Bandit optimization with large strategy sets

Alexandre Proutiere

Joint work with **Richard Combes**

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Outline

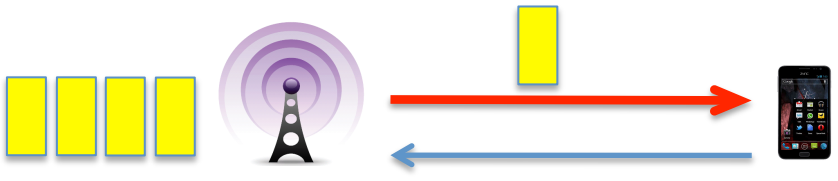
1. Motivation
2. Bandit optimization: background
3. Graphically unimodal bandits
4. Applications

1. Motivation

Rate adaptation in 802.11

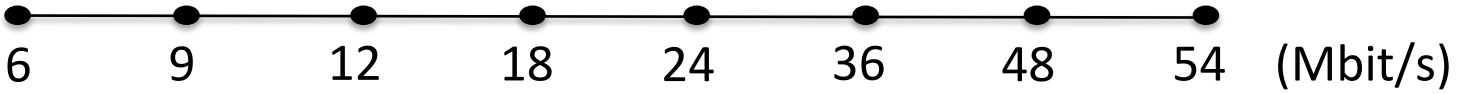
Adapting the modulation/coding scheme to the radio environment

- 802.11 a/b/g



rates	r_1	r_2	$\ldots$	r_N
Success probabilities	θ_1	θ_2	$\ldots$	θ_N
Throughputs	μ_1	μ_2	$\ldots$	μ_N

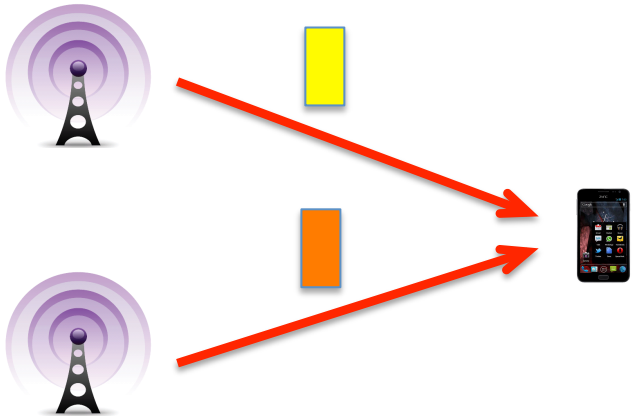
$$\mu_i = r_i \theta_i$$



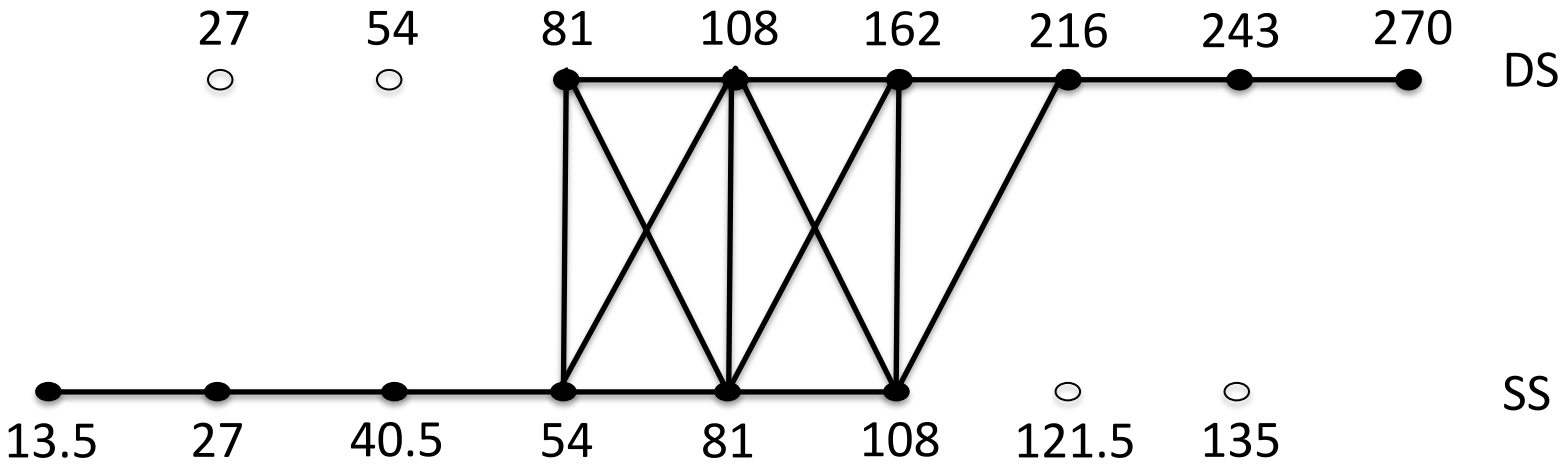
- Optimal sequential rate selection?

Rate adaptation in 802.11

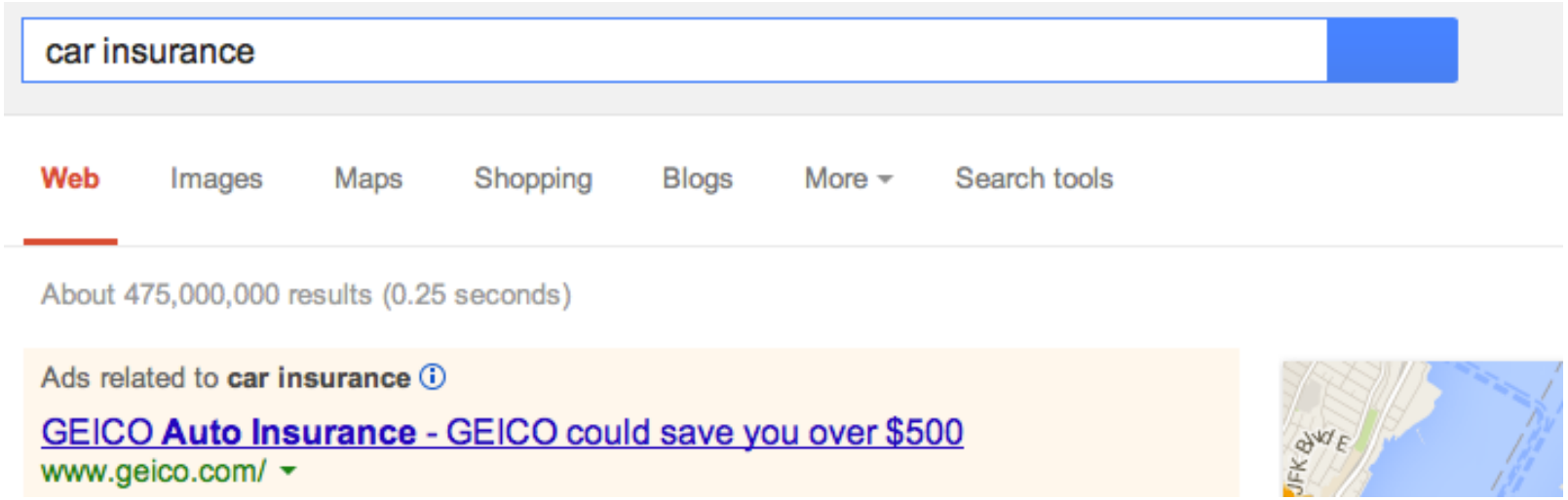
- 802.11 n/ac MIMO
  Rate + MIMO mode



- Example: two modes, single-stream (SS) or double-stream (DS)



CTR estimation in Ad Auctions

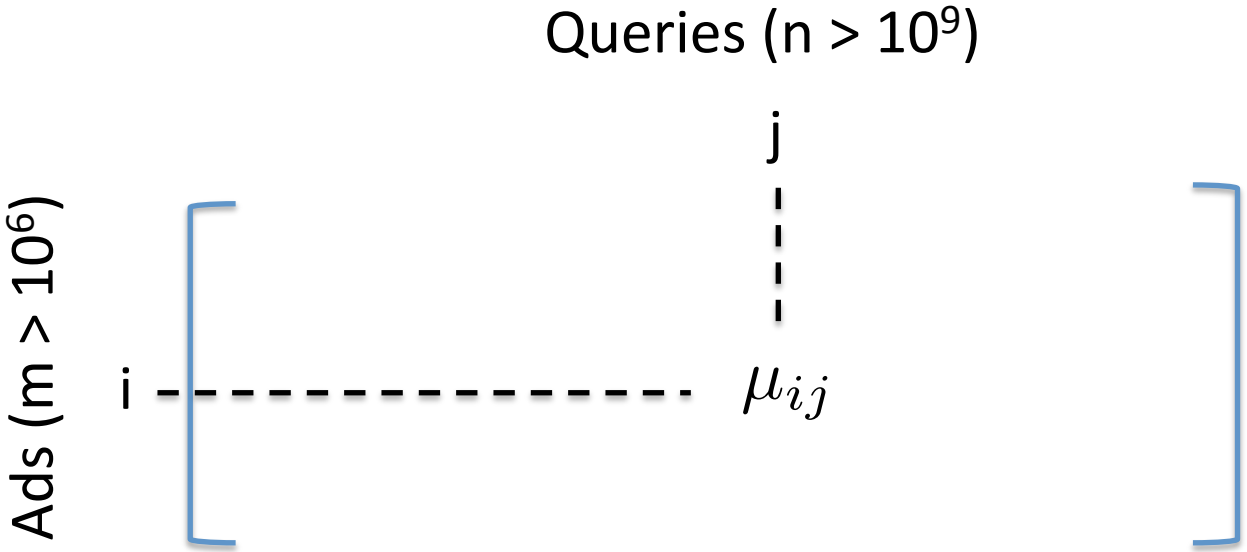


- Current practice:
  - ✓ Bayesian estimator (graphical models + EP)
  - ✓ Minimize the mean square error on CTRs
  - ✓ Underlying assumption: a static system

- A dynamic system (changing ads, changing CTRs, …)
- … but most importantly our goal is to maximize profit not minimize CTR estimation error!

CTR bandits

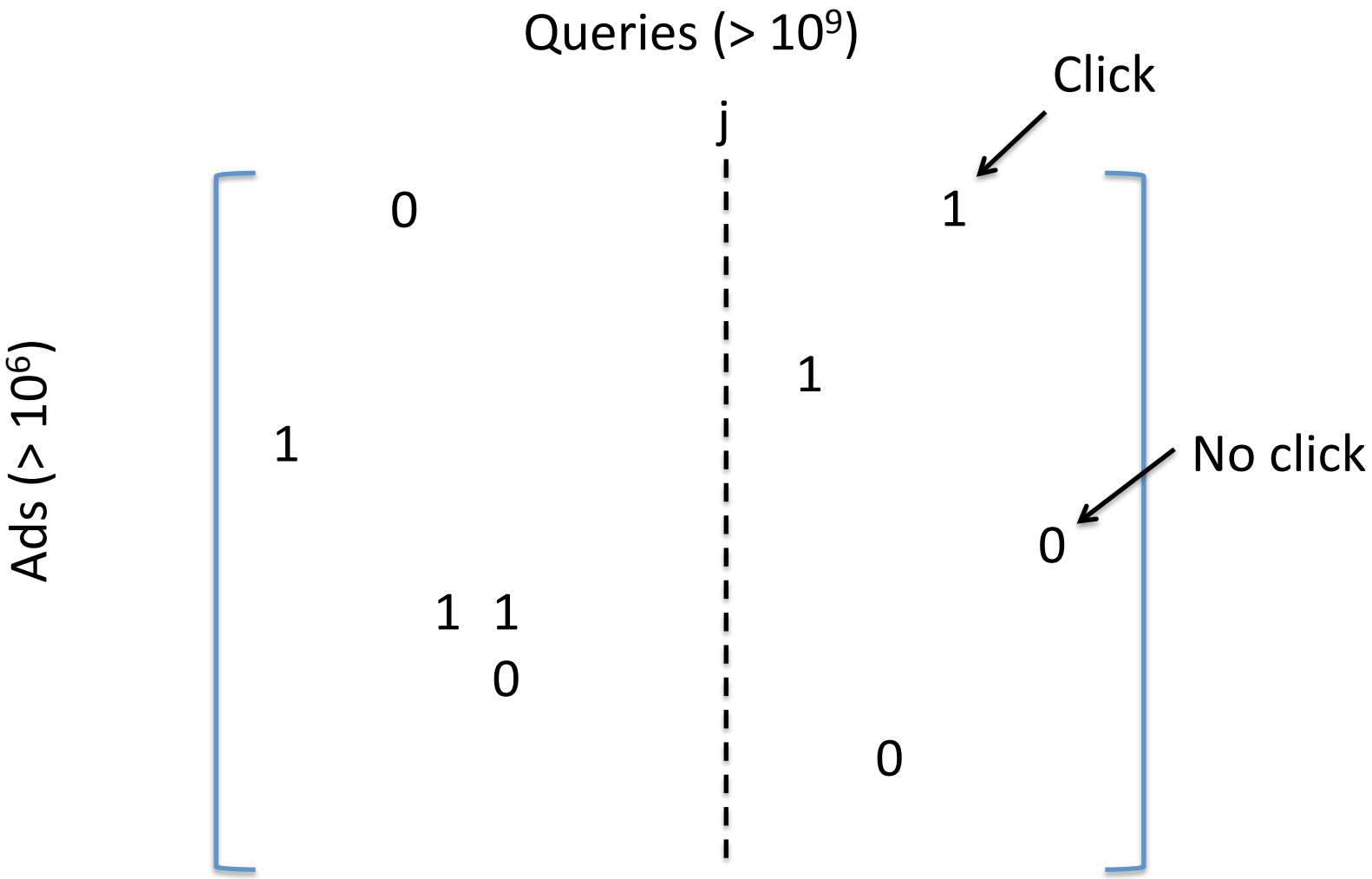
- CTR matrix:



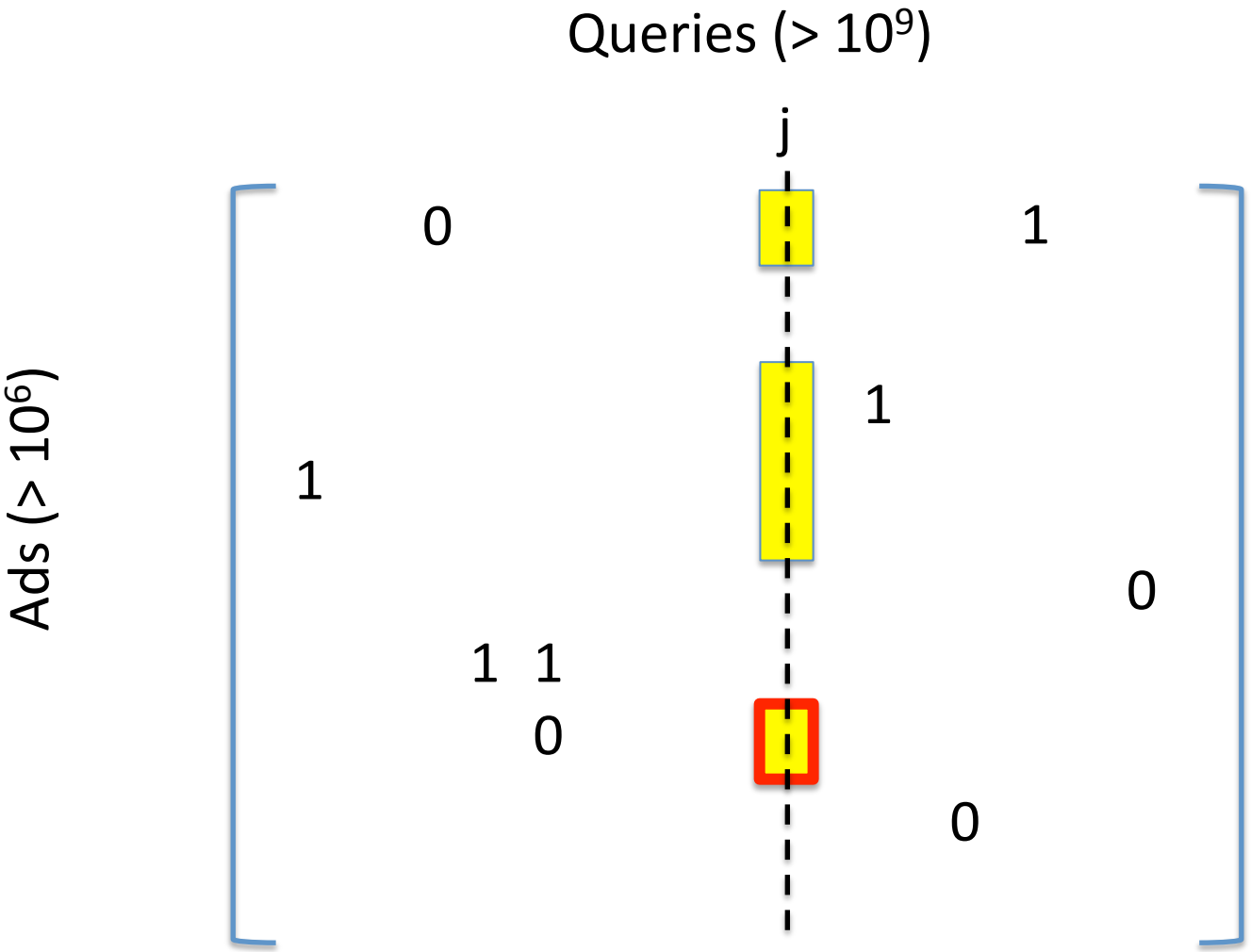
- Profit after *T* queries: $\text{profit}(T) = \sum_{t=1}^{T} b_{i(t)j(t)} X_{i(t)j(t)}$

$$X_{ij} \sim \text{Ber}(\mu_{ij})$$

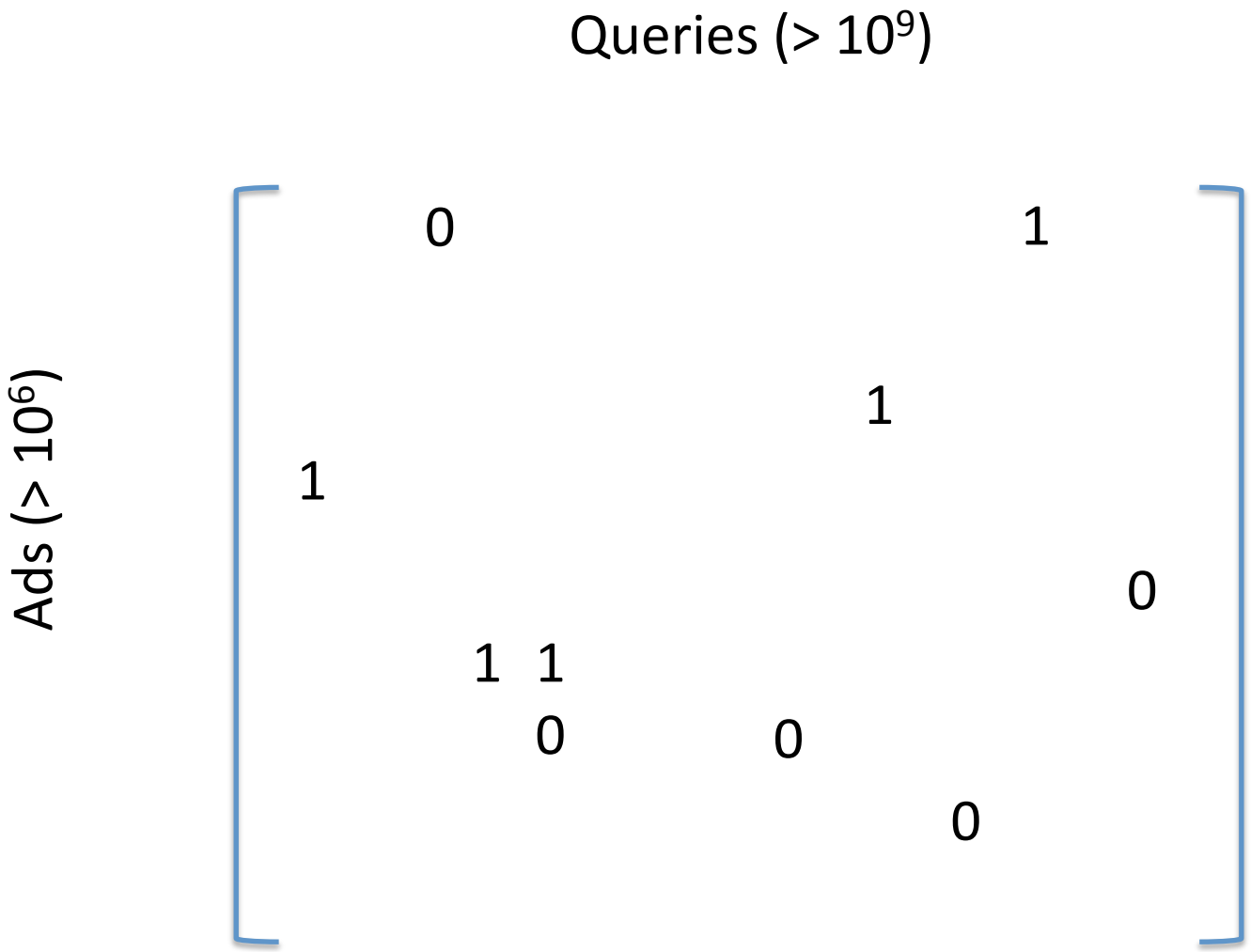
Online decision problem



Online decision problem



Online decision problem



2. Bandit optimization

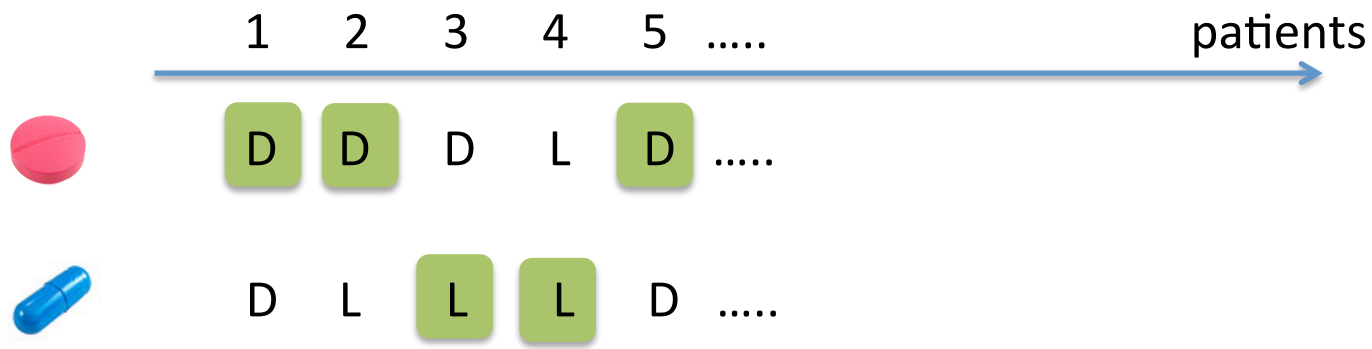
Bandit optimization

A sequential decision problem (**Thompson** 1933)

- A set of possible actions at each step
- Unknown sequence of rewards for each action

Bandit optimization

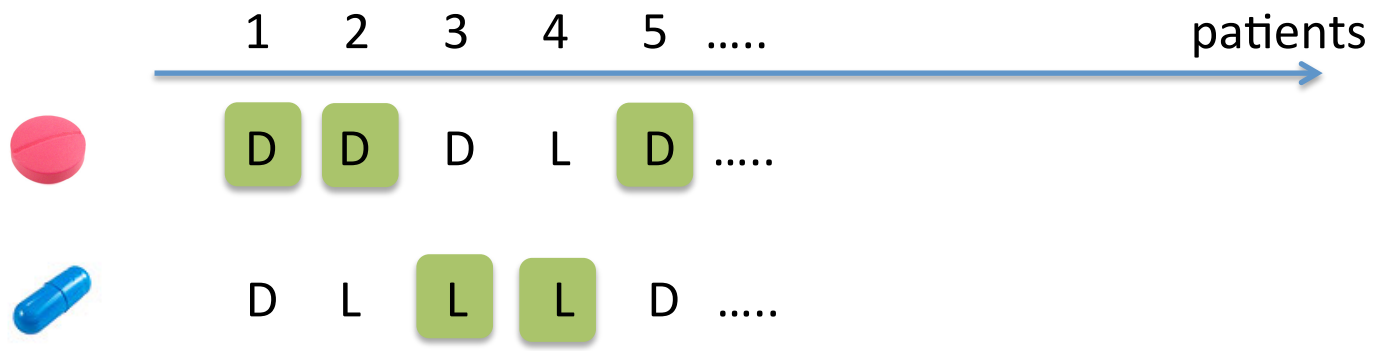
A sequential decision problem (**Thompson** 1933)



- A set of possible actions at each step
- Unknown sequence of rewards for each action
- Bandit feedback: only rewards of chosen actions are observed

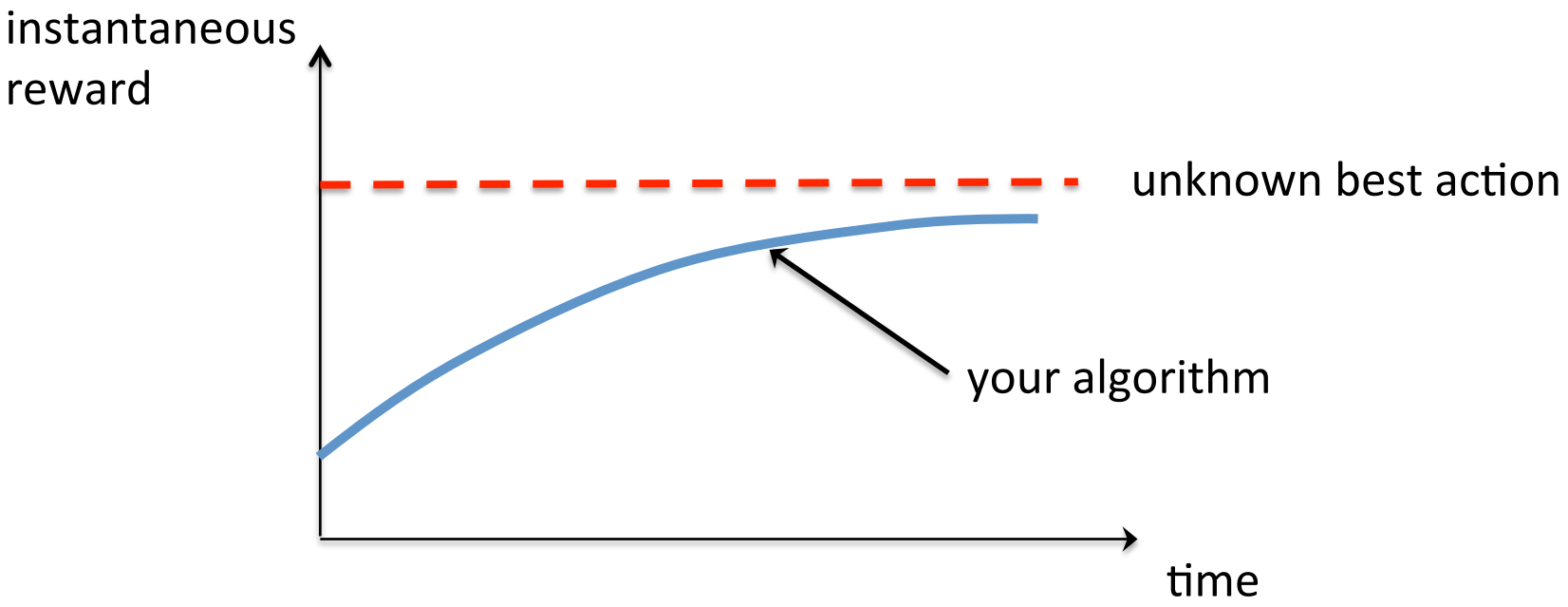
Bandit optimization

A sequential decision problem (**Thompson** 1933)

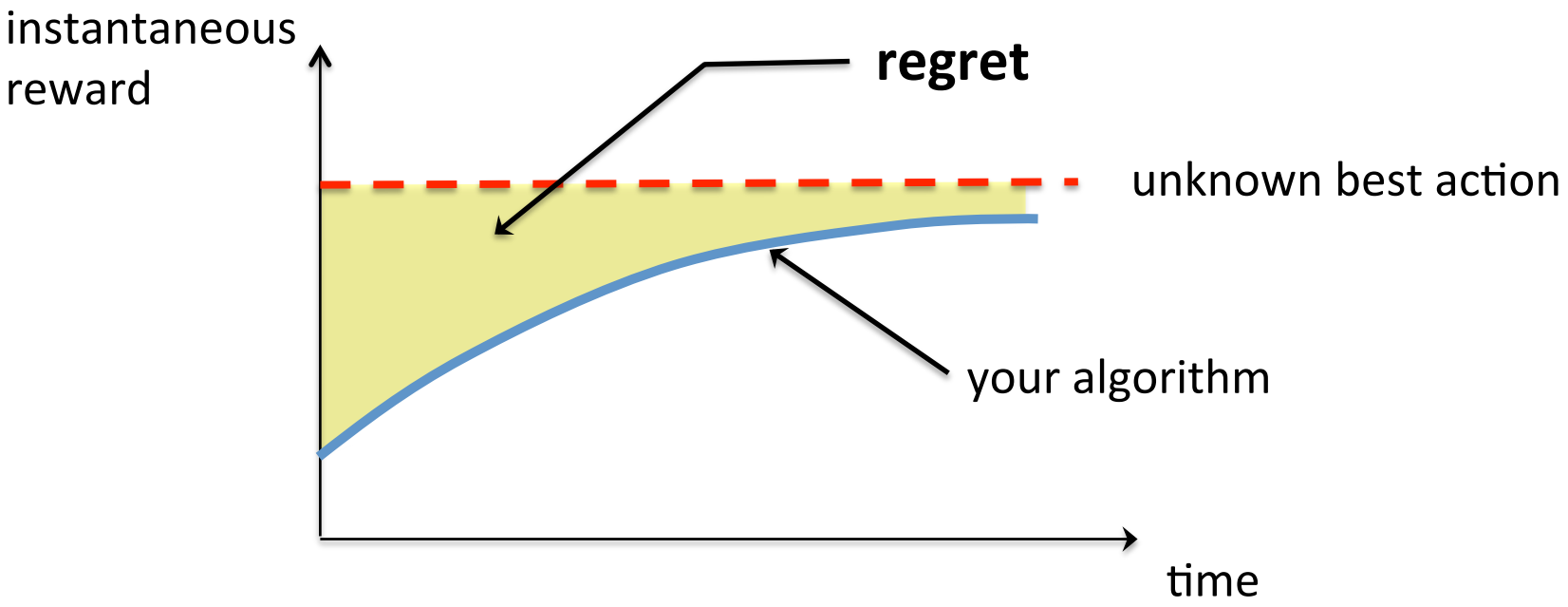


- A set of possible actions at each step
- Unknown sequence of rewards for each action
- Bandit feedback: only rewards of chosen actions are observed
- Goal: maximize the cumulative reward (up to step T), i.e., strike the optimal exploration-exploitation trade-off

Regret


instantaneous
reward
unknown best action
your algorithm
time

Regret



Objective: to identify the best action with minimum exploration, i.e., to minimize regret (to maximize the "convergence rate")

Particularly relevant when the best action evolves – for tracking problems

Stochastic Bandits

Robbins 1952

- *K* arms / decisions / actions
- Unknown i.i.d. rewards: $X_{i,t} \sim \mathrm{Ber}(\mu_i), \quad \mu^\star = \max_i \mu_i = \mu_{i^\star}$
- Lack of structure: $\mu_i \in [0,1], \quad \forall i \in \{1,\dots,K\}$
- Under online algorithm π, arm selected at time t:

  I_t^π function of history $(I_1^\pi, X_{I_1^\pi,1}, \dots, I_{t-1}^\pi, X_{I_{t-1}^\pi,t-1})$
- Regret up to time T:

$$R^\pi(T) = \max_{i=1,\dots,K} \mathbb{E} \sum_{t=1}^{T} X_{i,t} - \mathbb{E} \sum_{t=1}^{T} X_{I_t^\pi,t}$$

Stochastic Bandits

- Asymptotic regret lower bound (no algorithm can beat this performance)
- Uniformly good algorithm: $\mathbb{E}[t_i(T)] = o(T^\alpha), \quad \forall \alpha > 0, \forall \mu, \forall i \neq i^\star$

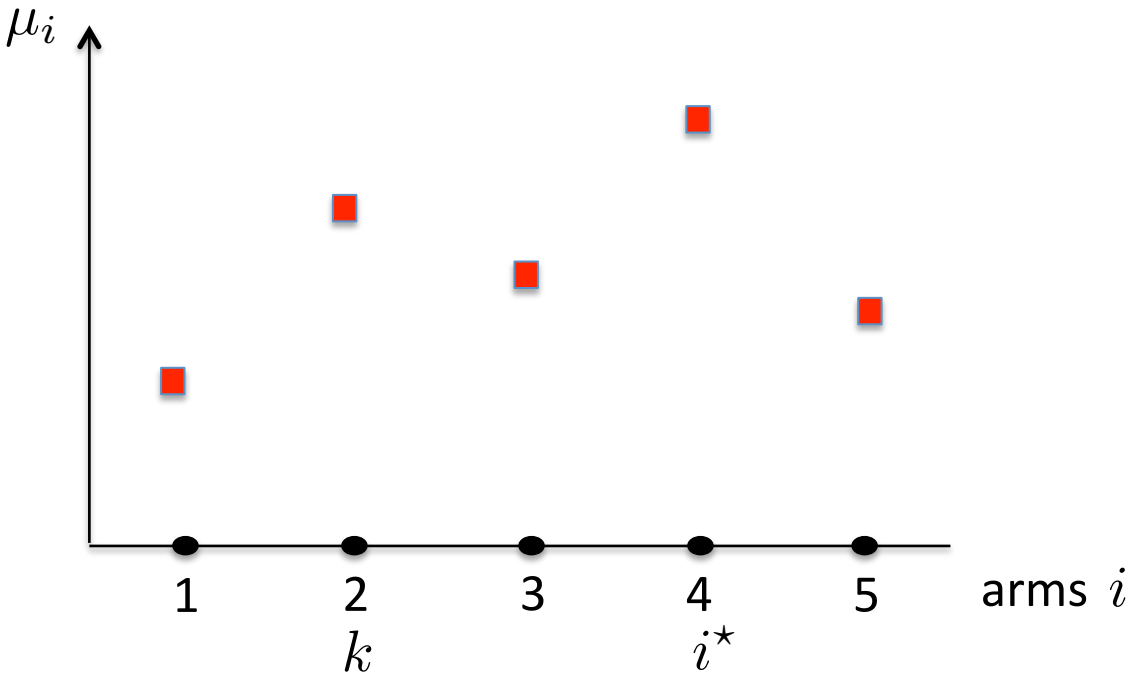
Theorem (Lai-Robbins 1985) For any uniformly good policy π

$$\liminf_{T\to\infty} \frac{R^\pi(T)}{\log(T)} \geq \sum_{i\neq i^\star} \frac{\mu^\star - \mu_i}{\mathrm{KL}(\mu_i, \mu^\star)}$$

KL divergence number: $KL(p,q) = p\log(\frac{p}{q}) + (1-p)\log(\frac{1-p}{1-q})$

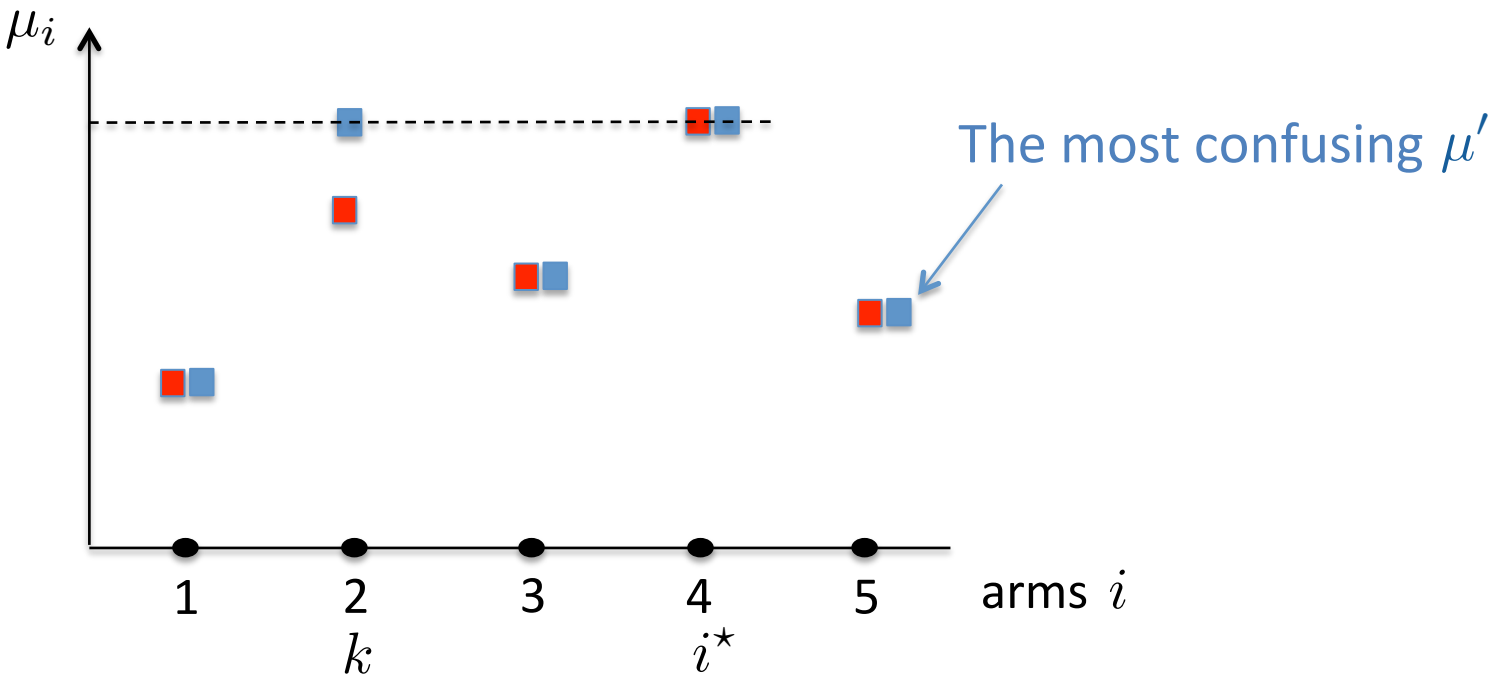
Regret linear in the number of arms, and proportional to $\frac{1}{(\mu^\star - \mu_i)}$

The change-of-measure argument



To identify the minimum number of times sub-optimal arm k must be played, find the most confusing parameters

The change-of-measure argument



To identify the minimum number of times sub-optimal arm k must be played, find the most confusing parameters

k played ~ $\dfrac{\log(T)}{\mathrm{KL}(\mu_k, \mu^\star)}$ times, yielding a regret $\dfrac{\mu^\star - \mu_k}{\mathrm{KL}(\mu_k, \mu^\star)} \log(T)$

Algorithms

- Optimal but complicated policy (**Lai-Robbins** 1985)
- Simpler and optimal algorithms (**Agrawal** 1995)
- ε-greedy algorithm, ε=1/t logarithmic regret
- UCB algorithm (**Auer et al.** 2002): a simple suboptimal index policy

$$\frac{1}{n_i(t)} \sum_{t=1}^{n_i(t)} X_{i,t} + \sqrt{\frac{2\log(t)}{n_i(t)}}$$

- KL-UCB (**Garivier-Cappe** 2011): claiming back optimality

  Index-based policy, index of arm *i*

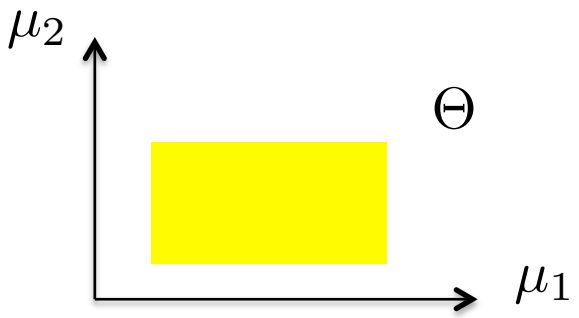
$$\max \left\{ q \leq 1 : n_i(t) \mathrm{KL}(\hat{\mu}_i(t), q) \leq \log(t) + 3\log\log(t) \right\}$$

Bandit classification

- **Unstructured bandits:** average rewards are not related

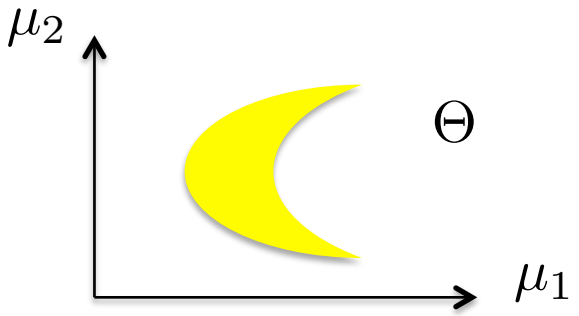
$$\mu = (\mu_1, \ldots, \mu_K) \in \Theta$$

$$\Theta = \prod_{i=1}^{K} [a_i, b_i]$$



- **Structured bandits:** the decision maker knows that average rewards are related

$$\Theta \neq \prod_{i=1}^{K} [a_i, b_i]$$

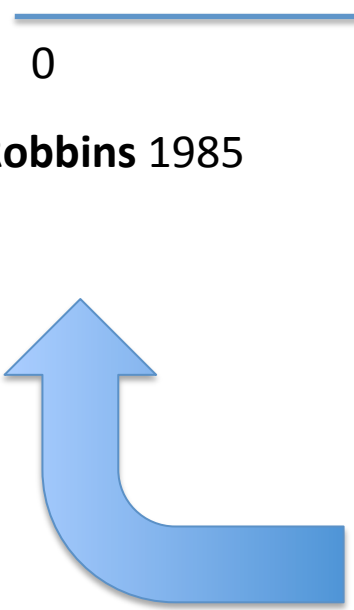


  - The rewards observed for a given action provide side-information about the average rewards of other actions
  - How can we exploit this side-information optimally?

Bandit classification

K/T

(Decision set/Time horizon)



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∞

Lai-Robbins 1985

Bandits with structure:
Combinatorial bandits
Continuous bandits:
- Linear bandits
- Convex bandits
- Unimodal bandits

Agrawal 1995
Kleinberg-Slivkins 2005
Bubeck et al. 2011 …

Infinite Bandits:
Berry et al. 1997
Wang et al. 2008
Bonald-P. 2013

Lai
Agrawal-Teneketzis-Anantharam 1989
Kumar 1985
Borkar-Varaiya 1979
Optimal control of Markov Chains

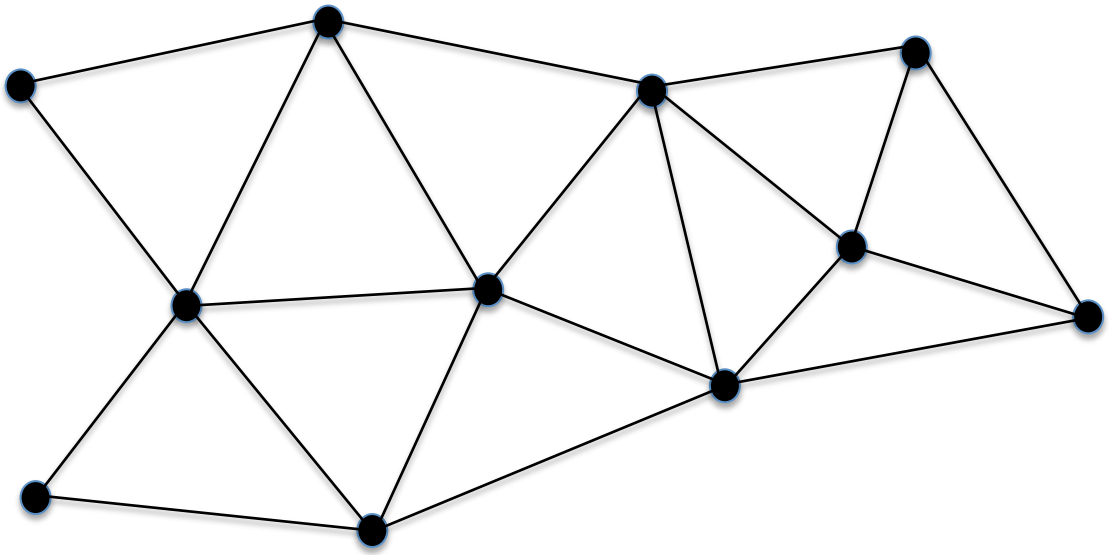
A few papers since 2005

[Abbasi-Yadkori, 2009] [Abernethy, Hazan, Rakhlin, 2008] [Abernethy, Bartlett, Rakhlin, Tewari, 2008] [Abernethy, Agarwal, Bartlett, Rakhlin, 2009] [Audibert, Bubeck, 2010] [Audibert, Munos, Szepesvári, 2009] [Audibert, Bubeck, Lugosi, 2011] [Auer, Ortner, Szepesvári, 2007] [Auer, Ortner, 2010] [Awerbuch, Kleinberg, 2008] [Bartlett, Hazan, Rakhlin, 2007] [Bartlett, Dani, Hayes, Kakade, Rakhlin, Tewari, 2008] [Bartlett, Tewari, 2009] [Ben-David, Pal, Shalev-Shwartz, 2009] [Blum, Mansour, 2007] [Bubeck, 2010] [Bubeck, Munos, 2010] [Bubeck, Munos, Stoltz, 2009] [Bubeck, Munos, Stoltz, Szepesvári, 2008] [Cesa-Bianchi, Lugosi, 2006] [Cesa-Bianchi, Lugosi, 2009] [Chakrabarti, Kumar, Radlinski, Upfal, 2008] [Chu, Li, Reyzin, Schapire, 2011] [Coquelin, Munos, 2007] [Dani, Hayes, Kakade, 2008] [Dorard, Glowacka, Shawe-Taylor, 2009] [Filippi, 2010] [Filippi, Cappé, Garivier, Szepesvári, 2010] [Flaxman, Kalai, McMahan, 2005] [Garivier, Cappé, 2011] [Grünewälder, Audibert, Opper, Shawe-Taylor, 2010] [Guha, Munagala, Shi, 2007] [Hazan, Agarwal, Kale, 2006] [Hazan, Kale, 2009] [Hazan, Megiddo, 2007] [Honda, Takemura, 2010] [Jaksch, Ortner, Auer, 2010] [Kakade, Shalev-Shwartz, Tewari, 2008] [Kakade, Kalai, 2005] [Kale, Reyzin, Schapire, 2010] [Kanade, McMahan, Bryan, 2009] [Kleinberg, 2005] [Kleinberg, Slivkins, 2010] [Kleinberg, Niculescu-Mizil, Sharma, 2008] [Kleinberg, Slivkins, Upfal, 2008] [Kocsis, Szepesvári, 2006] [Langford, Zhang, 2007] [Lazaric, Munos, 2009] [Li, Chu, Langford, Schapire, 2010] [Li, Chu, Langford, Wang, 2011] [Lu, Pàl, Pàl, 2010] [Maillard, 2011] [Maillard, Munos, 2010] [Maillard, Munos, Stoltz, 2011] [McMahan, Streeter, 2009] [Narayanan, Rakhlin, 2010] [Ortner, 2008] [Pandey, Agarwal, Chakrabarti, Josifovski, 2007] [Poland, 2008] [Radlinski, Kleinberg, Joachims, 2008] [Rakhlin, Sridharan, Tewari, 2010] [Rigollet, Zeevi, 2010] [Rusmevichientong, Tsitsiklis, 2010] [Shalev-Shwartz, 2007] [Slivkins, Upfal, 2008] [Slivkins, 2011] [Srinivas, Krause, Kakade, Seeger, 2010] [Stoltz, 2005] [Sundaram, 2005] [Wang, Kulkarni, Poor, 2005] [Wang, Audibert, Munos, 2008]

See ICML tutorial 2011: **Audibert-Munos**

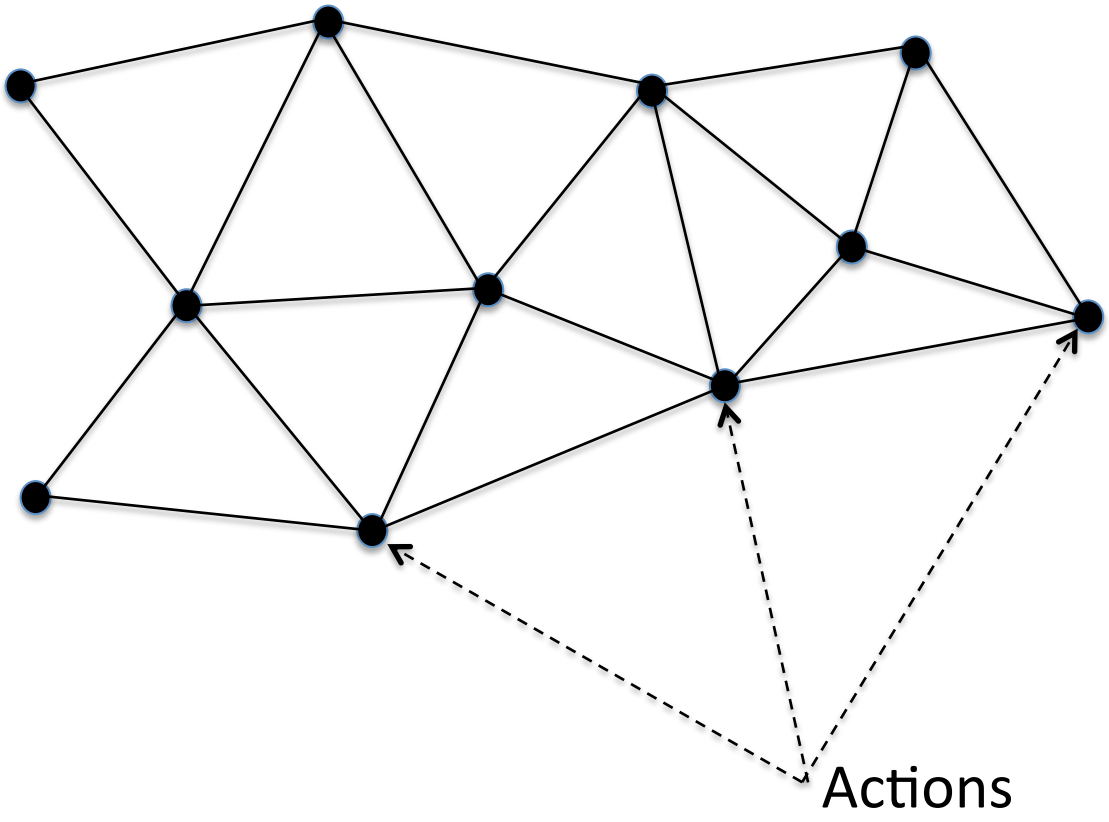
2. Graphically unimodal bandits

Actions and rewards



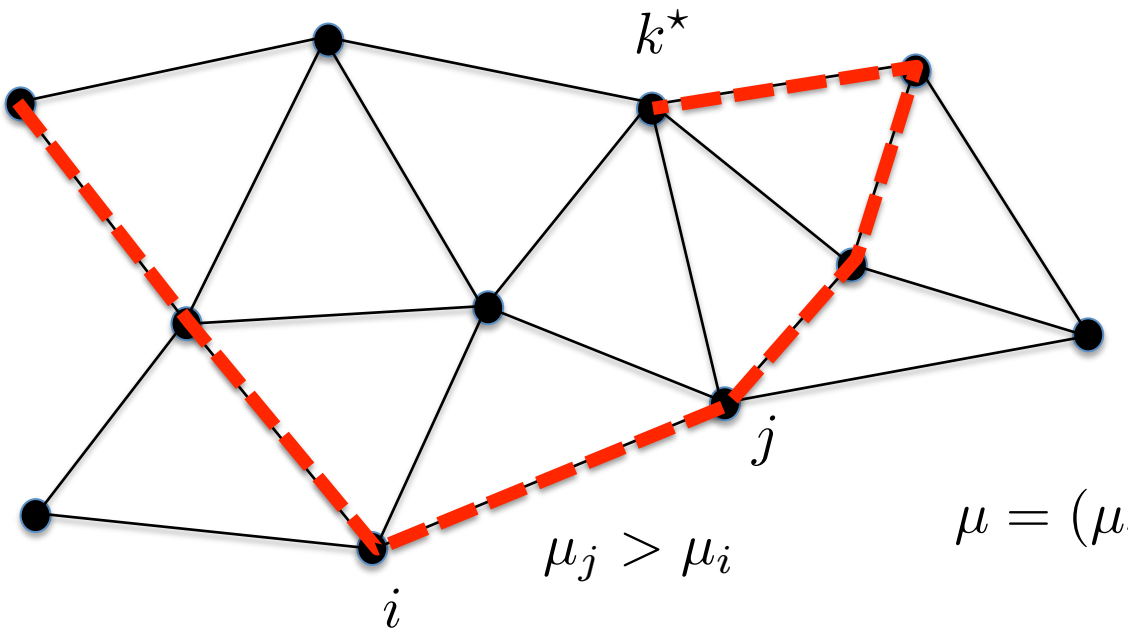
$G = (V, E)$

Actions and rewards



$$G = (V, E)$$

Actions and rewards



$G = (V, E)$

$\mu = (\mu_i)_{i \in V} \in \mathcal{U}_G$

Graphical unimodality: from any vertex, there is a path with increasing rewards to the best vertex.

Related work: **Yu-Mannor** 2011 (sub-optimal algorithms)

Average rewards

- Linear structure: at time t, action i yields a reward $r_i X_{i,t}$
  $X_{i,t} \sim \mathrm{Ber}(\theta_i)$

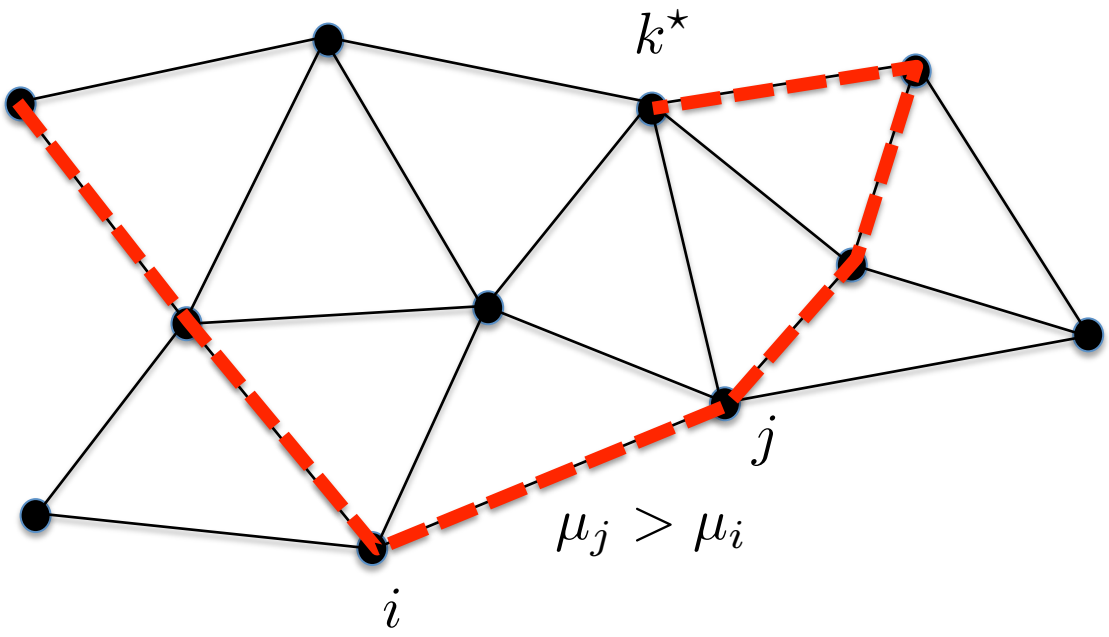
$$
\begin{array}{llll}
r_1 & r_2 & \dots & r_N \quad \text{known} \\
\theta_1 & \theta_2 & \dots & \theta_N \quad \text{unknown } \in [0,1] \\
\mu_1 & \mu_2 & \dots & \mu_N \quad \text{Average rewards}
\end{array}
$$

$$\mu_i = r_i \theta_i$$

Optimal action: $k^\star, \quad \mu^\star = \mu_{k^\star}$

- Graphical unimodality w.r.t. some known graph G

Graphically unimodal bandits

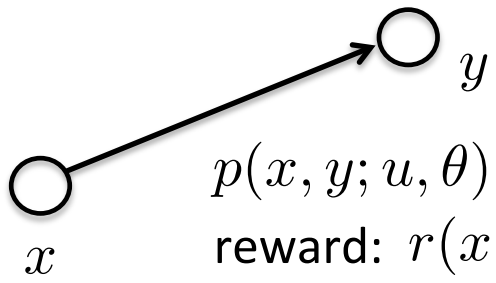


$G = (V, E)$

How can the reward structure be optimally exploited?
How does regret scale with the graph size and topology?

Controlled Markov Chains

Kumar, Lai, Borkar, Varaiya, …



- Finite state and action spaces $\theta \in \Theta$
- Unknown parameter

  Θ : compact metric space

- Control: finite set of irreducible control laws $g : \mathcal{X} \to \mathcal{U}$

$$\mu_g(\theta) = \sum_{x \in \mathcal{X}} \pi_\theta^g(x) r(x, g(x))$$

- Optimal control law: $g^\star$
- Regret: $R^\pi(T) = T\mu_{g^\star}(\theta) - \mathbb{E} \sum_{t=1}^{T} r(X_t, g^\pi(X_t))$

Regret lower bound

- KL number under policy g:

$$I^g(\theta,\lambda)=\sum_{x,y}\pi_\theta^g(x)p(x,y;g(x),\theta)\log\frac{p(x,y;g(x),\theta)}{p(x,y;g(x),\lambda)}$$

- Bad parameter set:

$$B(\theta)=\{\lambda\in\Theta:g^\star \text{ not opt.}, I^{g^\star}(\theta,\lambda)=0\}$$

- Lower bound (Graves-Lai'97): $\liminf_{T\to\infty}\frac{R^\pi(T)}{\log(T)}\geq c(\theta)$

$$c(\theta)=\inf\sum_{g\neq g^\star}c_g(\mu_{g^\star}(\theta)-\mu_g(\theta))$$

$$\text{s.t.}\quad\inf_{\lambda\in B(\theta)}\sum_{g\neq g^\star}c_gI^g(\theta,\lambda)\geq 1$$

Application to graphically unimodal bandits

- State space: set of possible rewards
- Control laws: constant mappings to the set of actions, e.g. $g = k$
- Transitions (i.i.d. process):

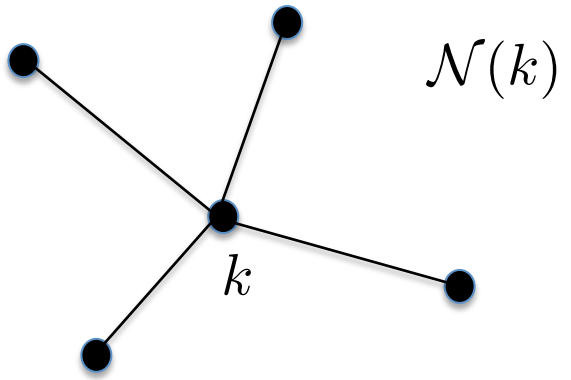
$$p(x, y : k, \theta) = \begin{cases} \theta_k & \text{if } y = r_k \\ 1 - \theta_k & \text{if } y = 0 \end{cases}$$

- Average rewards: $g = k$

$$\mu_g(\theta) = r_k \theta_k = \mu_k$$

Fundamental performance limit

$$N(k) = \{l \in \mathcal{N}(k) : r_k \theta_k \leq r_l\}$$



Theorem: For any uniformly good algorithm π

$$\liminf_{T\to\infty} \frac{R^\pi(T)}{\log(T)} \geq c_G(\theta) \qquad c_G(\theta) = \sum_{k \in N(k^\star)} \frac{r_{k^\star}\theta_{k^\star} - r_k \theta_k}{\mathrm{KL}(\theta_k, \frac{r_{k^\star}\theta_{k^\star}}{r_k})}$$

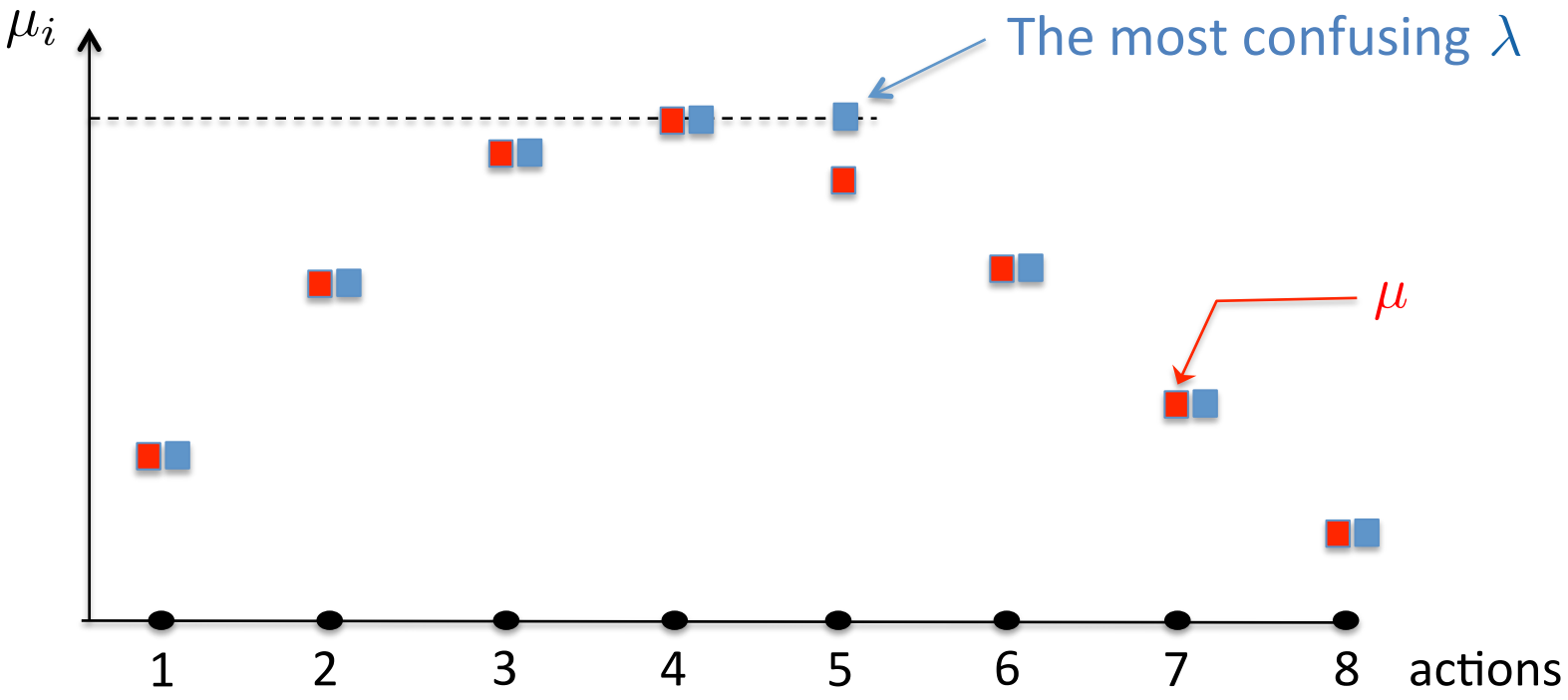
The performance limit does not depend on the size of the decision space! Structure could really help.

Proof

$$\inf \sum_{g \neq g^\star} c_g(\mu_{g^\star}(\theta) - \mu_g(\theta))$$

$$\text{s.t.} \quad \inf_{\lambda \in B(\theta)} \sum_{g \neq g^\star} c_g I^g(\theta, \lambda) \geq 1$$

Example: classical unimodality



Optimal Action Sampling

- Empirical average reward: $\hat{\mu}_k(n) = \frac{1}{t_k(n)} \sum_{s=1}^{t_k(n)} r_k X_k(s)$
- Leader at time *n*: $L(n) \in \arg\max_k \hat{\mu}_k(n)$
- Number of times *k* has been the leader: $l_k(n) = \sum_{s=1}^{n} 1_{L(s)=k}$
- Index of *k*: $b_k(n) = \max \left\{ q \in [0, r_k] : t_k(n) \mathrm{KL}\left(\frac{\hat{\mu}_k(n)}{r_k}, \frac{q}{r_k}\right) \right.$
$\left. \leq \log(l_{L(n)}(n)) + c \log\log(l_{L(n)}(n)) \right\}$

Optimal Action Sampling

Algorithm – Optimal Action Sampling (OAS)

For $n = 1, \ldots, K$, select action $k(n) = n$

For $n \geq K + 1$, select action $k(n)$:

$$k(n) = \begin{cases} L(n) & \text{if } (l_{L(n)}(n) - 1)/(\gamma + 1) \in \mathbb{N}, \\ \arg \max_{k \in N(L(n))} b_k(n) & \text{otherwise.} \end{cases}$$

Theorem: For any $\mu \in \mathcal{U}_G$, $\limsup_{T \to \infty} \frac{R^{OAS}(T)}{\log(T)} \leq c_G(\theta)$.

Proof

$$R^{OAS}(T) \leq r_K \sum_{k \neq k^\star} \mathbb{E}[l_k(T)]$$
$$+ \sum_{k \in N(k^\star)} (r_{k^\star}\theta_{k^\star} - r_k\theta_k)\mathbb{E}[\sum_{t=1}^{T} 1_{L(t)=k^\star, k(t)=k}]$$

First term $\leq O(\log\log(T))$

Second term $\leq (1+\epsilon)c(\theta)\log(T) + O(\log\log(T))$

Deviation bounds (refined concentration inequalities) + further decomposition of the set of events
Finite time analysis

Non-stationary environments

- Average rewards may evolve over time: $\theta(t)$
- Best decision at time *t*: $k^\star(t)$
- Goal: track the best decision
- Regret:

$$R^\pi(T) = \sum_{t=1}^{T} \left(r_{k^\star}(t)\theta_{k^\star(t)}(t) - r_{k^\pi(t)}\theta_{k^\pi(t)}(t) \right)$$

- Sub-linear regret cannot be achieved (**Garivier-Moulines** 2011)
- Assumptions: $\theta(t)$ σ-Lipschitz, and

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} \sum_{k,k' \in N(k)} 1_{|r_k\theta_k(n) - r_{k'}\theta_{k'}(n)| \geq \Delta} \leq \phi(K)\Delta$$

OAS with Sliding Window

- SW-OAS (applies OAS over a sliding window of size τ)
- Graphical unimodality holds at any time
- Parameters:

$$\tau = \sigma^{-3/4}\log(1/\sigma)/8, \quad \Delta = \sigma^{1/4}\log(1/\sigma)$$

Theorem: Under π = SW-OAS

$$\limsup_T \frac{R^\pi(T)}{T} \leq C\phi(K)\sigma^{\frac{1}{4}}\log(1/\sigma)(1+Ko(1)), \quad \sigma \to 0^+$$

OAS with Sliding Window

- Analysis made complicated by the smoothness of the rewards vs. time (previous analysis by **Garivier-Moulines** assumes separation of rewards at any time)
- Sub-logarithmic terms are essential in the regret analysis
- Upper bound on regret per time unit:
  - Tends to zero when the evolution of average rewards gets smoother

$$\sigma^{1/4}\log(1/\sigma) \to 0, \quad \text{as } \sigma \to 0^{+}$$

  - Does not depend on the size of the decision space if $\phi(K) \leq C$

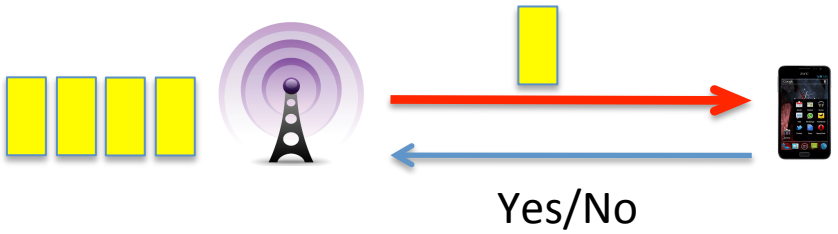
3. Rate adaptation in 802.11

with R. Combes, D. Yun, J. Ok, Y. Yi

Rate adaptation in 802.11

Adapting the modulation/coding scheme to the radio environment

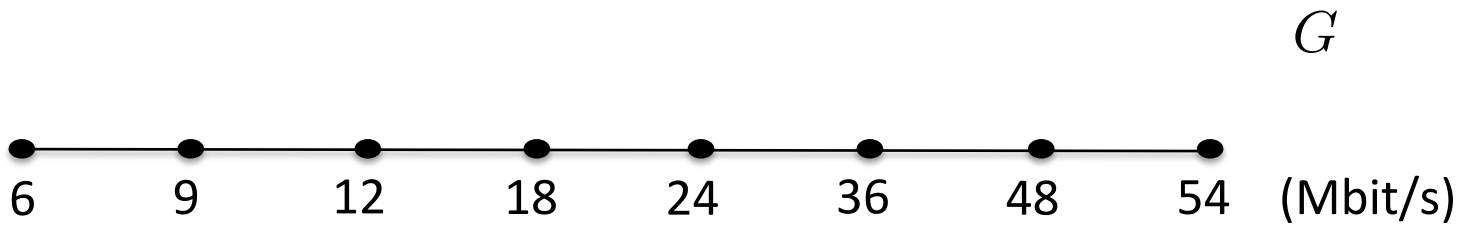
- 802.11 a/b/g



rates	r_1	r_2	$\ldots$	r_N
Success probabilities	θ_1	θ_2	$\ldots$	θ_N
Throughputs	μ_1	μ_2	$\ldots$	μ_N

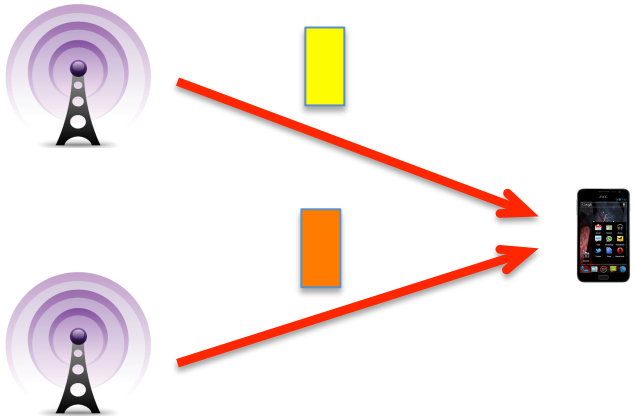
$\mu_i = r_i \theta_i$

- Structure: unimodality + $\theta_1 > \theta_2 > \ldots > \theta_N$

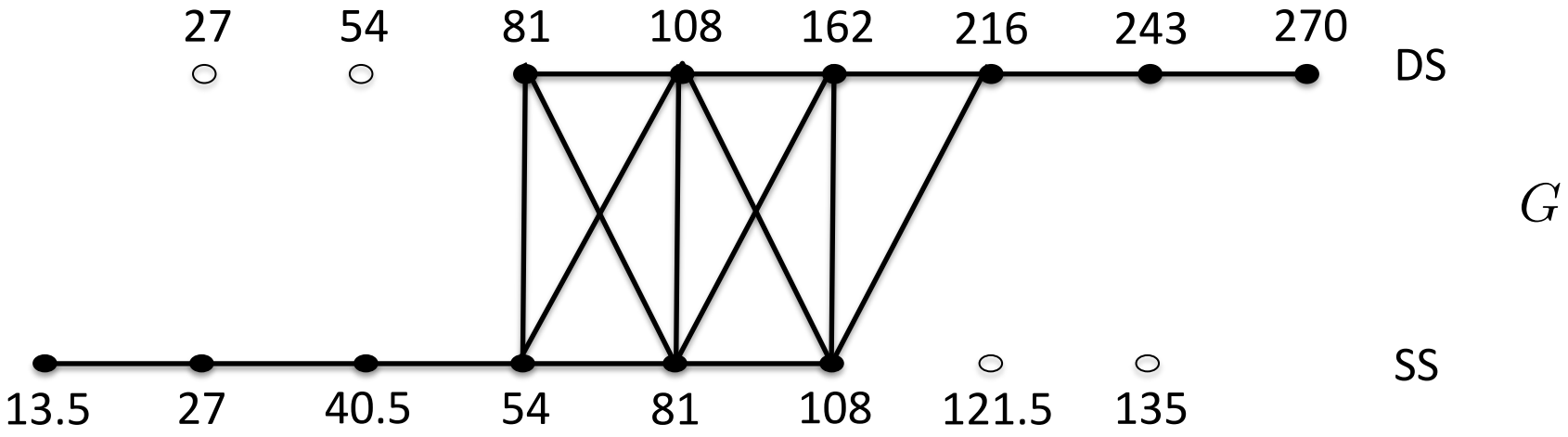


Rate adaptation in 802.11

- 802.11 n/ac MIMO
  Rate + MIMO mode
  (32 combinations in n)



- Example: two modes, single-stream (SS) or double-stream (DS)



State-of-the-art

- ARF (Auto Rate Fallback): after *n* successive successes, probe a higher rate; after two consecutive failures reduce the rate
- AARF: vary *n* dynamically depending on the speed at which the radio environment evolves
- SampleRate: based on achieved throughputs over a sliding window, explore a new rate every 10 packets
- Measurement based approaches: Map SNR to packet error rate (does not work – OFDM): RBAR, OAR, CHARM, …
- 802.11n MIMO: MiRA, RAMAS, …

All existing algorithms are heuristics.

Rate adaptation design: a graphically unimodal bandit with large strategy set

Optimal Rate Sampling

Algorithm – Optimal Rate Sampling (ORS)
For $n = 1, \ldots, K$, select action $k(n) = n$
For $n \geq K + 1$, select action $k(n)$:

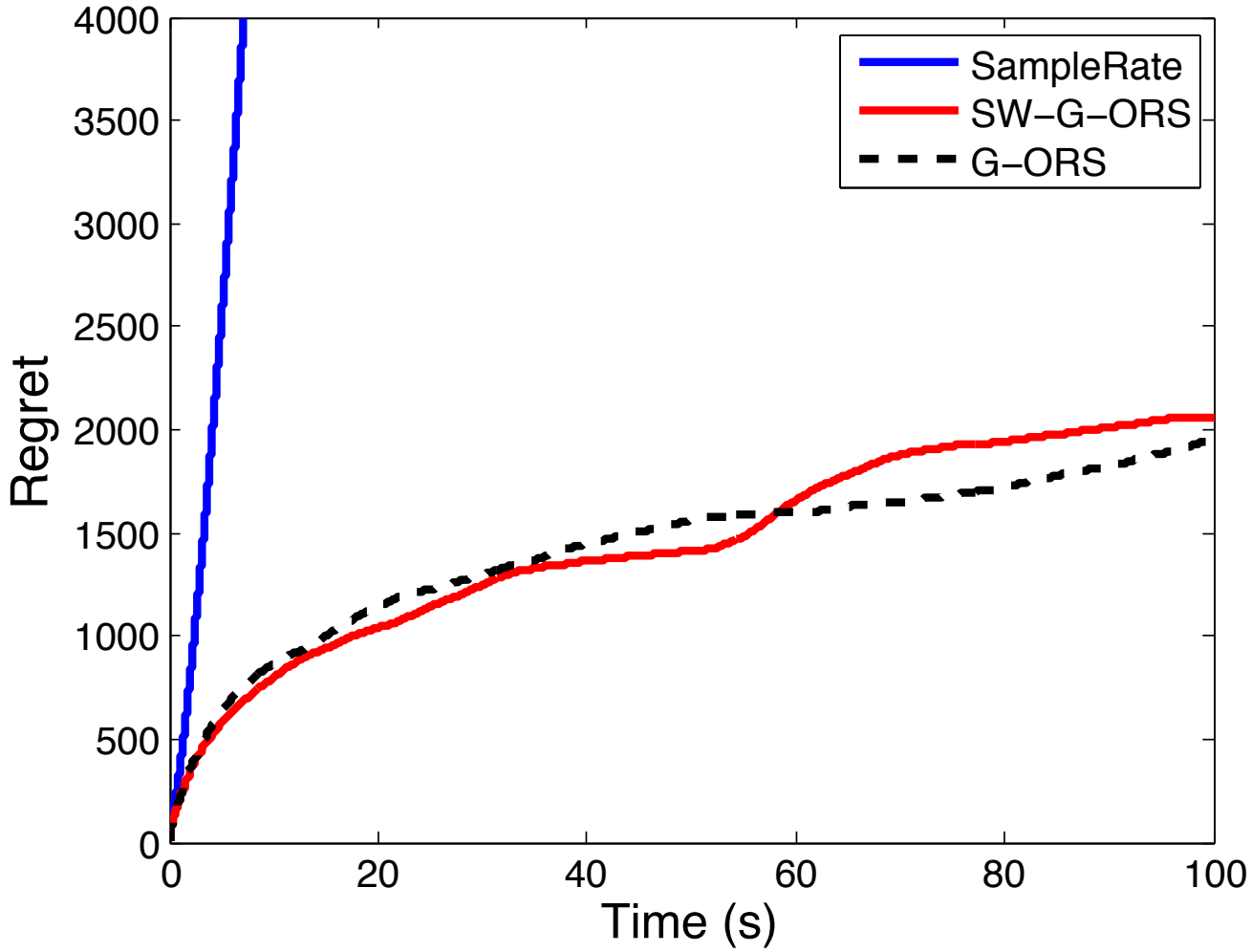
$$k(n) = \begin{cases} L(n) & \text{if } (l_{L(n)}(n) - 1)/(\gamma + 1) \in \mathbb{N}, \\ \arg \max_{k \in N(L(n))} b_k(n) & \text{otherwise.} \end{cases}$$

ORS is asymptotically optimal (minimizes regret)
Its performance does not depend on the number of possible rates!

For non-stationary environments: SW-ORS (ORS with sliding window)

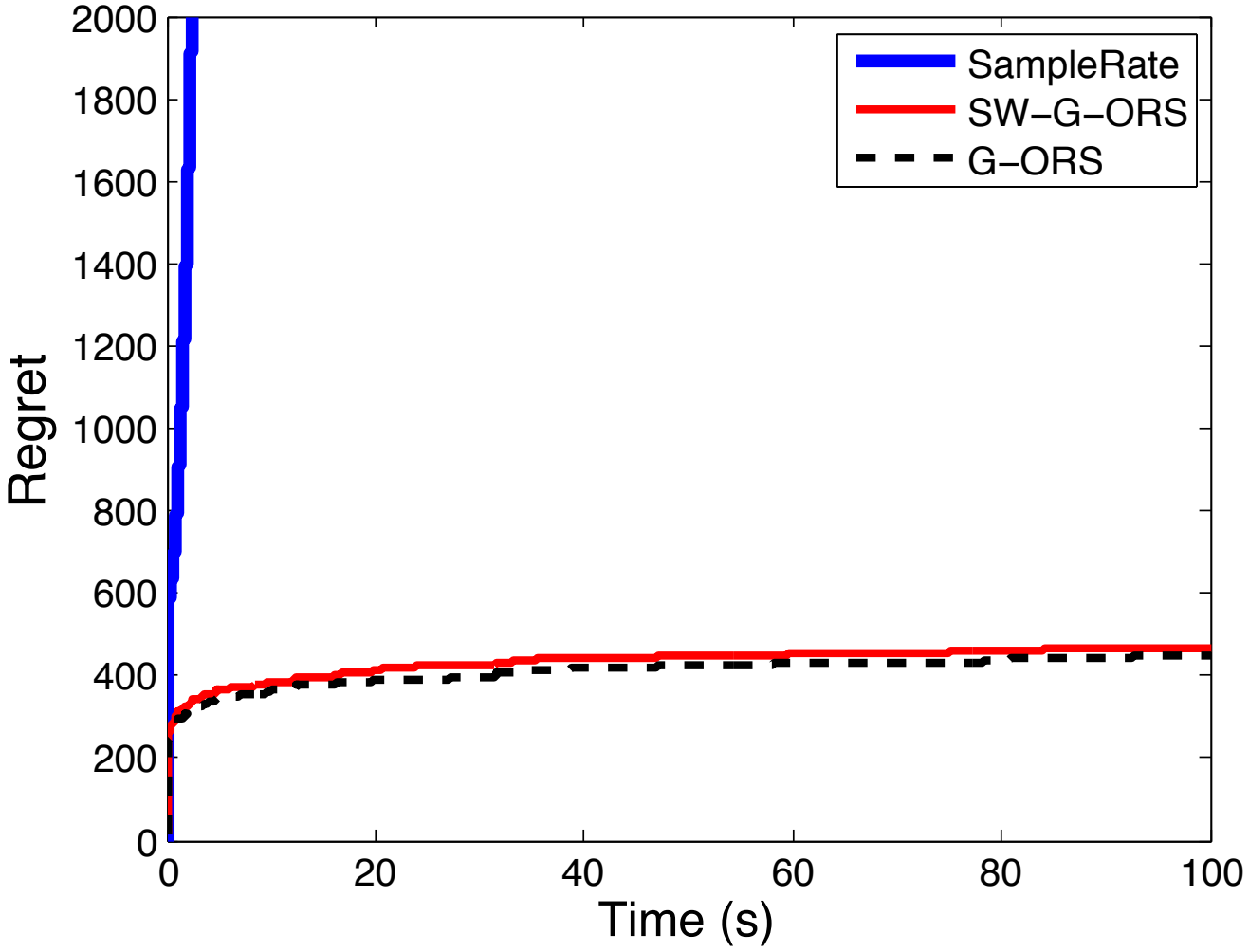
802.11g – stationary environment

GRADUAL (success prob. smoothly decreases with rate)



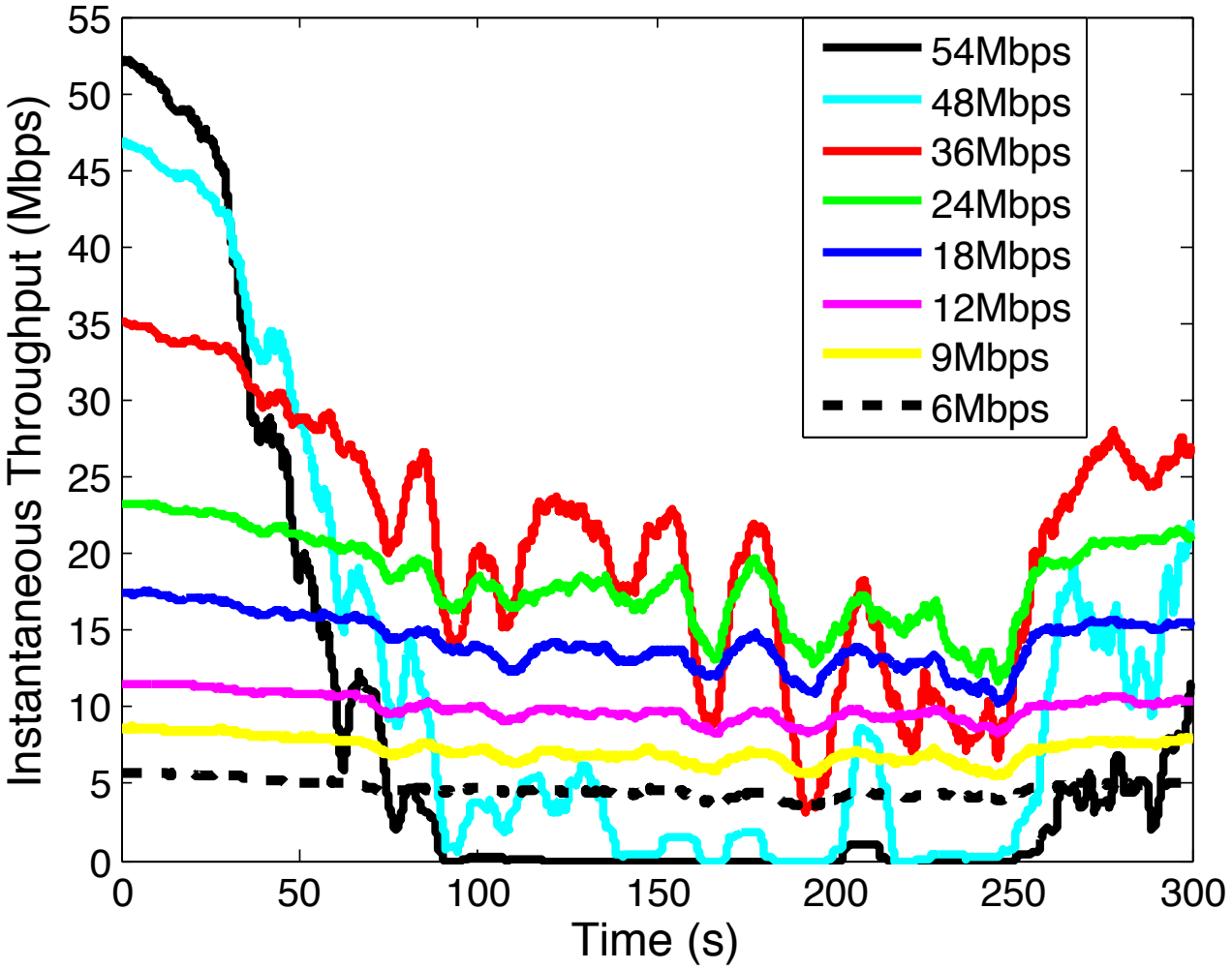
802.11g – stationary environment

STEEP (success prob. is either close to 1 or to 0)



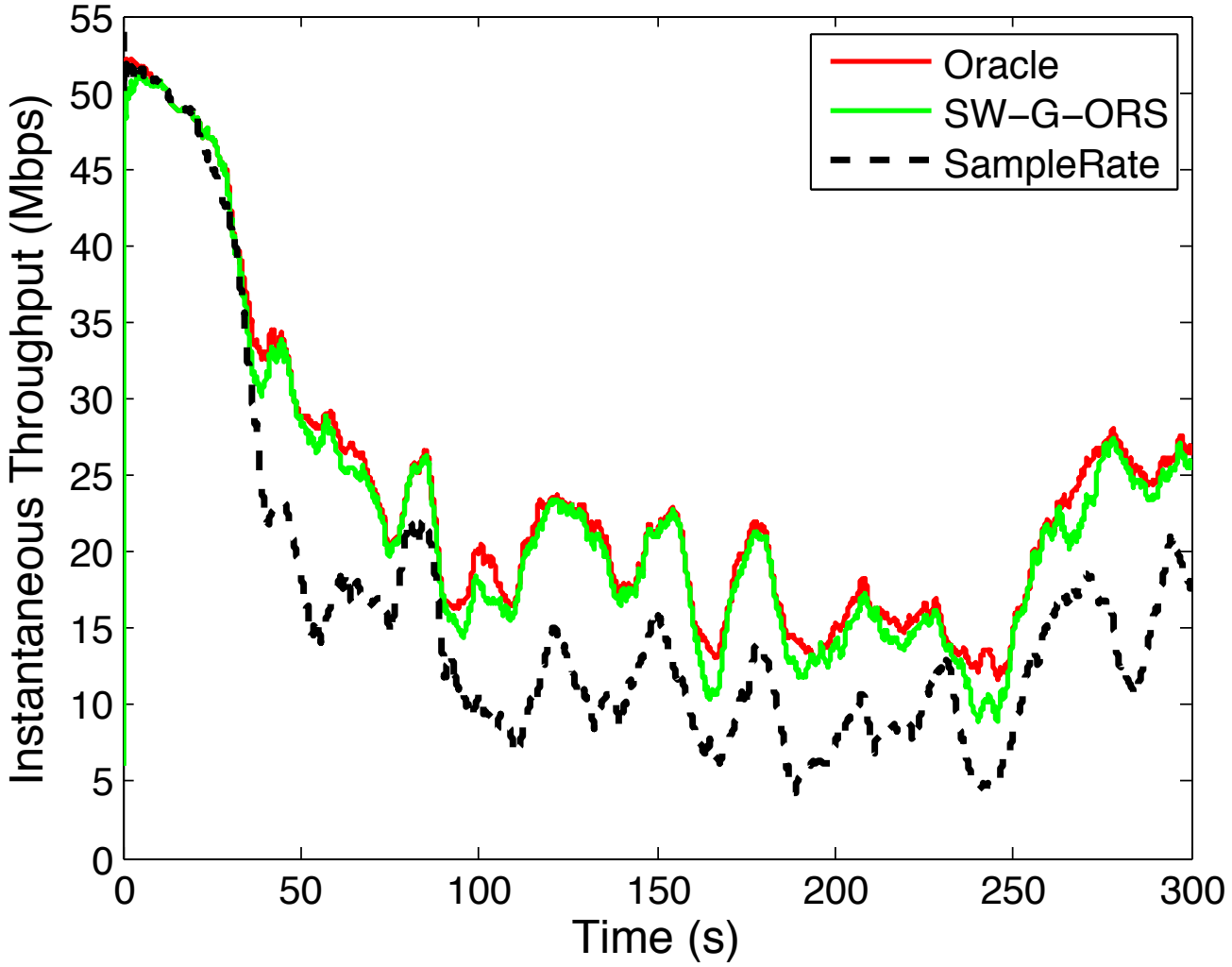
802.11g – non-stationary environment

TRACES



802.11g – non-stationary environment

RESULTS



Summary

- Regret minimization: the right objective for tracking optimal operating points in changing environments

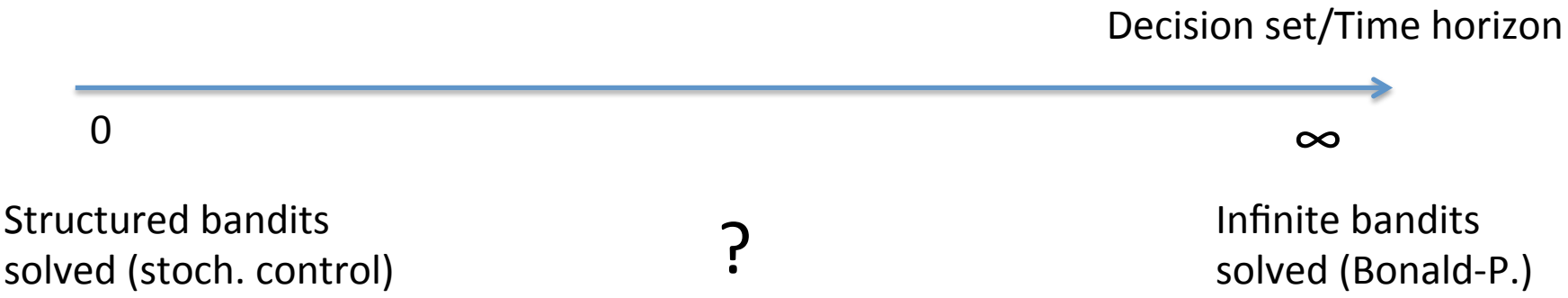
- A solution to graphically unimodal bandit problems, an important class of structured bandits
  - Lower bound on regret
  - Asymptotically optimal and simple algorithms
  - Non-stationary analysis

- Application to rate adaptation in 802.11 systems
  - An optimal algorithm: ORS
  - The regret under ORS does not depend on the number of available decisions (crucial!)
  - Numerical validation using simulations and test-beds

Current work

- Other applications of structured bandits
  - Wireless systems: large MIMO, scheduling, spectrum sharing
  - Online clustering
  - Recommendation systems
  - Pricing
  - …

Future work



- What can we do when the size of the decision space and the time horizon are comparable?

rcombes@kth.se
http://sites.google.com/site/
richardcombesresearch/

alepro@kth.se
http://people.kth.se/~alepro