On Moment Problems in Robust Control, Spectral Estimation, Image Processing and System Identification

Anders Lindquist
Shanghai Jiao Tong University, China, and the
Royal Institute of Technology, Stockholm, Sweden

University of Maryland
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Special recognition

Christopher Byrnes

Tryphon Georgiou

+ Sergei Gusev, Alexei Matveev, Alexandre Megretski, Giorgio Picci, Per Enqvist, Johan Karlsson, Ryozo Nagamune, Anders Blomqvist, Vanna Fanizza, Enrico Avventi, Axel Ringh

What is the talk about

- A classical problem – the moment problem – with a decidedly non-classical twist motivated by engineering applications.
- What is new are certain rationality constraints imposed by applications that alter the mathematical problem and make it nonlinear.
- A global-analysis approach that studies the class of solutions as a whole.
- A powerful paradigm for smoothly parameterizing, comparing, and shaping solutions to specifications.

The moment problem

Given $c_0, c_1, \ldots, c_n$, find $d\mu$ such that

$$\int_a^b \alpha_k(t) d\mu = c_k, \quad k = 0, 1, \ldots, n$$

$d\mu \in \mathcal{M}_+$ space of positive measures

- Power moment problem: $\alpha_k(t) = t^k$
- Trigonometric moment problem: $\alpha_k(t) = e^{ikt}, \quad [a, b] = [-\pi, \pi]$
- Nevanlinna-Pick interpolation: $\alpha_k(t) = \frac{e^{it}+z_k}{e^{it}-z_k}, \quad [a, b] = [-\pi, \pi]$

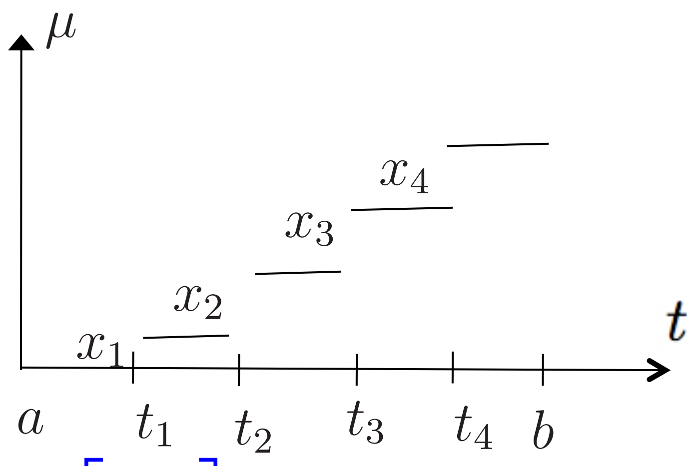
Chebyshev

Markov

Lyapunov

$$\int_a^b \alpha_k(t)\,d\mu = c_k, \quad k = 0, 1, \ldots, n$$

Ex. 1 μ step function



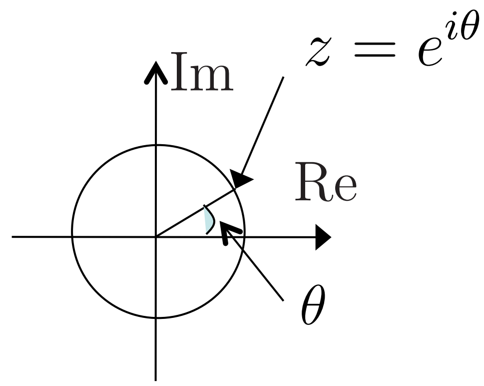
$$\sum_{j=1}^{N} \alpha_k(t_j) x_j = c_k, \quad k = 0, 1, \ldots, n$$

$$\begin{bmatrix} \alpha_0(t_1) & \alpha_0(t_2) & \cdots & \alpha_0(t_N) \\ \alpha_1(t_1) & \alpha_1(t_2) & \cdots & \alpha_1(t_N) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n(t_1) & \alpha_n(t_2) & \cdots & \alpha_n(t_N) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$N > n$

infinitely many solutions

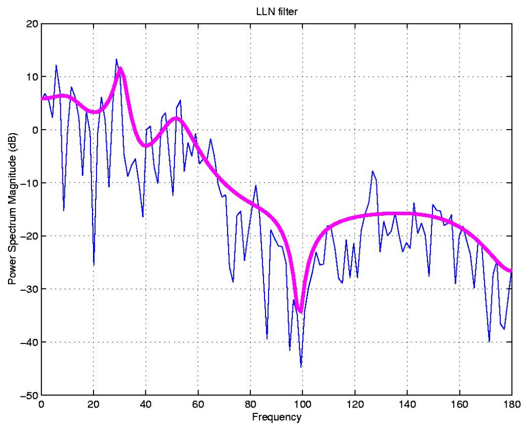
Ex. 2 $d\mu = \Phi(e^{i\theta}) \dfrac{d\theta}{2\pi}$ spectral density

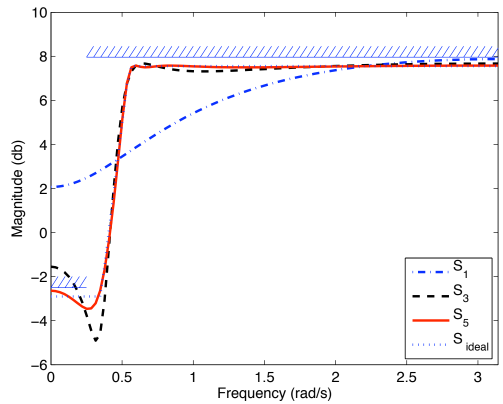


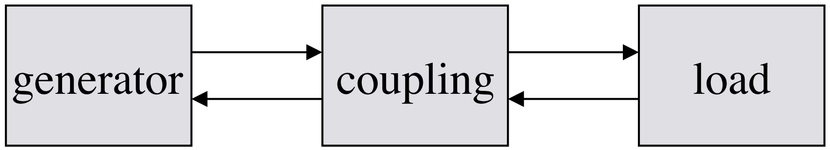
$$\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} = c_k, \quad k = 0, 1, \ldots, n$$

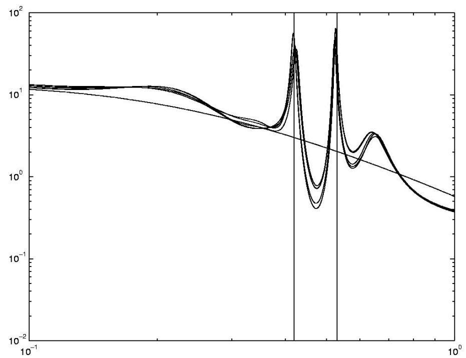
Where do we find moment problems in applications?

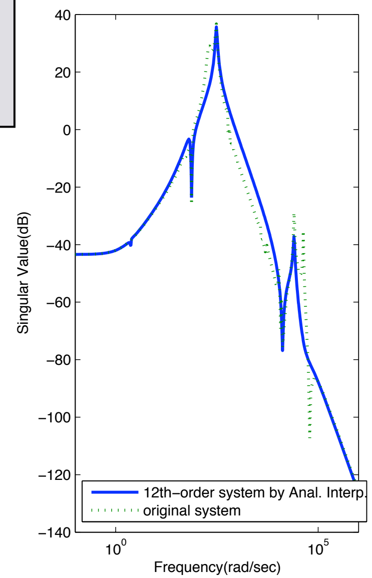
- spectral estimation
- speech synthesis
- system identification
- image processing
- optimal control
- robust control
- model reduction
- model matching problems
- simultaneous stabilization
- optimal power transfer





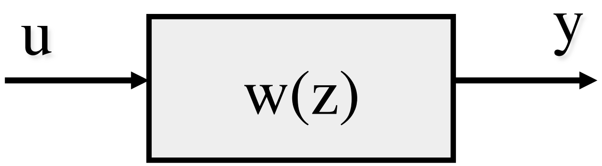






… and why do these problems require a nonclassical approach?

- Solution must be of bounded complexity (such as rational of a bounded degree) so that one can realize it by a finite-dimensional device



$$y(t) = \sum_{k=-\infty}^{t} w_{t-k} u(k)$$

System is finite-dimensional iff $w(z) := \sum_{k=0}^{\infty} w_k z^{-k}$ is rational

- Classical theory does not provide natural parameterizations of rational solutions of bounded degree

Prototype problem: Covariance extension

$c_k = \mathrm{E}\{y(t+k)y(t)\}$, $k = 0, 1, 2, \ldots,$
where y stationary stochastic process

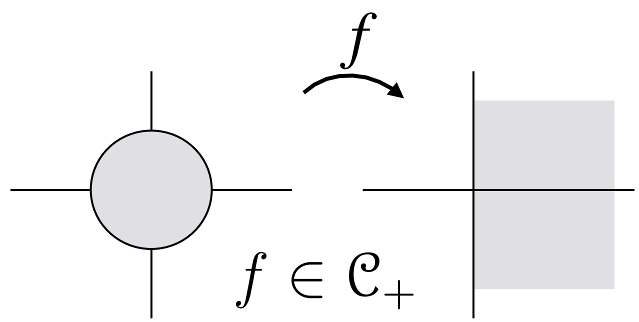
Given $c_0, c_1, \ldots, c_n$, find an infinite extension $c_{n+1}, c_{n+2}, \ldots$ such that

Carathéodory
Schur

$$f(z) = \tfrac{1}{2}c_0 + c_1 z + \cdots + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

(i) is a Carathéodory function

(ii) is rational of degree at most n



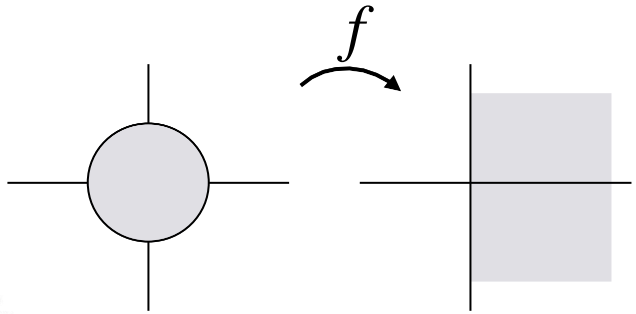
Kalman

Trigonometric moment problem

$$f(z) = \frac{1}{2}c_0 + c_1 z + \cdots + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

$$\begin{aligned} \Phi(e^{i\theta}) &= 2\mathrm{Re}\{f(e^{i\theta})\} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \geq 0 \end{aligned}$$

$$c_{-k} = \bar{c}_k$$

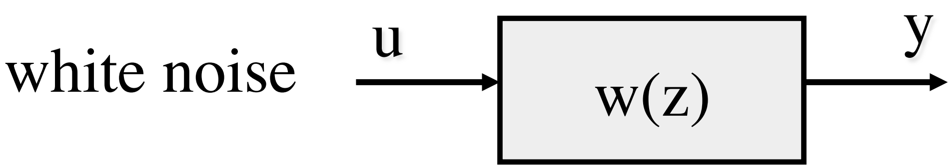


MOMENT PROBLEM: Find Φ of degree at most 2n such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta = c_k, \quad k = 0, 1, \ldots, n$$

$$\Phi(z) = \frac{P(z)}{Q(z)}, \quad P, Q \text{ trigonometric polynomials of degree } n$$

Spectral estimation by covariance extension



stationary process
with spectral density
$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta = E\{y(t+k)y(t)\}$$

$$= \lim_{N\to\infty} \frac{1}{N+1}\sum_{t=0}^{N-k} y_{t+k}y_t$$

where $y_0, y_1, y_2, \dots, y_N$

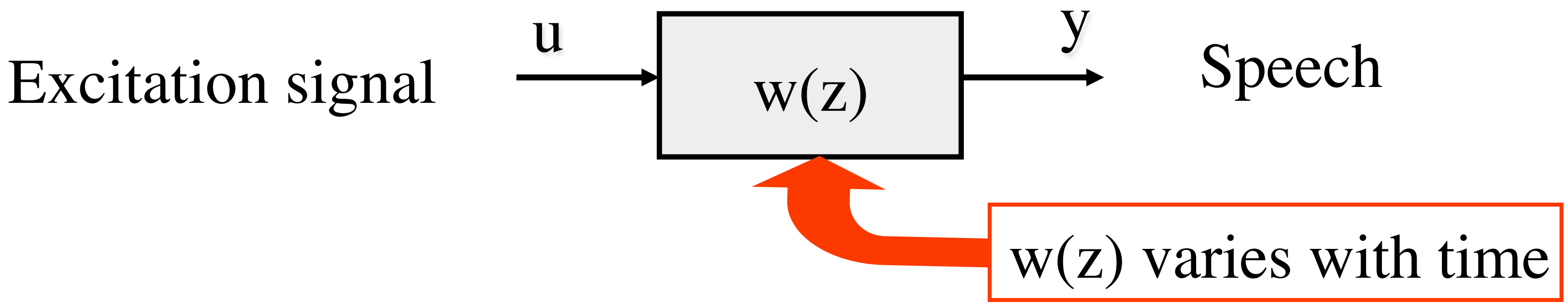
observed data

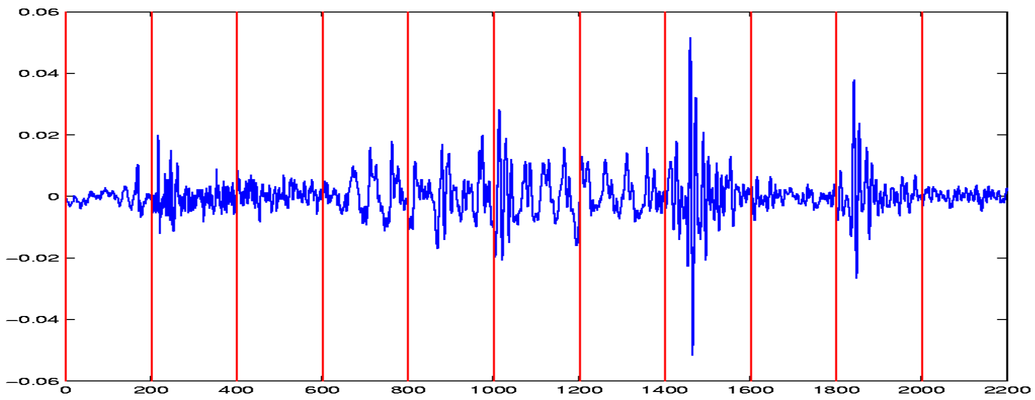
Since $N < \infty$, we use
ergodic estimate

$$c_k = \frac{1}{N+1}\sum_{t=0}^{N-k} y_{t+k}y_t$$

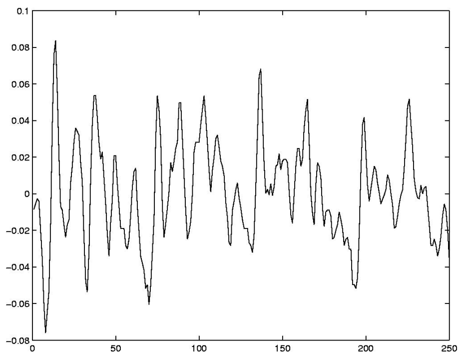
Hence, we can only estimate
$c_0, c_1, \dots, c_n, \quad n << N$
Remains to determine
$c_{n+1}, c_{n+2}, c_{n+3}, \dots$

Modeling speech

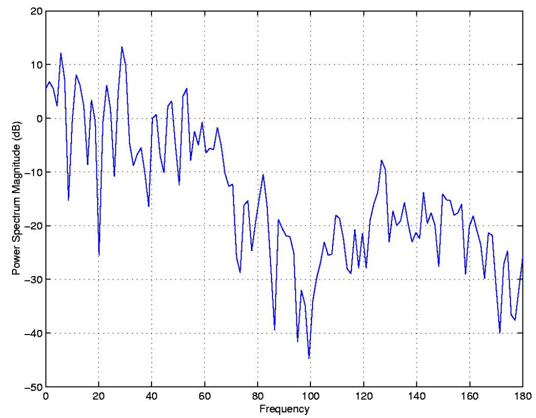




w(z) constant on each (30 ms) subinterval

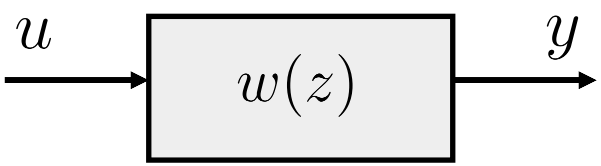


A 30 ms frame of speech for
the voiced nasal phoneme [ng]

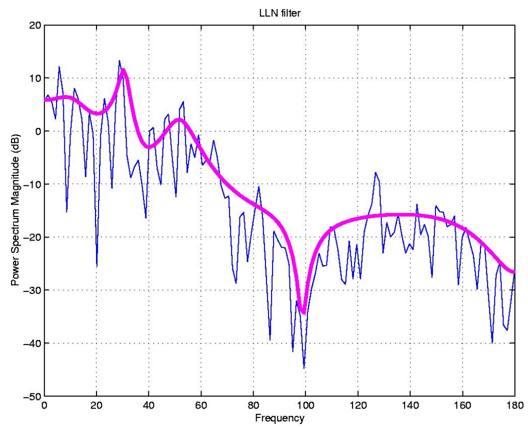


Periodogram (FFT) of
voiced nasal phoneme [ng]

Construct a rational filter with a rational $w(z)$
of low degree, modeling the window of speech



filter determined by
few parameters



Linear Predictive (LPC) Filtering

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} \varphi_{nn} \\ \varphi_{n,n-1} \\ \vdots \\ \varphi_{n1} \end{bmatrix} = \begin{bmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_1 \end{bmatrix}$$

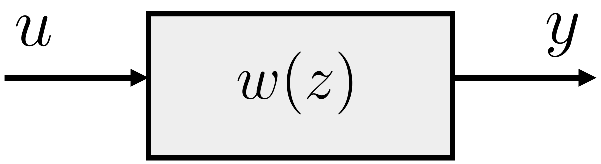
yields Szegö polynomial

$$\varphi_n(z) = z^n + \varphi_{n1} z^{n-1} + \cdots + \varphi_{nn}$$

and modeling filter

$$\boxed{w(z) = \frac{\sqrt{\rho_n} z^n}{\varphi_n(z)}} \qquad \text{where} \qquad \rho_n = \sum_{j=0}^{n} c_j \varphi_{nj}$$

Linear Predictive (LPC) Filtering



$$w(z) = \sqrt{\rho_n}\,\frac{z^n}{\varphi_n(z)} \qquad n = 10$$

Choose an excitation signal u from a code book with $1024 = 2^{10}$ entries (10 bits)

Send coefficients in $\varphi_{10}(z)$ and number of “best” signal

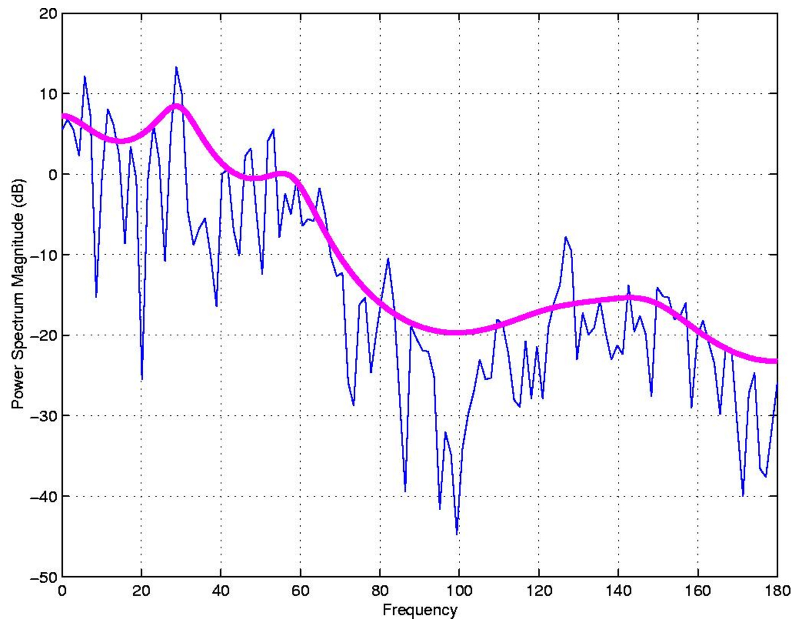
However, can we construct the general solution?

$$w(z) = \frac{\sigma(z)}{a(z)}$$

$$w(z) = \frac{\sigma(z)}{a(z)}$$

Cellular telephone:

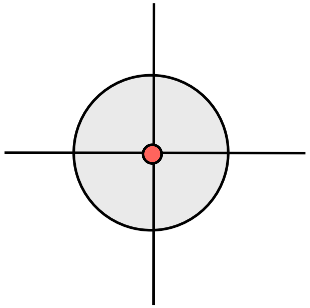
$$w(z) = \sqrt{\rho_n} \frac{z^n}{\varphi_n(z)}$$



FFT in blue   envelope in purple

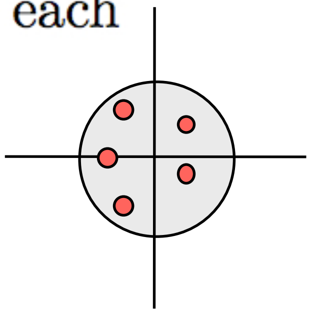
$$\sigma(z) = \sqrt{\rho_n} z^n$$

spectral zeros



Is there a solution $d\mu$ for each choice of spectral zeros?

YES (Georgiou 1983)



Unique? (Georgiou's conjecture)

Well-posed?

YES (Byrnes, Lindquist Gusev, Matveev 1993)

All the solutions

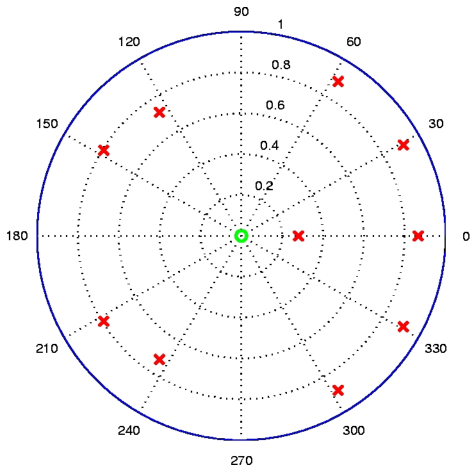
THEOREM. The solutions of the rational covariance extension problem are completely parameterized by the zeros of the corresponding shaping filter, i.e., given an arbitrary monic stable polynomial $\sigma(z)$ there is one and only one stable polynomial $a(z)$ such that

$$w(z) = \frac{\sigma(z)}{a(z)}$$

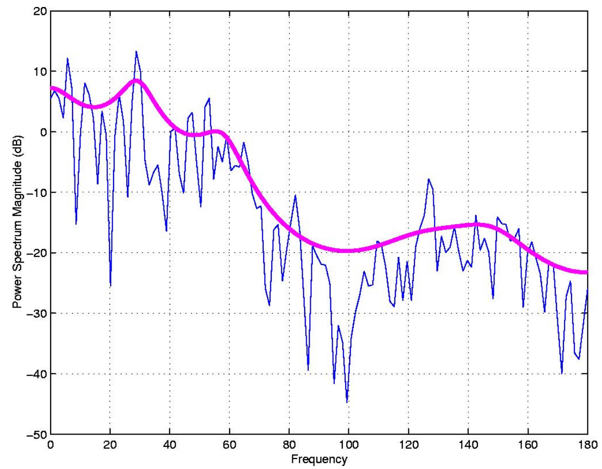
is a shaping filter for $c_0, c_1, \ldots, c_n$. The correspondence is a diffeomorphism.

- Existence proved by Georgiou 1983; conjectured uniqueness
- The rest proved by Byrnes, Lindquist, Gusev, Matveev 1993
- These first proofs were nonconstructive, but there are now constructive proofs based on optimization

$$w(z) = \frac{\sqrt{\rho_n} z^n}{\varphi_n(e^{it})}$$

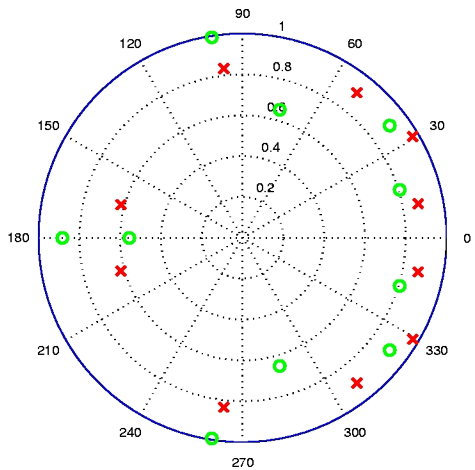


zeros/poles

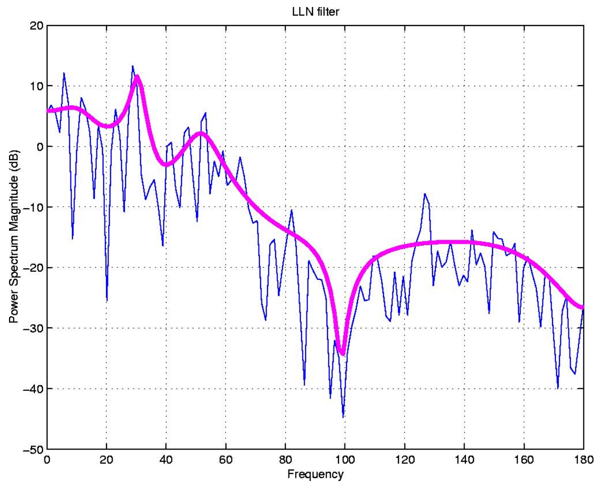


envelope

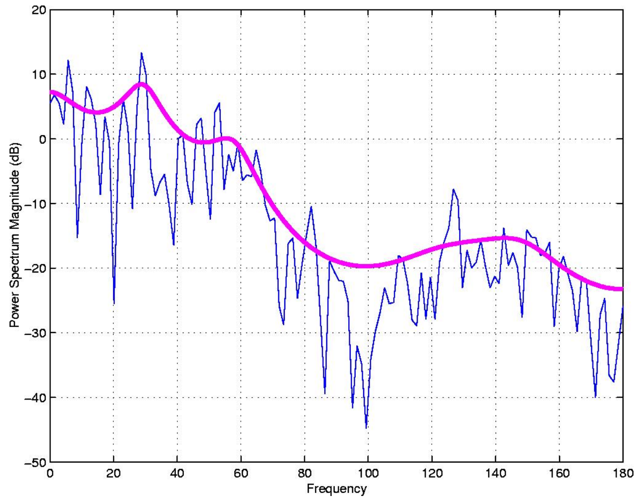
A $w(z)$ with other spectral zeros, but with the same degree

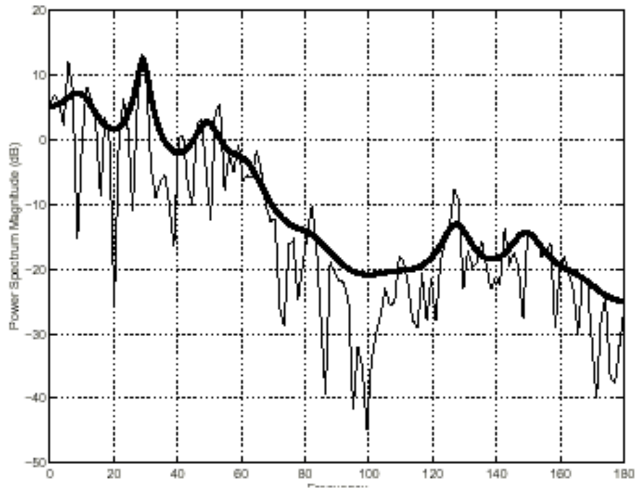


zeros/poles



envelope

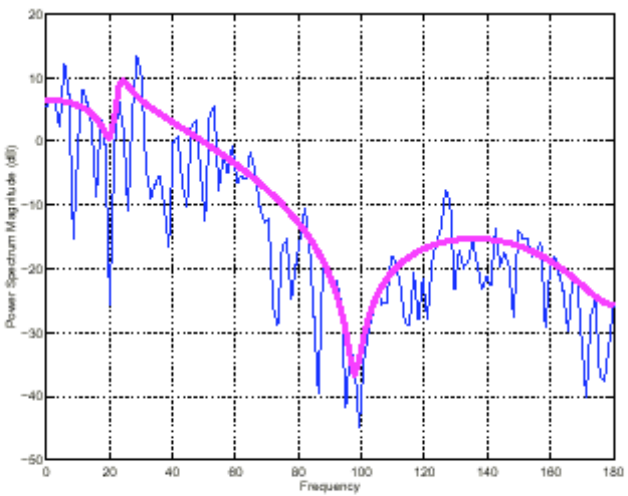


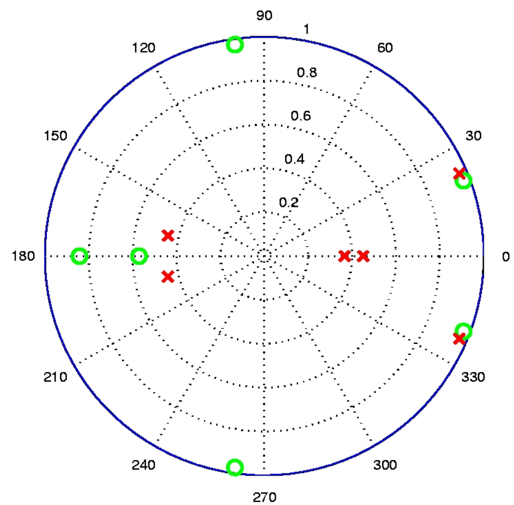


10th degree LPC

20th degree LPC

6th degree filter with appropriate zeros





Optimization approach

Given $\sigma(z)$ and c, minimize

$$\mathbb{J}_\sigma(q) = c_0 q_0 + c_1 q_1 + \cdots + c_n q_n$$
$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \log Q(e^{i\theta}) d\theta$$

over all q such that

$$Q(e^{i\theta}) := q_0 + q_1 \cos\theta + \cdots + q_n \cos n\theta > 0$$

THEOREM. There is a unique minimum.

Then

$$f(z) = \frac{b(z)}{a(z)}$$

$$w(z) = \frac{\sigma(z)}{a(z)}$$

where $|a(e^{i\theta})|^2 = Q(e^{i\theta})$

$$a(z)b(z^{-1}) + a(z^{-1})b(z) = \sigma(z)\sigma(z^{-1})$$

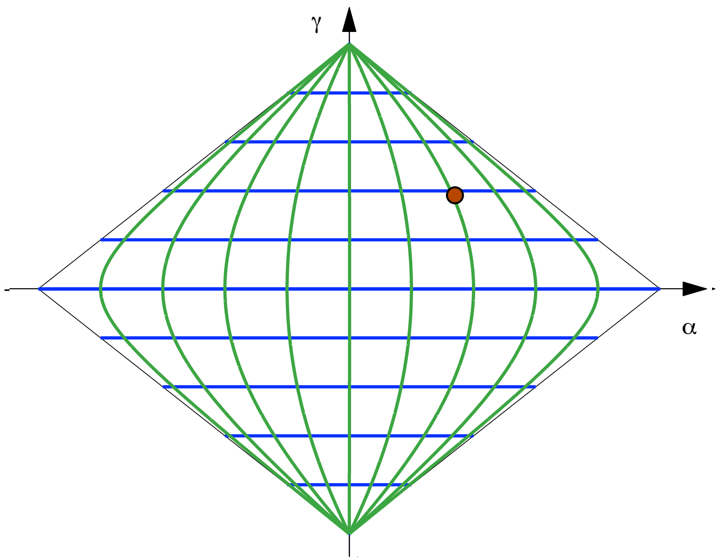
Nonlinear coordinates

Normalize
so that $c_0 = 1$

The space of all rational Carathéodory functions f of degree at most n is a $2n$-dimensional manifold.

A foliation with one leaf for each choice of σ (fast Kalman filtering)

A foliation with one leaf for each choice of $c = (c_1, c_2, \ldots, c_n)$



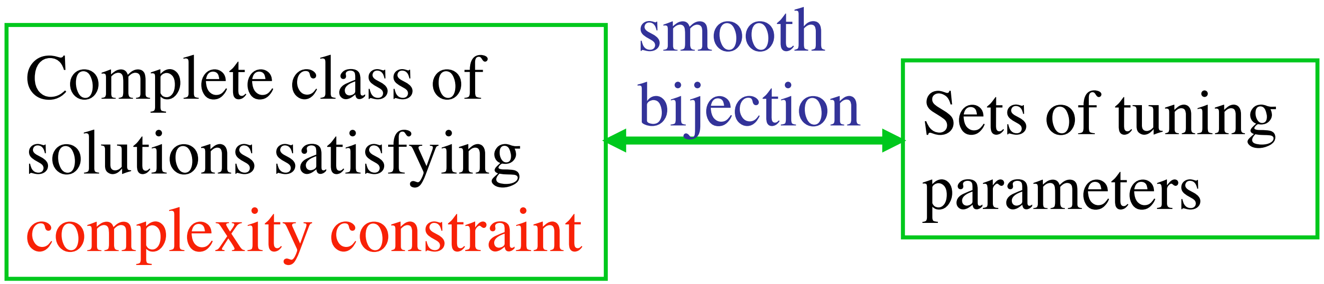
THEOREM. The two foliations intersect transversely so that each leaf in one meets each leaf in the other in exactly one point.

$$\min_q \mathbb{J}_\sigma(q)$$

unique solution $\Phi = \frac{|\sigma|^2}{Q}$

A global analysis approach

- Find complete parameterization



- For any choice of tuning parameters, determine the corresponding solution by convex optimization

- Choose a solution that best satisfies additional design specifications (without increasing the complexity)

Other applications leading to moment problem with rationality constraints

Given $c_0, c_1, \ldots, c_n$,
find $d\mu$ such that

$$\int_a^b \alpha_k(t) d\mu = c_k, \quad k = 0, 1, \ldots, n$$

Nevanlinna-Pick interpolation

$$\alpha_k(t) = \frac{e^{it} + z_k}{e^{it} - z_k}, \quad k = 0, 1, \ldots, n$$

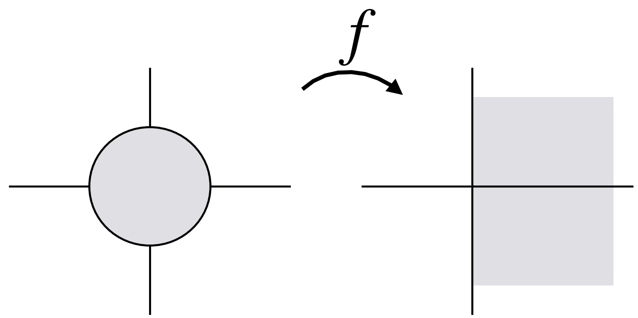
$z_0, z_1, \ldots, z_n \in \mathbb{D}$
(distinct)

Given $z_0, z_1, \ldots, z_n \in \mathbb{D}$ (distinct), find a Carathéodory function f such that

$$f(z_k) = c_k, \quad k = 0, 1, \ldots, n$$

⇕

$$\int_{-\pi}^{\pi} \alpha_k(t) \mathrm{Re}\{f(e^{it})\} dt = c_k, \quad k = 0, 1, \ldots, n$$

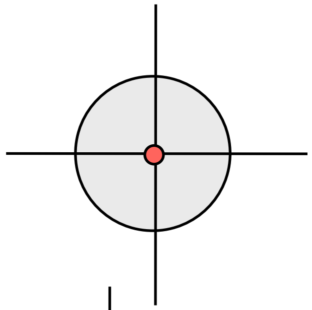


analytic in $\mathbb{D}$

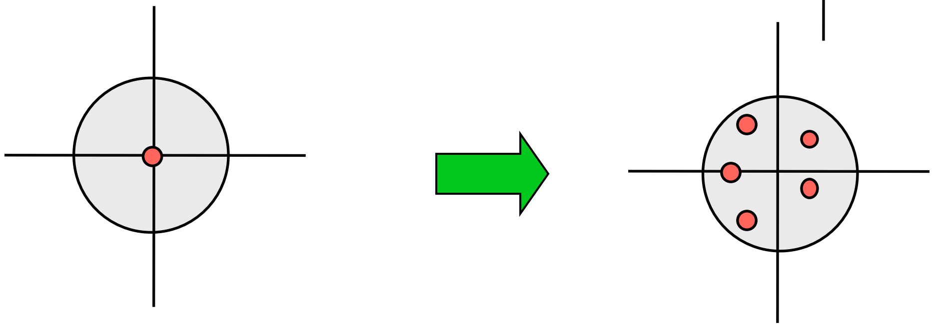
$\mathrm{Re}\{f(z)\} \geq 0$ in $\mathbb{D}$

Tuning by moving interpolation points

Covariance extension corresponds to N-P interpolation with all interpolation points at the origin ($z = 0$)

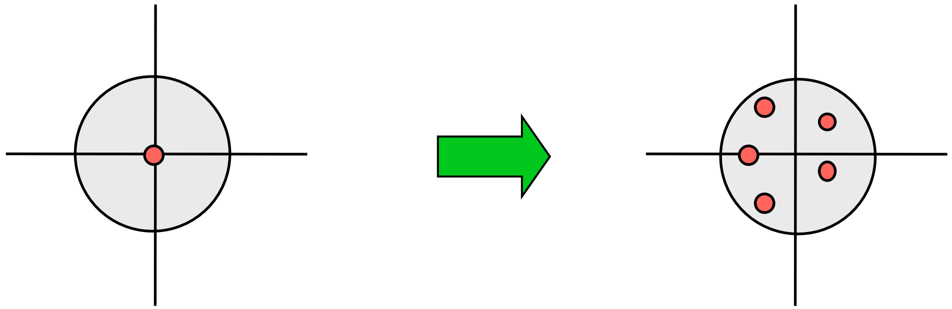


We moved the spectral zeros from the origin closer to the unit circle



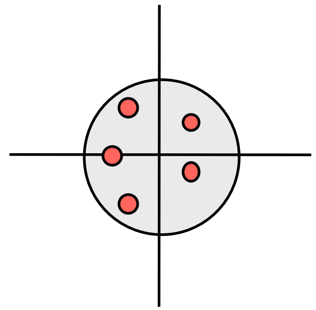
maximum entropy (LPC)

Next we tune by moving the interpolation points closer to the unit circle

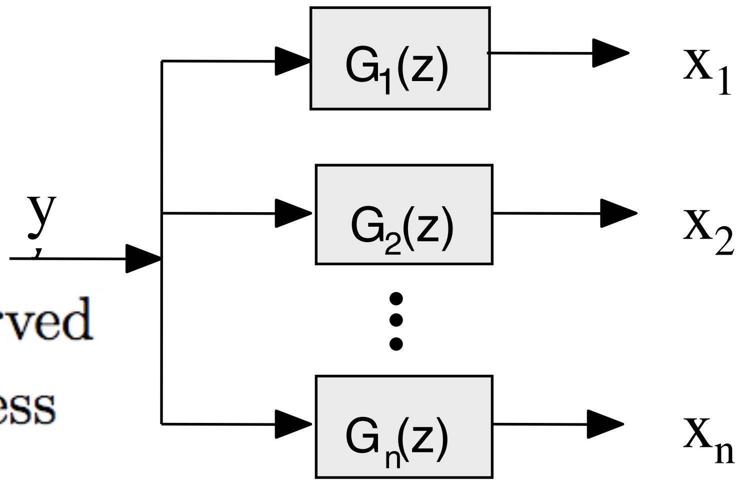


A tunable high resolution spectral estimator (THREE)

Zoom into a selected spectral band by moving interpolation points from the origin closer to the unit circle.



Byrnes-Georgiou-L



observed process

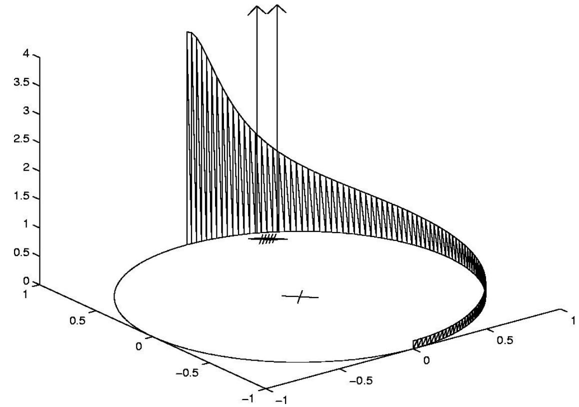
$$G_k(z) = \frac{1}{1 - \bar{z}_k z}$$

$$\boxed{f(z_k) = w_k := \tfrac{1}{2}(1 - z_k^2)E\{x_k^2\}}$$

$$E\{x(t)x(t)^{\mathsf{T}}\} = \left[\frac{w_k + \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n}$$

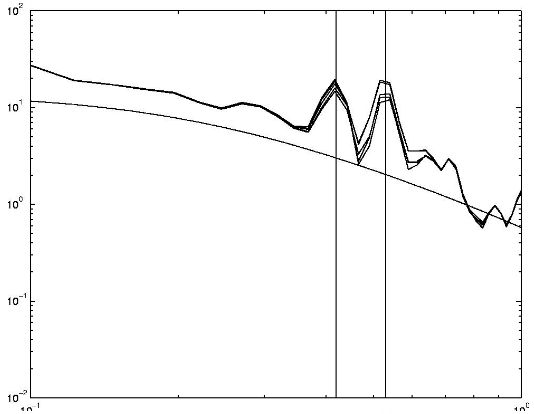
Two sets of tuning parameters:
- filter bank poles
- spectral zeros (P)

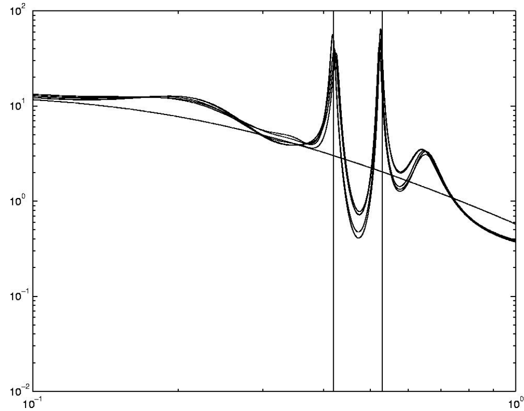
Estimation of spectral lines in colored noise



separation between spectral lines = 0.11

five runs superimposed

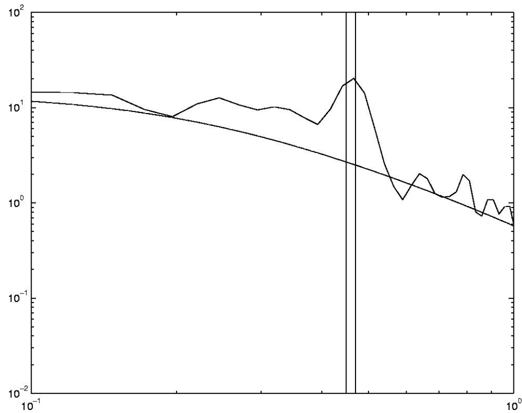


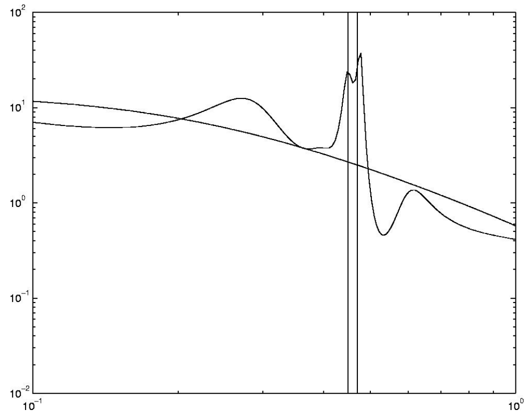


Periodogram (FFT)

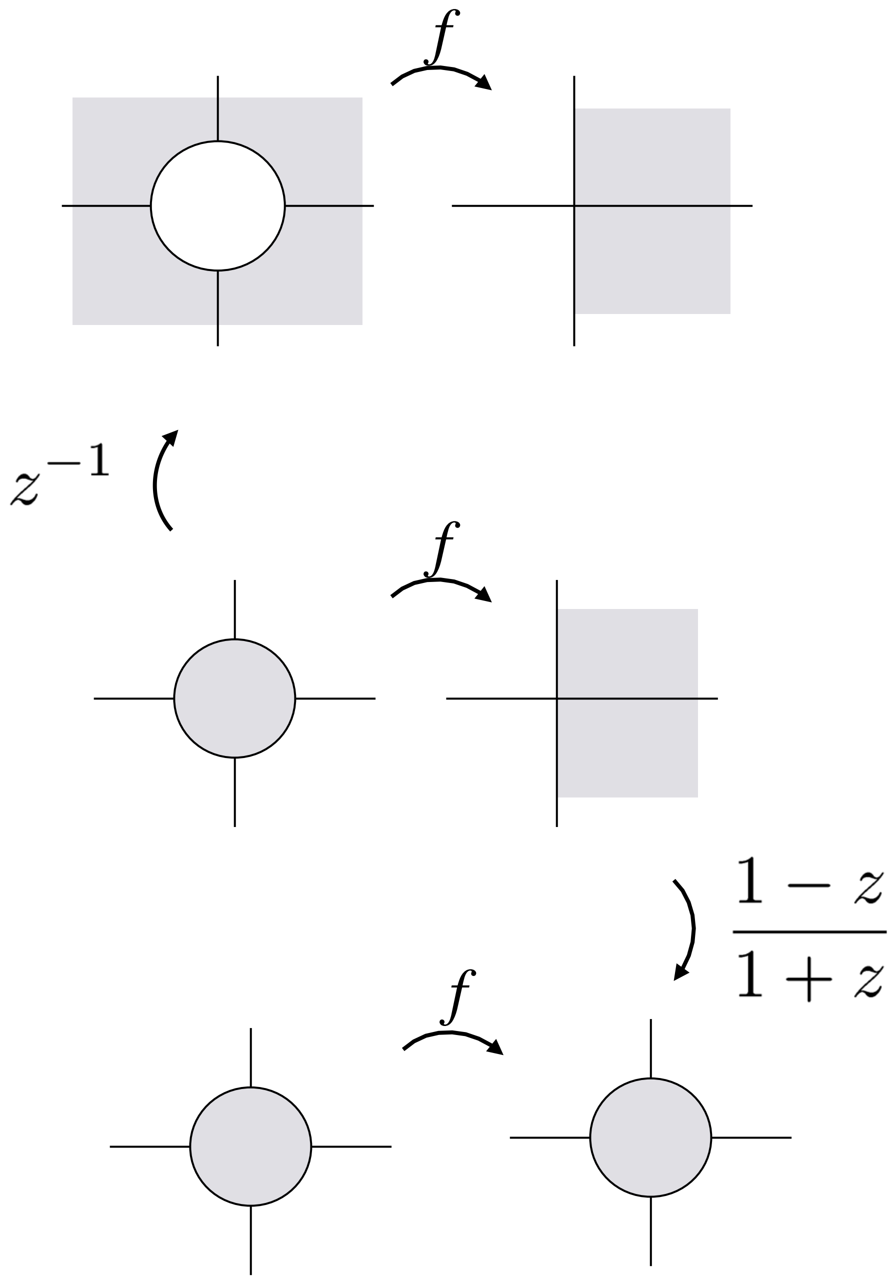
THREE (default setting)

separation between spectral lines = 0.02



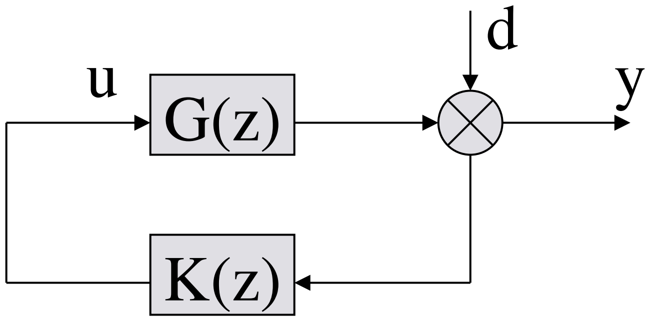


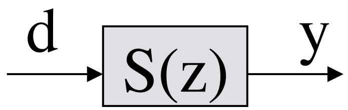
Robust control



- positive real function
  - f analytic for $|z| \geq 1$
  - $\mathrm{Re}\{f(z)\} > 0$ for $|z| \geq 1$
- Carathéodory function
  - f analytic for $|z| \leq 1$
  - $\mathrm{Re}\{f(z)\} > 0$ for $|z| \leq 1$
- Schur function
  - f analytic for $|z| \leq 1$
  - $|f(z)| < 1$ for $|z| \leq 1$

Loop shaping in robust control





$S = (1 - GK)^{-1}$

Sensitivity function

- Internal stability requires

  S analytic in $\mathbb{D}^c := \{z \mid |z| > 1\}$

  $S(z_k) = 0$ at all unstable poles of G

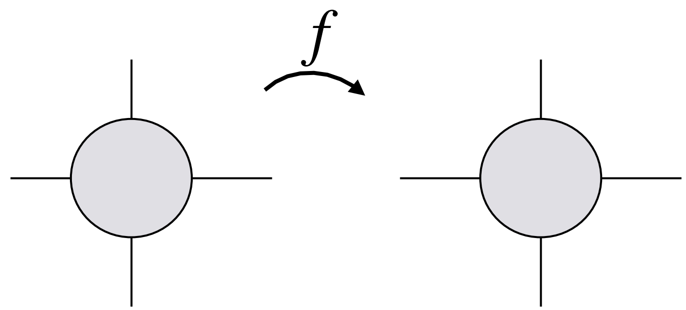
  $S(z_j) = 1$ at all zeros of G in $\mathbb{D}^c$

- Disturbance attenuation requires

  $\|S\|_\infty \leq \gamma$

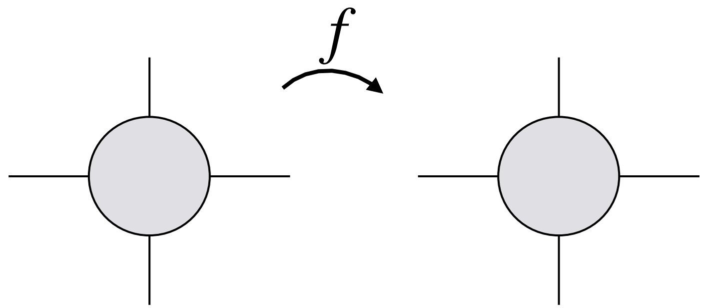
- We want $\deg S$ to be small

There is a minimum bound γ_{opt} but we choose $\gamma > \gamma_{\text{opt}}$ and define $f(z) := \frac{1}{\gamma} S(z^{-1})$



Nevanlinna-Pick interpolation
for Schur functions

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n$$



class of Schur functions $\mathcal{S}$

The interpolants of degree at most n are parmeterized by the spectral zeros (σ) in a $1-1$ fashion

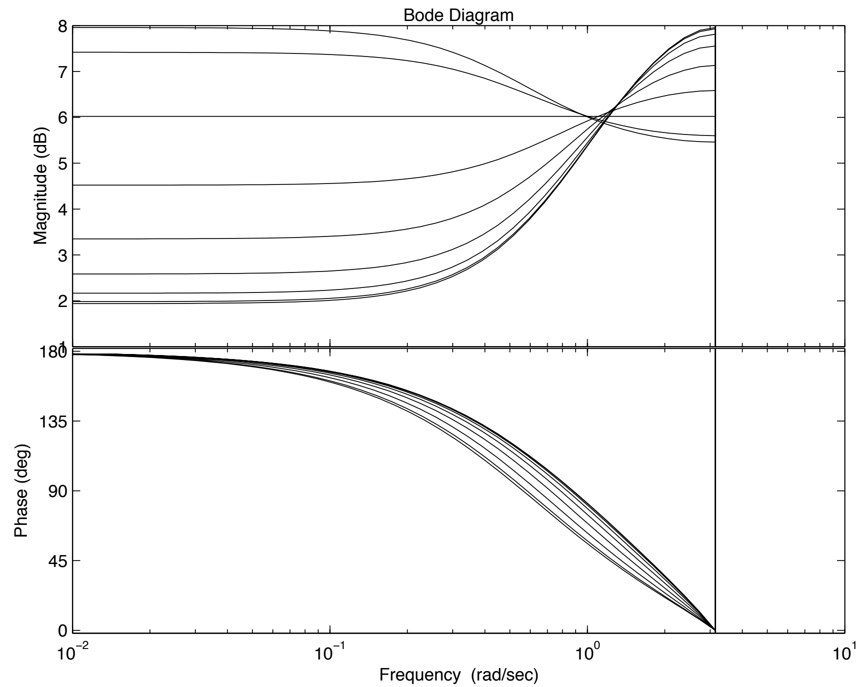
Example $\quad G(z) = \dfrac{1}{z-2}$

$$S(2) = 0, S(\infty) = 1$$

Find all S of degree at most $n = 1$.

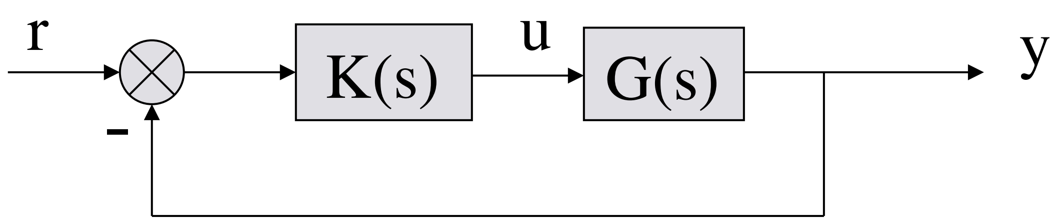
$$S(z) = \frac{z-2}{z-a}, \quad -1 < a < 1$$

$$\gamma = 2.5 > \gamma_{\text{opt}} = \min \|S\|_\infty = 2$$



Example: Sensitivity shaping

step
reference



$$G(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}$$

Doyle, Francis
Tannenbaum

Design a strictly proper *K* so that the closed-loop system is

- internally stable and
- satisfies the specifications:

- settling time at most 8 seconds
- overshoot at most 10%
- $|u(t)| \leq 0.5$

Try to achieve $$S_{ideal} = \frac{s(s+1.2)}{s^2+1.2s+1}$$ (DFT)

Internal stability requires $\deg S \leq 4$

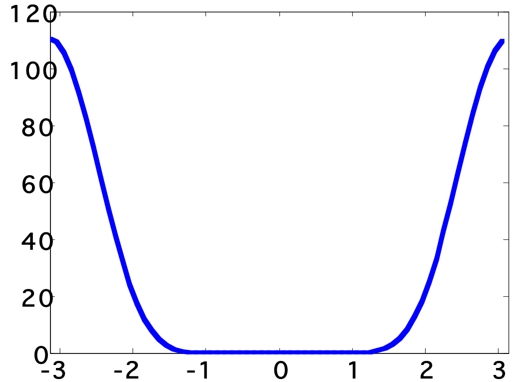
$S(0) = S'(\infty) = 0$ (unstable plant poles)

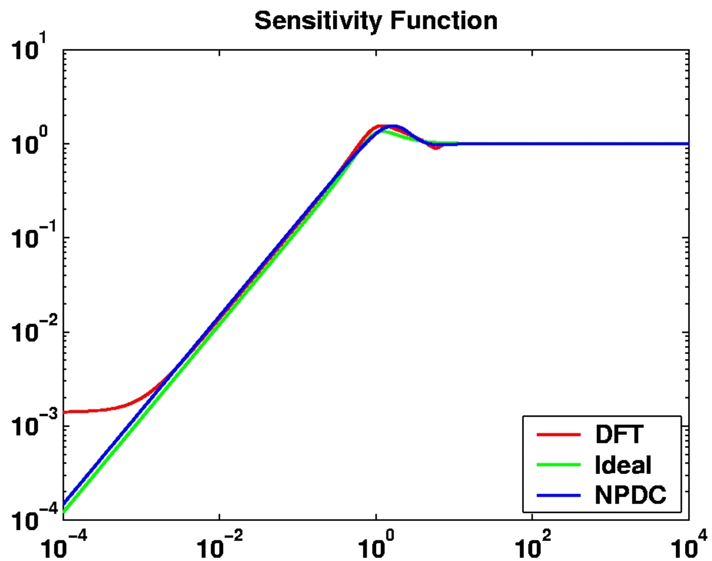
$S(\infty) = S(5.5308) = 1$ (nonminimum-phase plant zeros)

$S''(\infty) = 0$ (strictly proper controller)

Choose:

$|\sigma|^2$ →

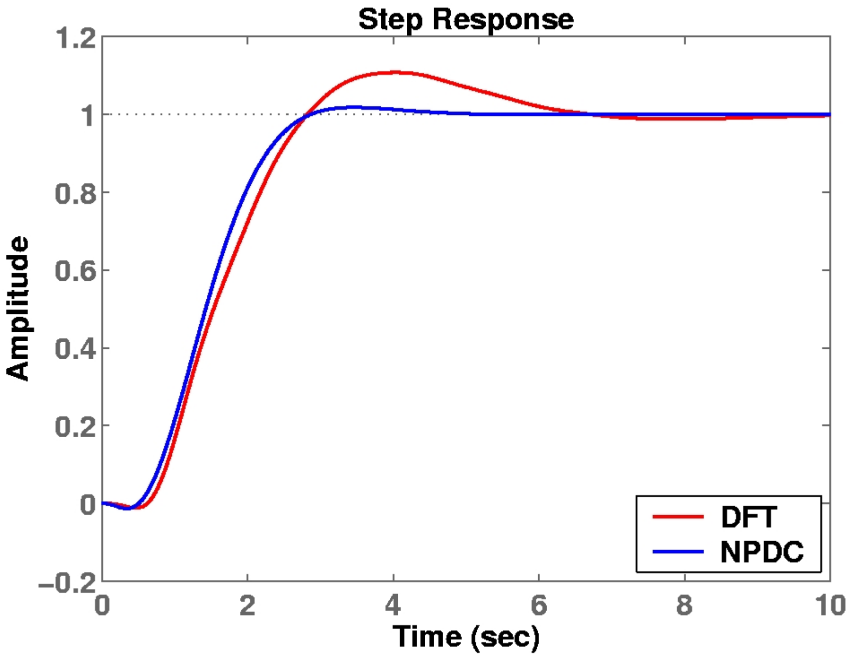


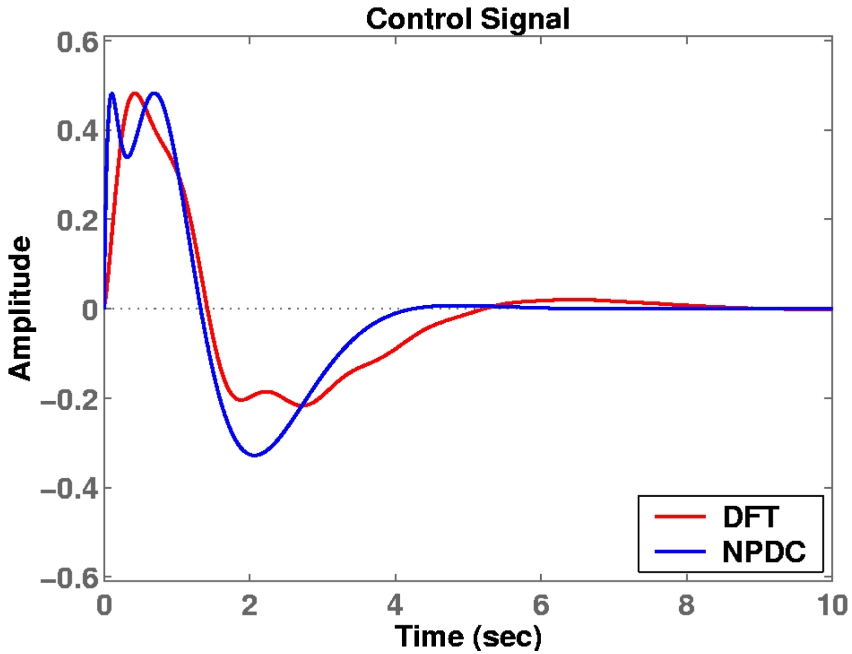


DFT: deg K = 8

NPDC: deg K = 4

A. Blomqvist and R. Nagamune





- settling time at most 8 seconds
- overshoot at most 10%
- $|u(t)| \leq 0.5$

DFT: deg C = 8

NPDC: deg C = 4

Multidimensional moment problems

$$\int_K \alpha_k d\mu = c_k, \quad k = 1, 2, \ldots, n$$

- $d\mu$ nonnegative measure on a compact subset K of $\mathbb{R}^d$
- $\alpha_1, \alpha_2, \ldots \alpha_n$ linearly independent basis functions defined on K

Image compression

Model reduction

Original system:

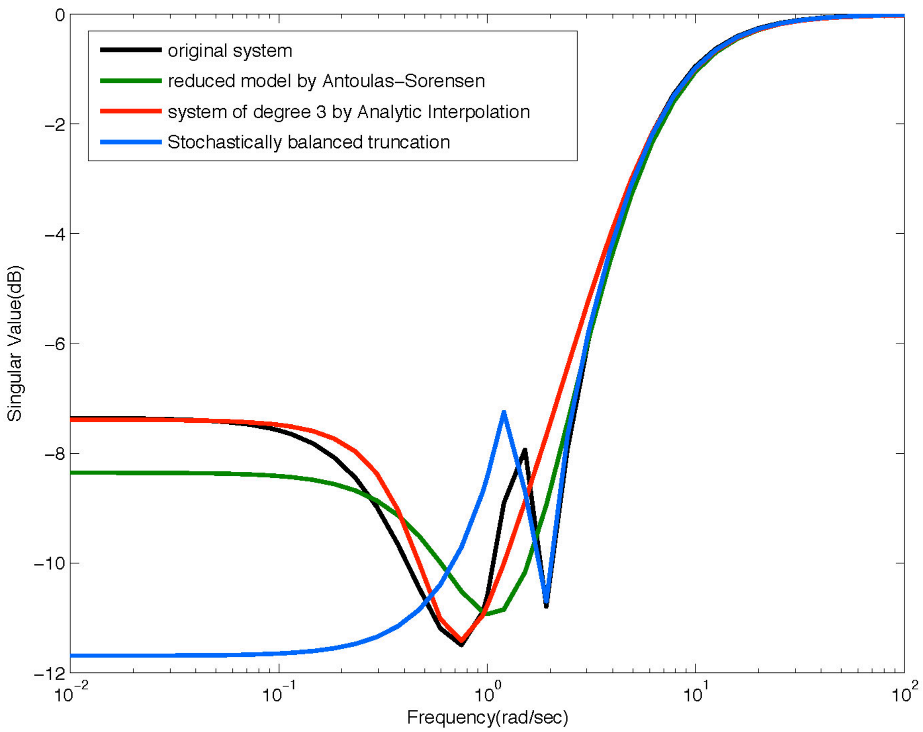
$$G(s) = \frac{s^5 + 3s^4 + 6s^3 + 9s^2 + 7s + 3}{s^5 + 7s^4 + 14s^3 + 21s^2 + 23s + 7}$$

Antoulas-Sorensen:

$$\hat{G}(s) = \frac{s^3 + 2.553s^2 + 2.906s + 1.173}{s^3 + 6.681s^2 + 8.459s + 3.07}$$

Global-analysis approach:

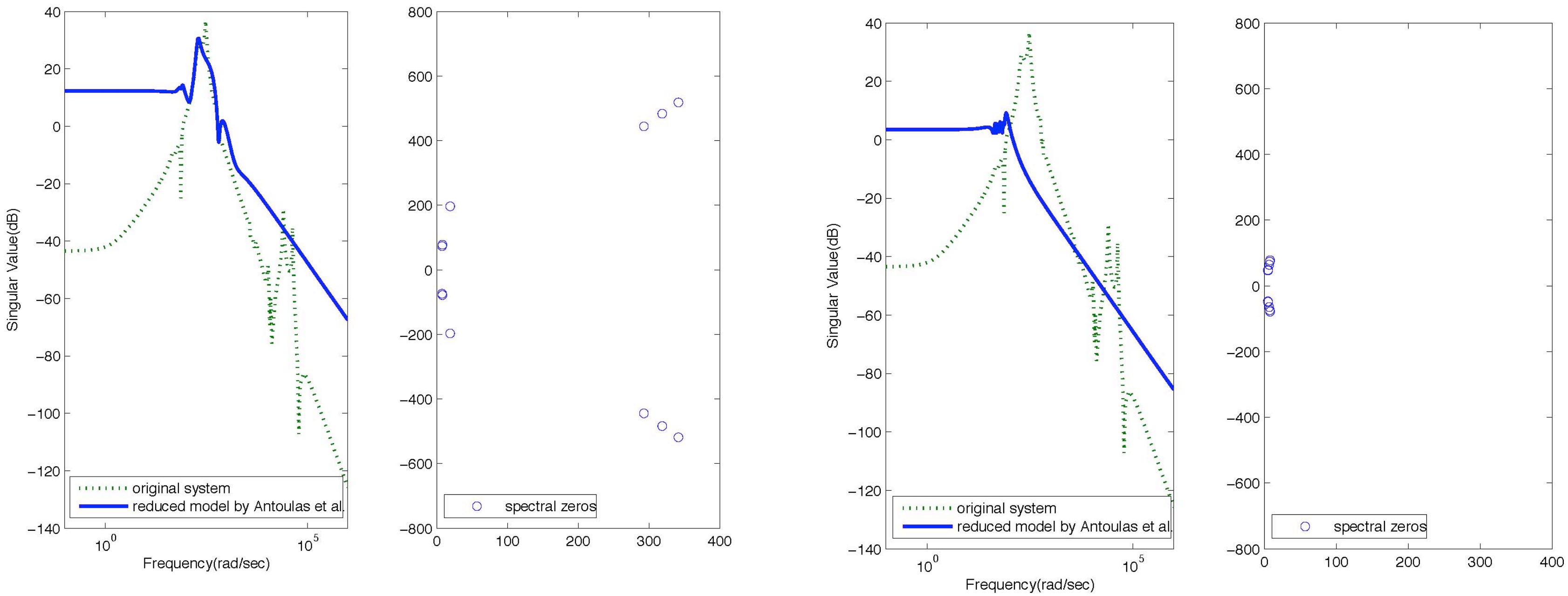
$$f(s) = \frac{1.002s^3 + 2.84s^2 + 1.927s + 0.8978}{s^3 + 7.298s^2 + 6.084s + 2.099}$$



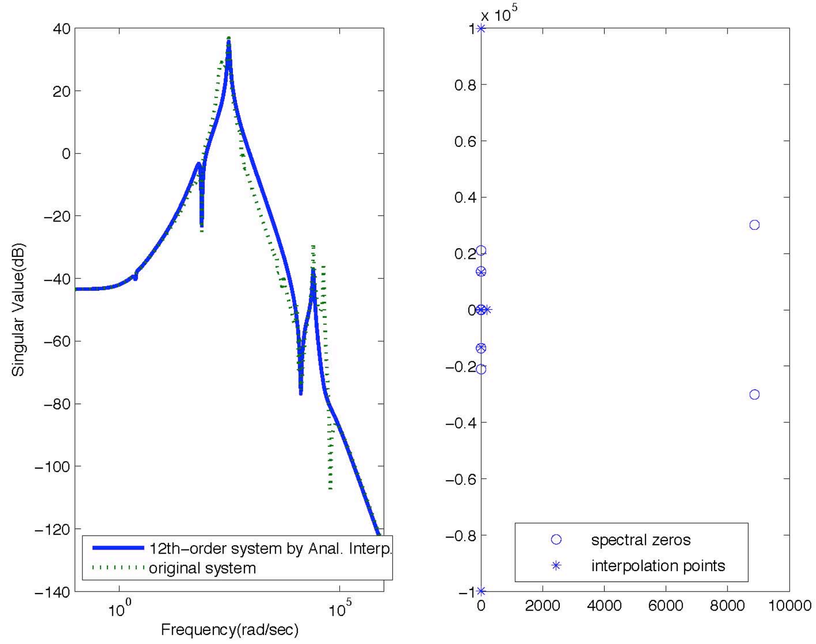
A large-scale problem: A CD player

Model reduction: $\deg G = 120 \longrightarrow \deg \hat{G} = 12$

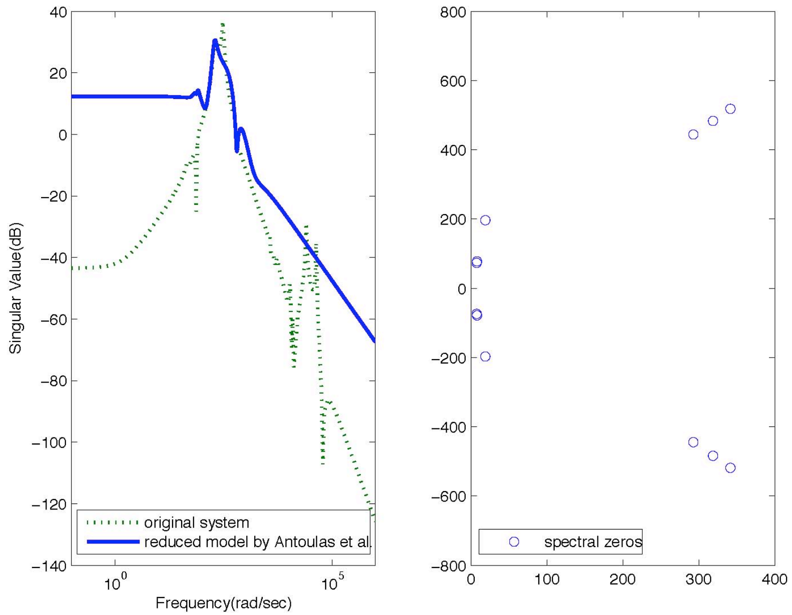
Antoulas-Sorensen solutions:

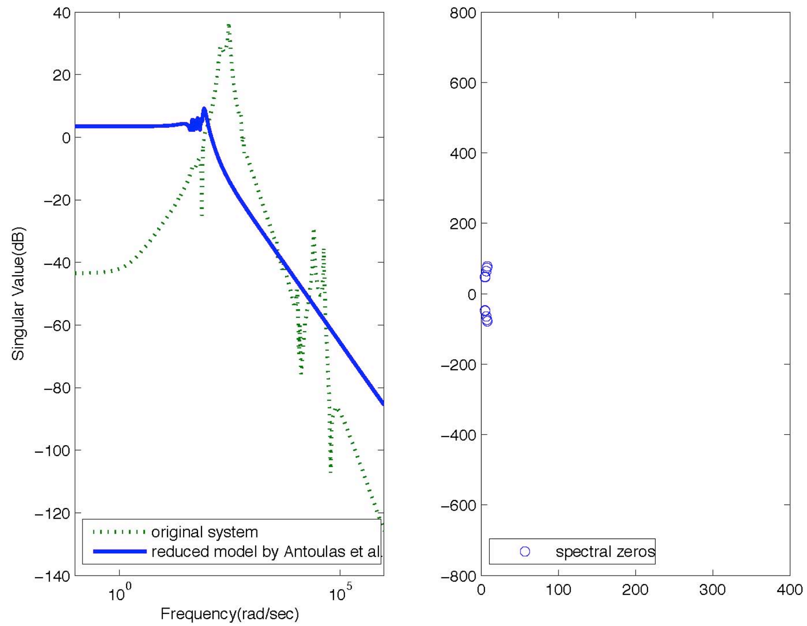


Global-analysis solution



Antoulas-Sorensen solutions





Conclusions

An enhanced theory for generalized moment problems that incorporates rationality constraints prescribed by applications.

- Complete parameterizations of solutions with smooth tuning strategies.
- A global analysis approach that studies the class of solutions as a whole.
- Convex optimization for determining solutions.