive application of parameterized tube optimal control,
  - RHC/MPC strategies,
  - Suitably tailored use of optimization and control synthesis,
  - Strong system–theoretic properties.
- Satisfaction of synthesis objectives:
  - Constraint Satisfaction (Invariance),
  - Stable process despite constraints and uncertainty (Stability),
  - Optimized performance in an adequate sense,
  - Computational efficiency.

Relevant Papers

- Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks, IEEE-TAC Regular Paper, (Published)
  (Raković and Barić),
- Parameterized Tube MPC, IEEE-TAC Regular Paper, (Accepted)
  (Raković, Kouvaritakis, Cannon, Panos and Findeisen),
- Fully Parameterized Tube MPC, IFAC 2011, (Published)
  (Raković, Kouvaritakis, Cannon, Panos and Findeisen),
- Fully Parameterized Tube MPC, IJRNC D. W. Clarke's Special Issue Paper, (Accepted)
  (Raković, Kouvaritakis, Cannon and Panos),
- Three More Surprise but Top Secret Papers ;-)
  (Raković and co–author/s),

Historical Remarks

- Dynamic Programming and Controllability Under Constraints and Uncertainty (Bertsekas, Schweepe, Witsenhausen, Kurzhanski, Krasovski, Pontryagin, La Salle, Hermes, Artstein, Aubin, Frankowska, Lasserre, Blanchini, Miani, ...),
- “Simplified” Tube Based Control Synthesis Under Constraints and Uncertainty:
  - Time–Varying Tube MPC
    (Blanchini, Kouvaritakis, Cannon, Lee, Chisci, Zappa, Rositer, ...),
  - Rigid Tube MPC
    (Mayne, Raković, Seron, Allgöwer, Teel, Astolfi, ...),
  - Homothetic Tube MPC
    (Raković, Kouvaritakis, and Cannon)
  - Parameterized and Fully Parameterized Tube MPC
    (Raković, Kouvaritakis, Cannon, and Panos)
- Min–Max Feedback MPC (Bertsekas, Mayne and Scoekert, Kerrigan and Maciejowski)
- Disturbance Affine Feedback Robust MPC (Ben–Tal, Loefberg, Goulart and Kerrigan and Maciejowski)

That is all folks!

Thank you for patience! & Any questions?