Opinion Dynamics over Signed Social Networks

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Social Networks

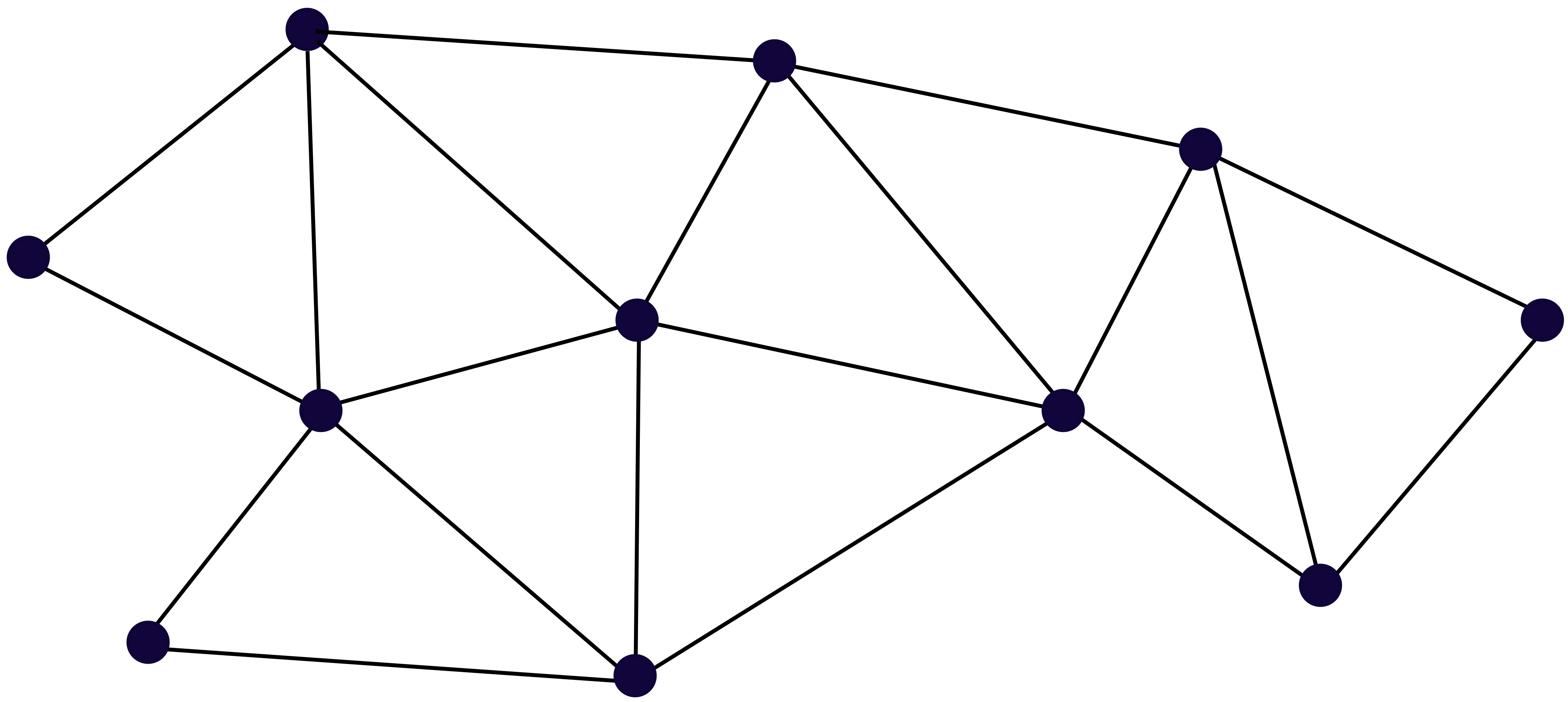
Opinions

iPhone, Blackberry, or Samsung? Republican or Democrat? Sell or buy AAPL? The rate of economic growth this year? Social cost of carbon?

French 1956; Benerjee 1992; Galam 1996

Dynamics of Opinions

$$x_i(k+1) = f_i\Big(x_i(k); x_j(k), j \in \mathcal{N}_i\Big)$$



De Groot Social Interactions

$$\mathbf{x}(k+1) = \mathbf{P}\mathbf{x}(k)$$

- $\mathbf{P}$ is a stochastic matrix

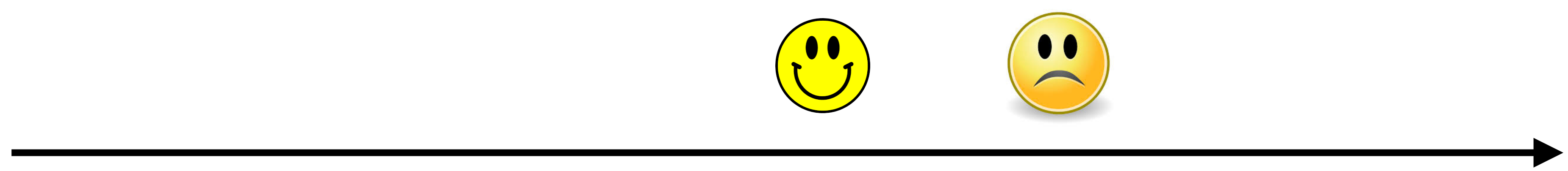
DeGroot 1974

De Groot Social Interactions

Before:

$$x_i(k+1) = x_i(k) + \alpha(x_j(k) - x_i(k))$$

After:



$$0 < \alpha < 1$$

DeGroot 1974

De Groot Social Interactions

$$\mathbf{x}(k+1) = \mathbf{P}\mathbf{x}(k)$$

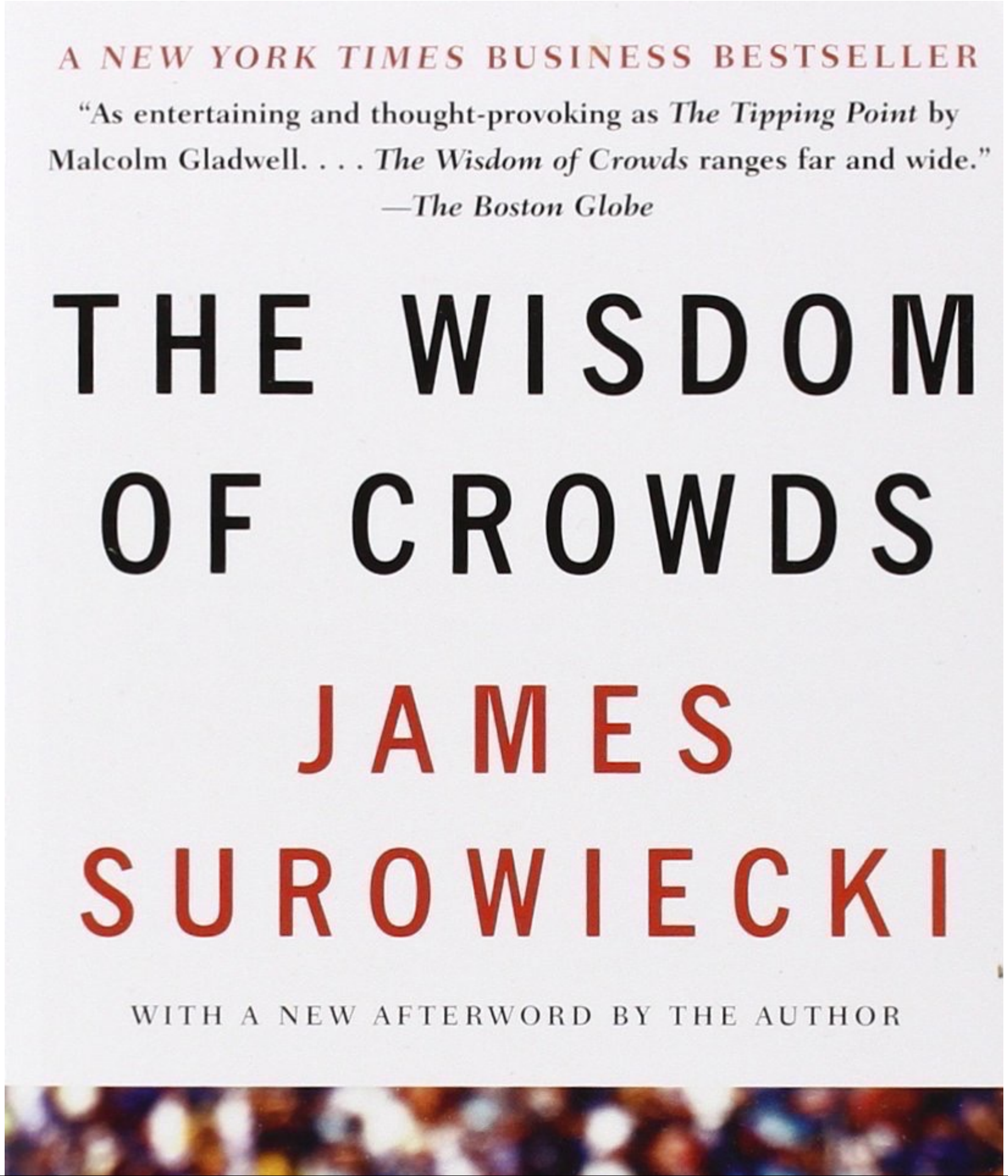
An agreement is achieved if $\mathbf{P}$ is ergodic in the sense that

$$\lim_{k\to\infty} x_i(k) = \mathbf{v}^\top \mathbf{x}(0)$$

Trust and cooperation lead to social consensus!

Wisdom of Crowds (with Trust)

Golub and Jackson 2010



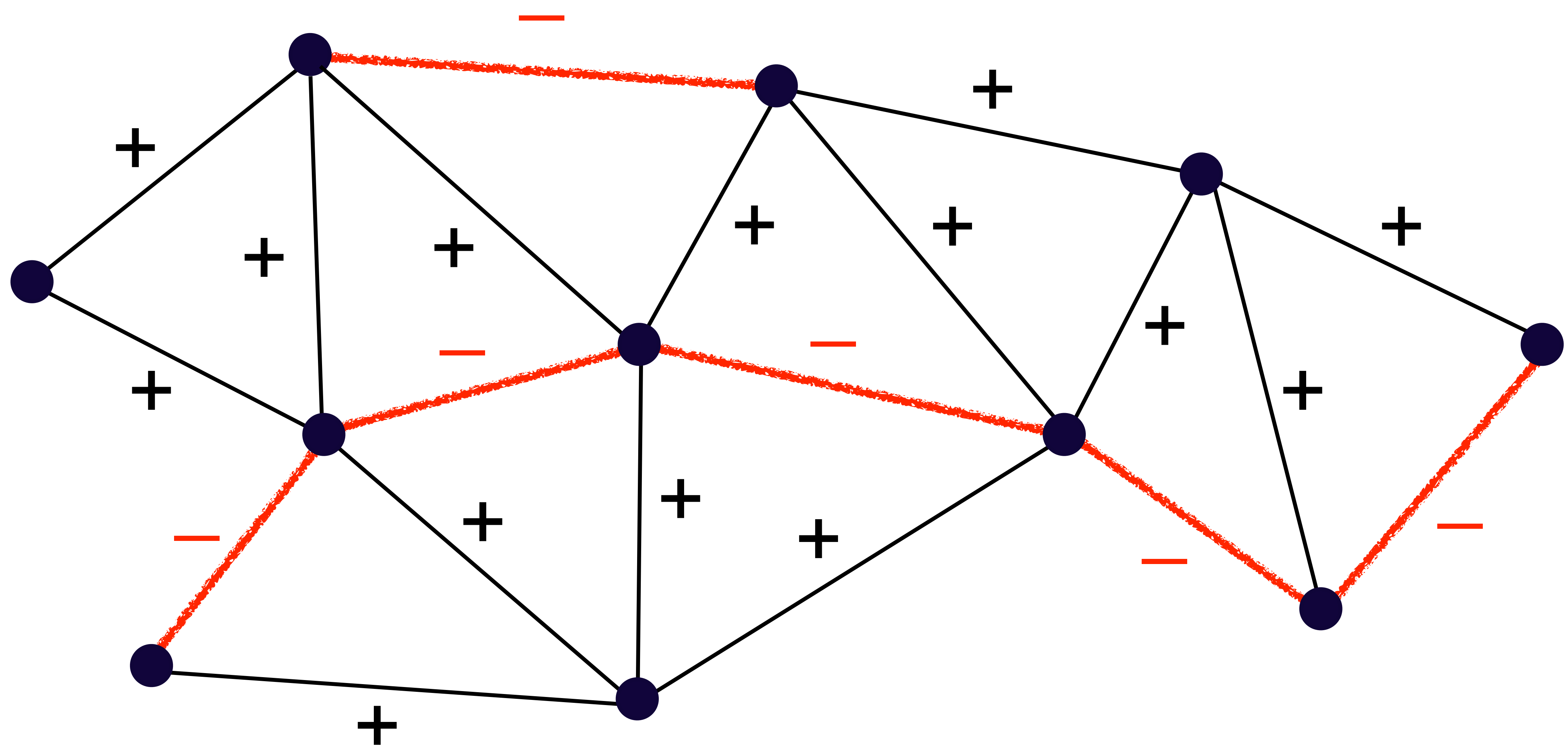
Disagreement Models

- Memory of initial values
  - Friedkin and Johnsen (1999)
- Bounded confidence
  - Krause (1997); Hegselmann-Krause (2002); Blondel et al. (2011); Li et al. (2013)
- Stubborn agents
  - Acemoglu et al. (2013)
- Homophily
  - Dandekar et al. (2013)

This Talk

A model and theory for opinion dynamics over social networks with friendly and adversarial interpersonal relations coexisting.

Friends and Adversaries

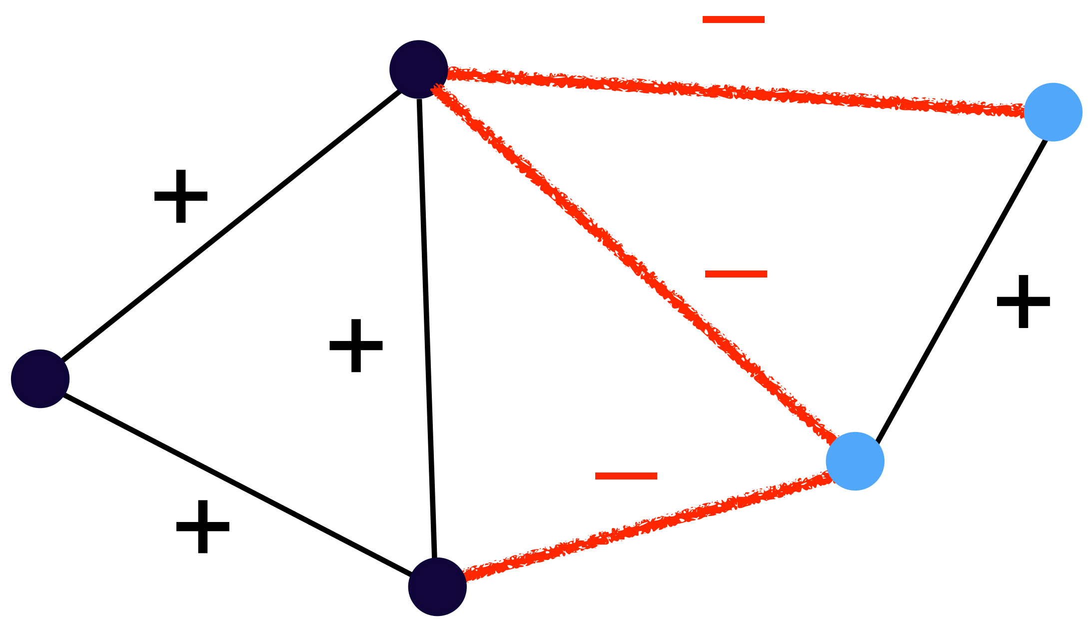


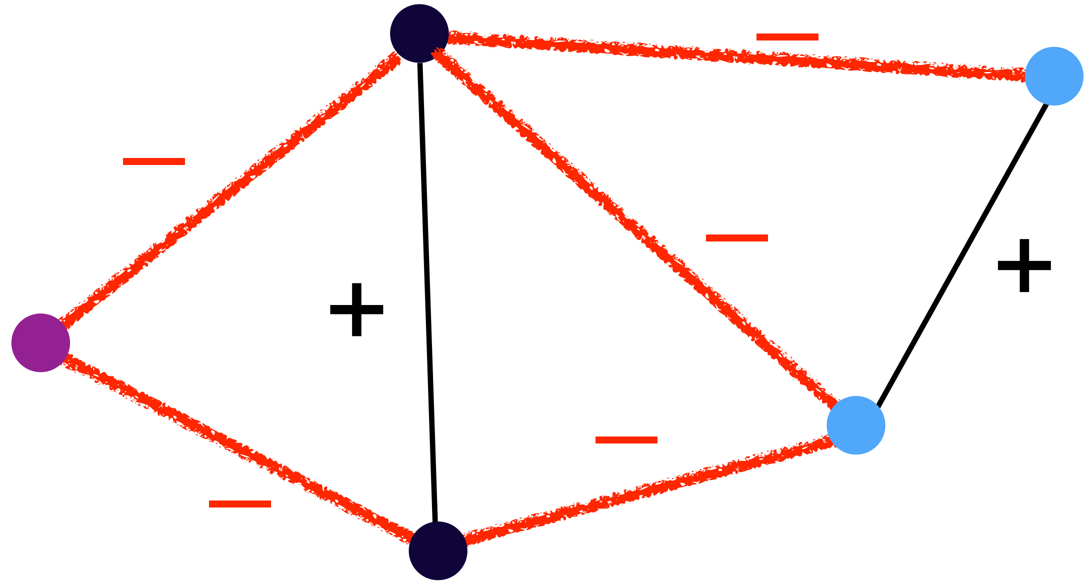
Structural Balance Theory

- **Strongly balanced** if the node set can be divided into two disjoint subsets such that negative links can only exist between them;
- **Weakly balanced** if such a partition contains maybe more than two subsets.

Heider (1947), Harary (1953), Cartwright and Harary (1956), Davis (1963)

Structural Balance





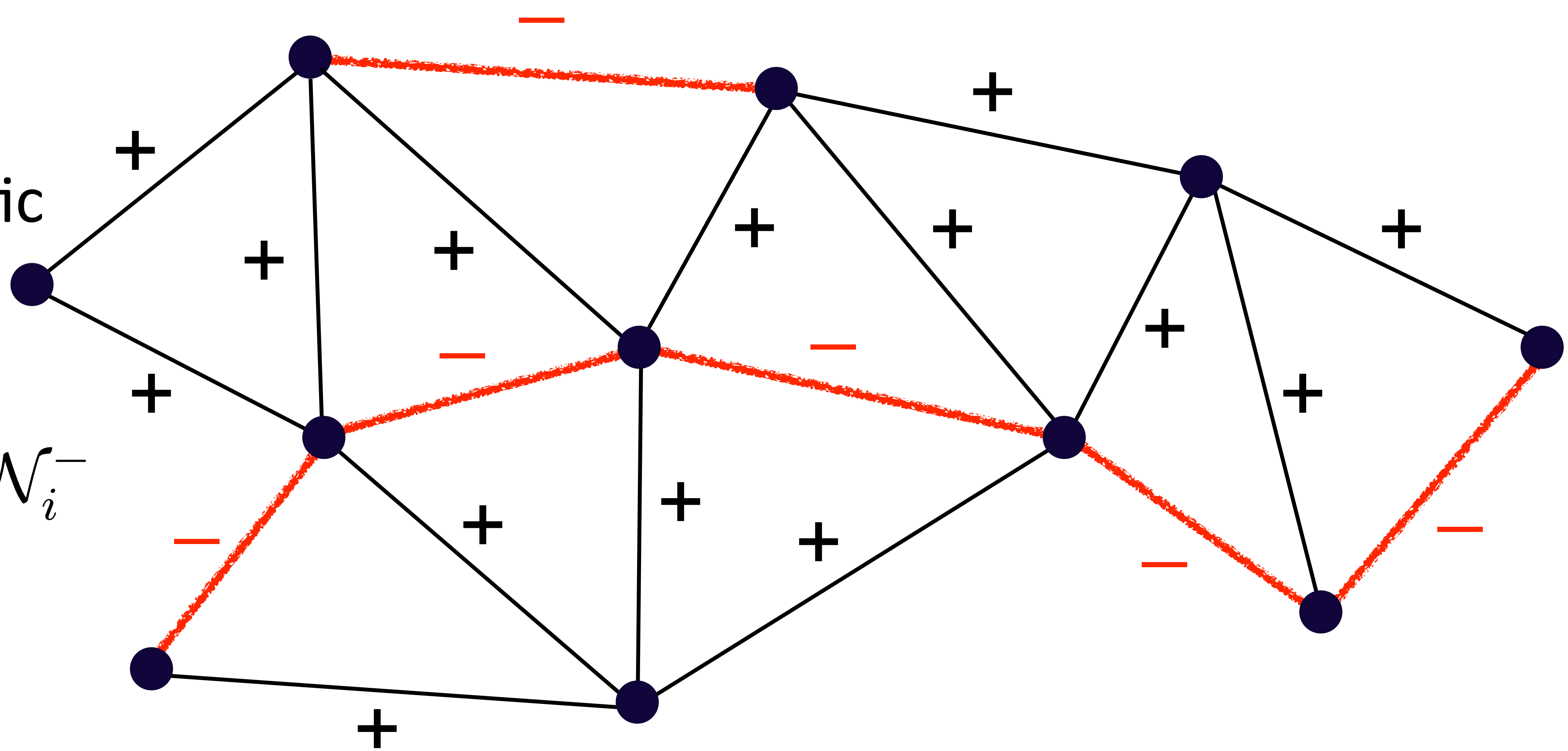
When do nodes interact?

Underlying World

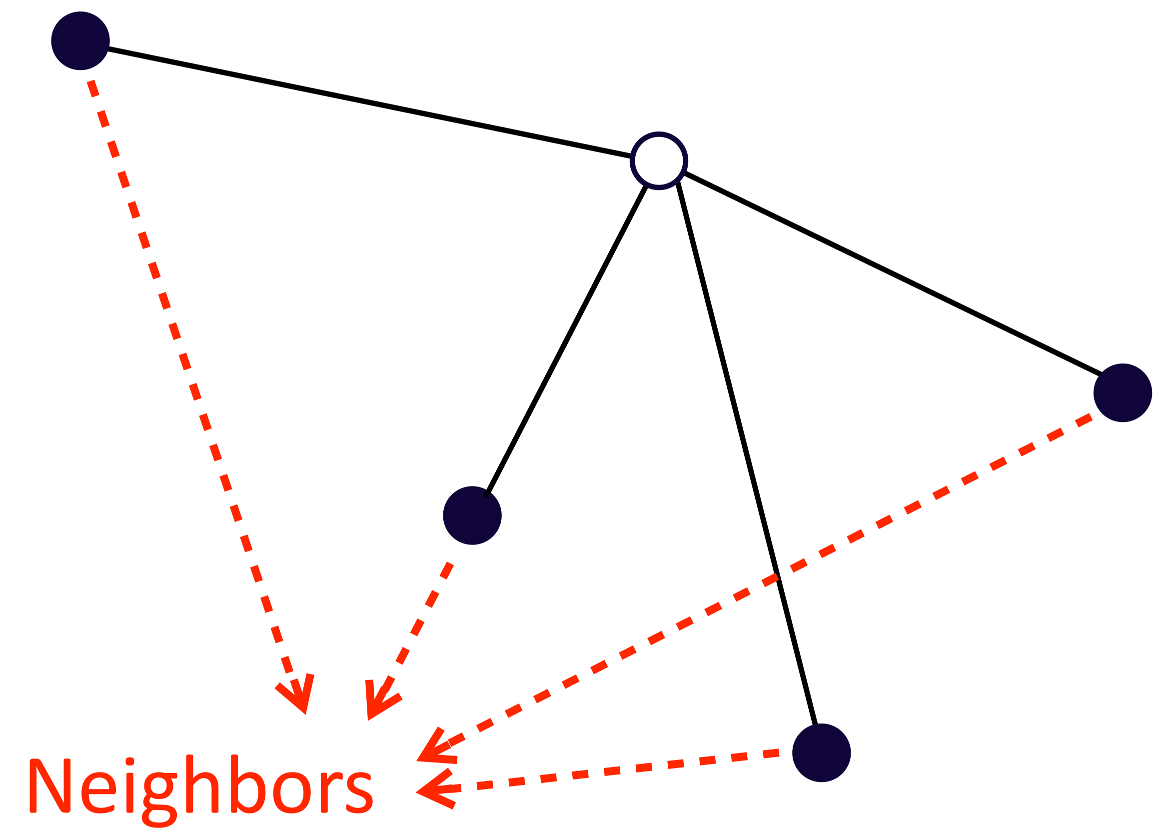
$$\mathrm{G} = \mathrm{G}^+ \cup \mathrm{G}^-$$

- Fixed
- Undirected
- Deterministic
- Connected

$$\mathcal{N}_i = \mathcal{N}_i^+ \cup \mathcal{N}_i^-$$

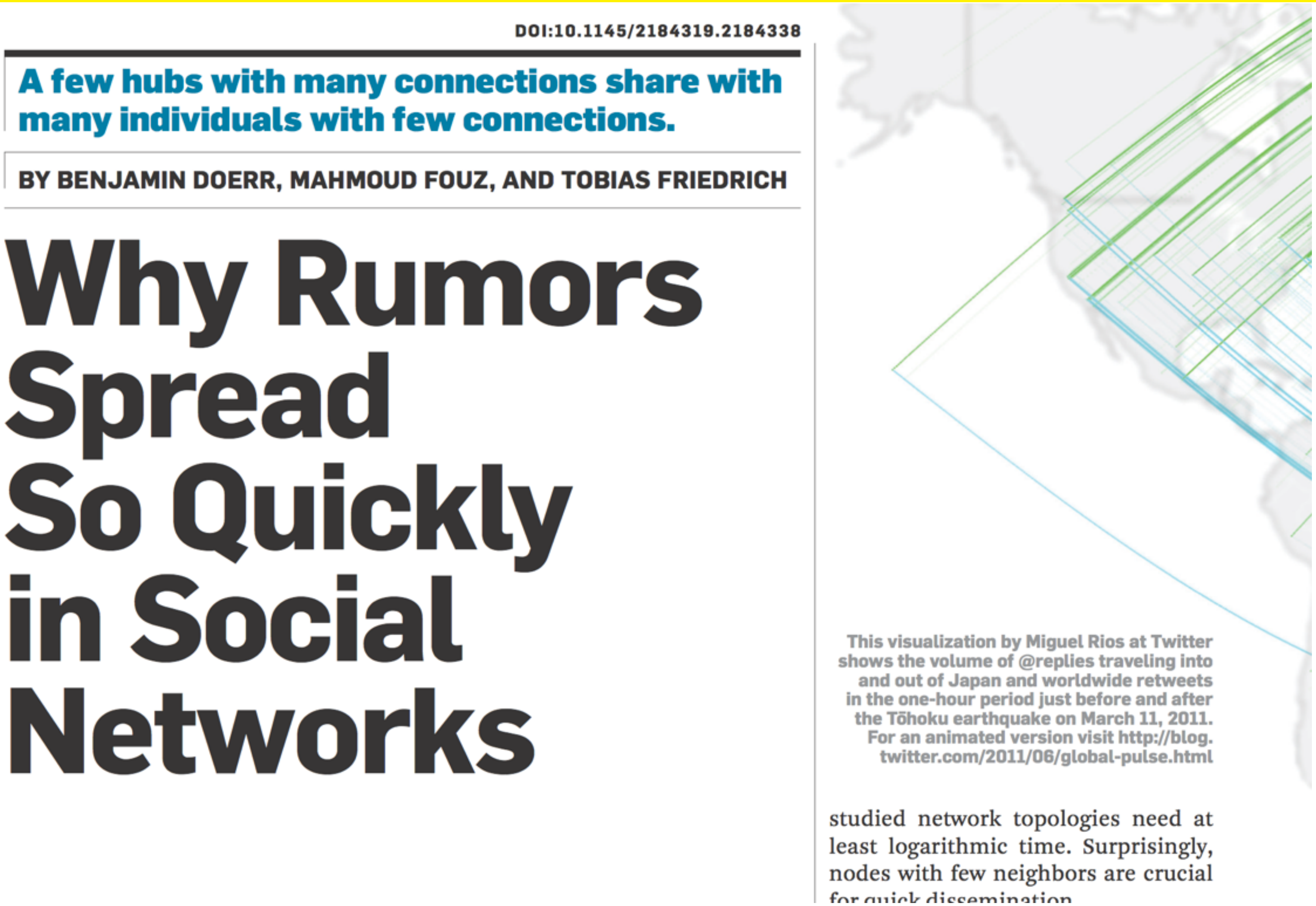


Gossip Model



Gossip Model

Independent with other time and node states, at time k,
(i) A node i is drawn with probability $1/N$;
(ii) Node i selects one of its neighbor j with probability $1/|\mathcal{N}_i|$.


DOI:10.1145/2184319.2184338

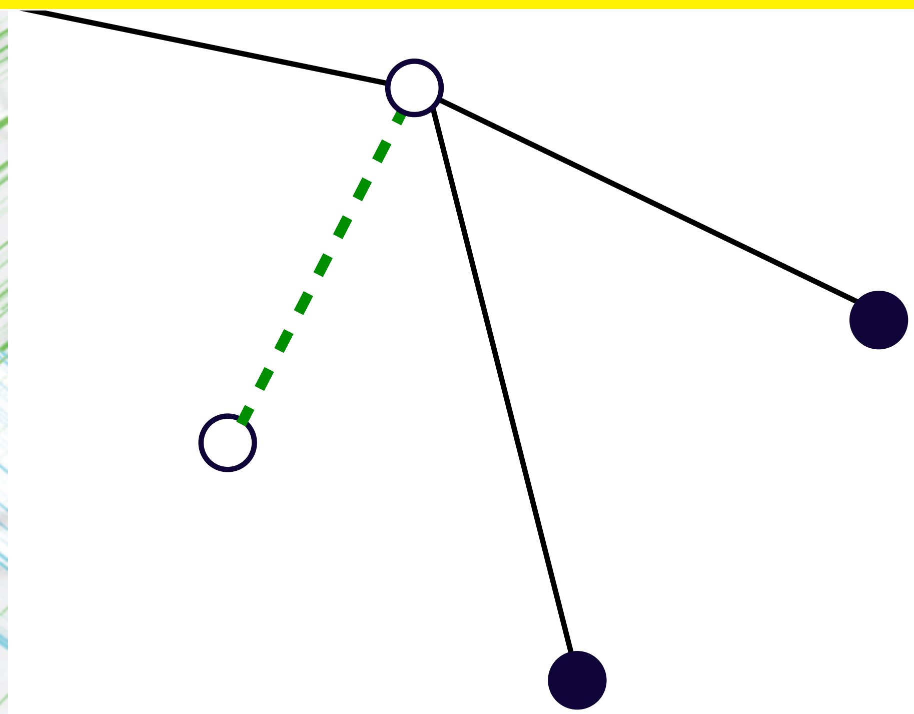
A few hubs with many connections share with many individuals with few connections.

BY BENJAMIN DOERR, MAHMOUD FOUZ, AND TOBIAS FRIEDRICH

Why Rumors Spread So Quickly in Social Networks

This visualization by Miguel Rios at Twitter shows the volume of @replies traveling into and out of Japan and worldwide retweets in the one-hour period just before and after the Tōhoku earthquake on March 11, 2011. For an animated version visit http://blog.twitter.com/2011/06/global-pulse.html

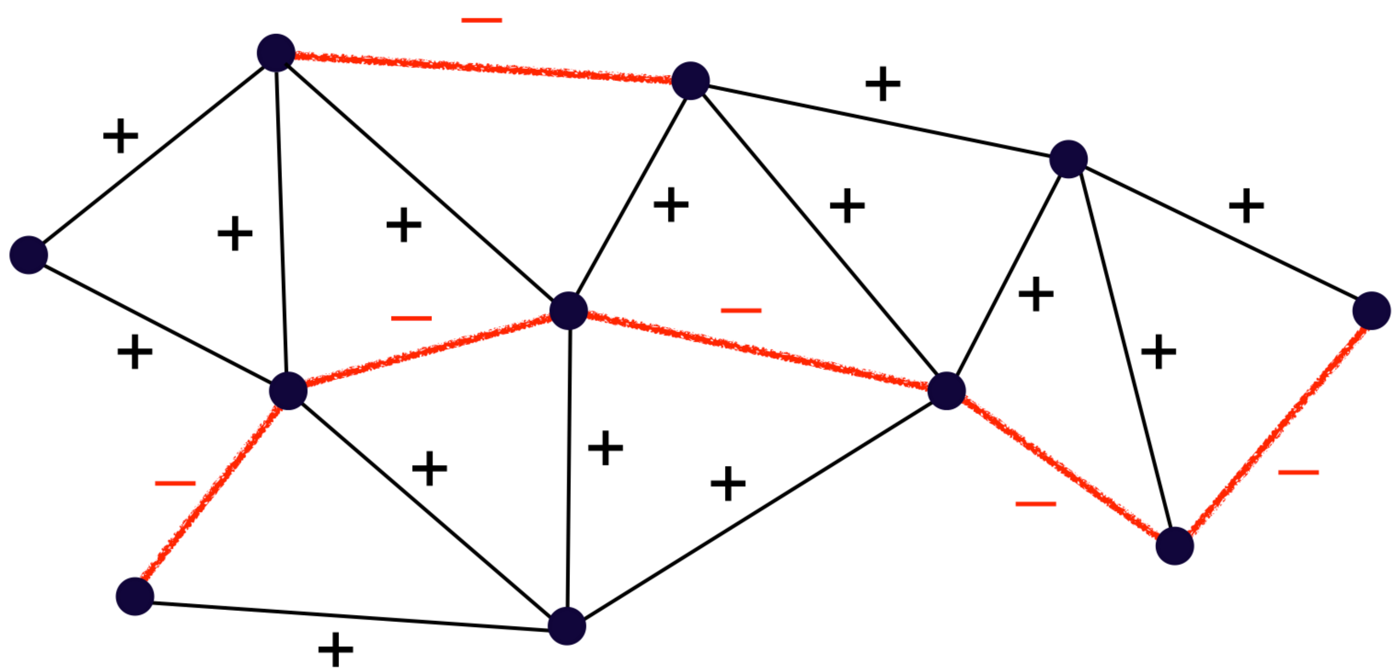
studied network topologies need at least logarithmic time. Surprisingly, nodes with few neighbors are crucial

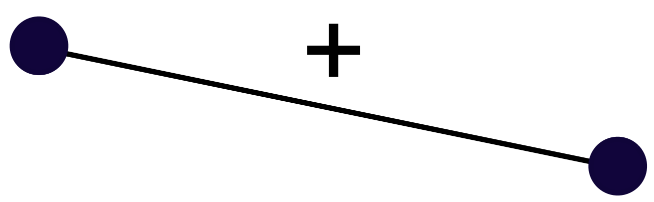


How a pair of nodes interacts with each other when they meet?

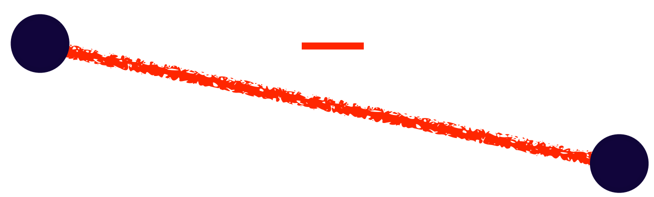
Positive and Negative Interactions

A pair (i,j) is randomly selected.
The two selected nodes update.





$$x_i(k+1) = x_i(k) + \alpha\big(x_j(k) - x_i(k)\big) \qquad 0 < \alpha < 1$$

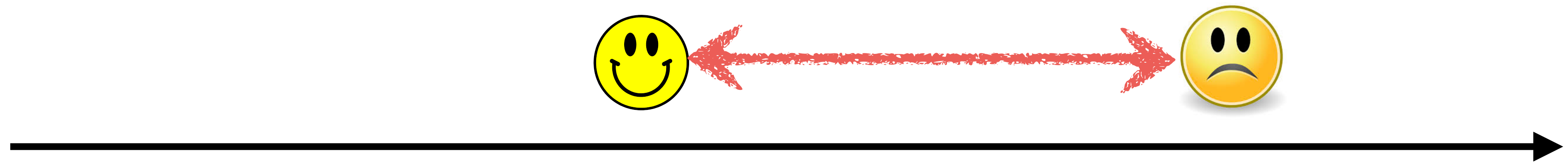


$$x_i(k+1) = x_i(k) + \beta\big(x_i(k) - x_j(k)\big) \qquad \beta > 0$$

Altafini 2013

Positive and Negative Interactions

Before:



$$x_i(k+1) = x_i(k) + \alpha(x_j(k) - x_i(k))$$

After:

$$x_i(k+1) = x_i(k) - \beta\big(x_j(k) - x_i(k)\big)$$

After:

Mean/Mean-Square Evolution

Relative-State-Flipping Model

$$\mathbf{x}(k+1) = W(k)\mathbf{x}(k)$$

$$\mathbb{E}\{W(k)\} = I - \alpha L_{\text{pst}}^{\dagger} + \beta L_{\text{neg}}^{\dagger}.$$

- It's an eigenvalue perturbation problem!

Phase Transition

Theorem. Suppose G^+ is connected and G^- is non-empty. Then there exists β_* such that

(i) $\lim_{k\to\infty} \mathbb{E}\{x_i(k)\} = \sum_{i=1}^{N} x_i(0)/N$ for all $i = 1, \ldots, N$ if $\beta < \beta_*$;

(ii) $\lim_{k\to\infty} \max_{i,j} \|\mathbb{E}\{x_i(k)\} - \mathbb{E}\{x_j(k)\}\| = \infty$ if $\beta > \beta_*$.

- It's possible to prove that the expectation of the state transition matrix is *eventually positive*.

Phase Transition

Let G be the complete graph. Let G^{-} be the Erdos-Renyi random graph with link appearance probability p.

(i) If $p < \frac{\alpha}{\alpha+\beta}$, then

$$\mathbf{P}\Big(\text{Consensus in expectation}\Big) \to 1$$

as the number of nodes N tends to infinity;

(ii) If $p > \frac{\alpha}{\alpha+\beta}$, then

$$\mathbf{P}\Big(\text{Divergence in expectation}\Big) \to 1$$

as the number of nodes N tends to infinity.

Sample Path Behavior

Live-or-Die Lemma

Introduce

$$\mathscr{C}_{x^0} \doteq \Big\{ \limsup_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \Big\}, \quad \mathscr{D}_{x^0} \doteq \Big\{ \limsup_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \Big\}$$

$$\mathscr{C}^*_{x^0} \doteq \Big\{ \liminf_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \Big\}, \quad \mathscr{D}^*_{x^0} \doteq \Big\{ \liminf_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \Big\}$$

Lemma.
Suppose G^+ is connected. Then (i) $\mathbb{P}(\mathscr{C}_{x^0}) + \mathbb{P}(\mathscr{D}_{x^0}) = 1$; (ii) $\mathbb{P}(\mathscr{C}^*_{x^0}) + \mathbb{P}(\mathscr{D}^*_{x^0}) = 1$. As a consequence, almost surely, one of the following events happens:

$$\Big\{ \lim_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \Big\};$$

$$\Big\{ \lim_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \Big\};$$

$$\Big\{ \liminf_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = 0;\ \limsup_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \Big\}.$$

Zero-One Law

$$\mathscr{C} \doteq \Big\{ \limsup_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \text{ for all } x^0 \in \mathbb{R}^n \Big\},$$

$$\mathscr{D} \doteq \Big\{ \exists \text{ (deterministic) } x^0 \in \mathbb{R}^n, \quad \text{s.t. } \limsup_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \Big\}$$

Theorem. Both $\mathscr{C}$ and $\mathscr{D}$ are trivial events (i.e., each of them occurs with probability equal to either 1 or 0) and $\mathbb{P}(\mathscr{C}) + \mathbb{P}(\mathscr{D}) = 1$.

No-Survivor Theorem

Theorem. There always holds

$$\mathbb{P}\left(\liminf_{k\to\infty} \left|x_i(k) - x_j(k)\right| = \infty \;\middle|\; \liminf_{k\to\infty} \max_{i,j} \left|x_i(k) - x_j(k)\right| = \infty \right) = 1$$

for all $i \neq j$.

Phase Transition

Theorem.

(i) Suppose G^+ is connected. Then there is a $\beta_* > 0$ such that

$$\mathbb{P}\Big(\lim_{k\to\infty} x_i(k) = \sum_{i=1}^{N} x_i(0)/N \Big) = 1$$

for all i if $\beta < \beta_*$.

(ii) There is $\beta^* > 0$ such that

$$\mathbb{P}\Big(\liminf_{k\to\infty} \max_{i,j} \big\| x_i(k) - x_j(k) \big\| = \infty \Big) = 1$$

for all i if $\beta > \beta^*$.

Bounded State Model

Bounded States

- Let $A > 0$ be a constant and define $\mathscr{P}_A(\cdot)$ by $\mathscr{P}_A(z) = -A, z < -A$, $\mathscr{P}_A(z) = z, z \in [-A, A]$, and $\mathscr{P}_A(z) = A, z > A$.

- Define the function $\theta : \mathrm{E} \to \mathbb{R}$ so that $\theta(\{i,j\}) = \alpha$ if $\{i,j\} \in \mathrm{E}^+$ and $\theta(\{i,j\}) = -\beta$ if $\{i,j\} \in \mathrm{E}^-$.

Consider the following node interaction under relative-state flipping rule:

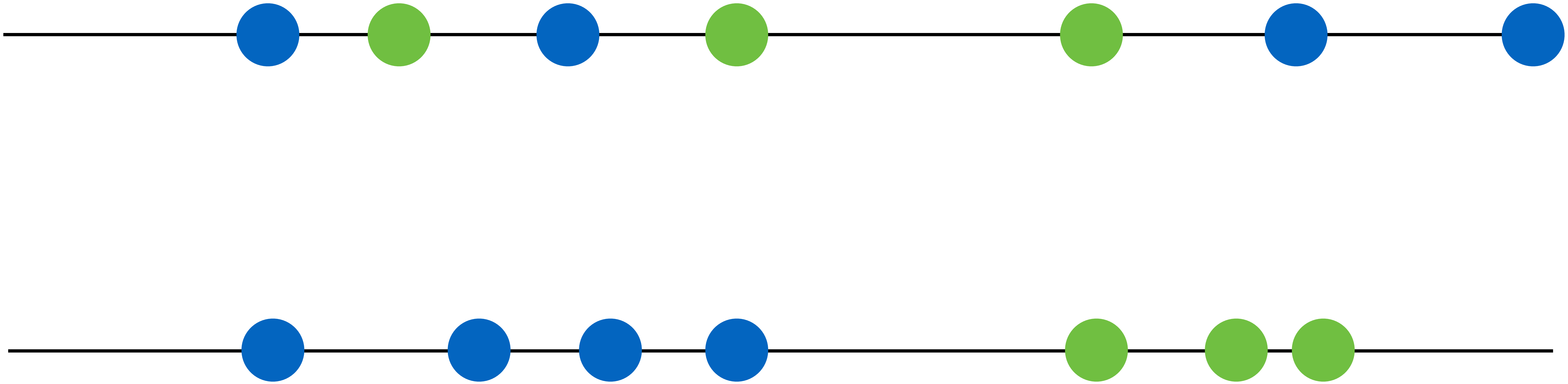
$$x_s(t+1) = \mathscr{P}_A\Big((1-\theta)x_s(t) + \theta x_{-s}(t)\Big), \ s \in \{i,j\}.$$

Clustering of Opinions

Theorem. Let $\alpha \in (0, 1/2)$. Assume that G is a weakly structurally balanced complete graph under the partition $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2 \cdots \cup \mathrm{V}_m$ with $m \geq 2$. Let $\alpha \in (0, 1/2)$. When β is sufficiently large, almost sure boundary clustering is achieved in the sense that for almost all initial value $\mathbf{x}(0)$ w.r.t. Lebesgue measure, there are there are m random variables, $l_1(\mathbf{x}(0)), \ldots, l_m(\mathbf{x}(0))$, each of which taking values in $\{-A, A\}$, such that:

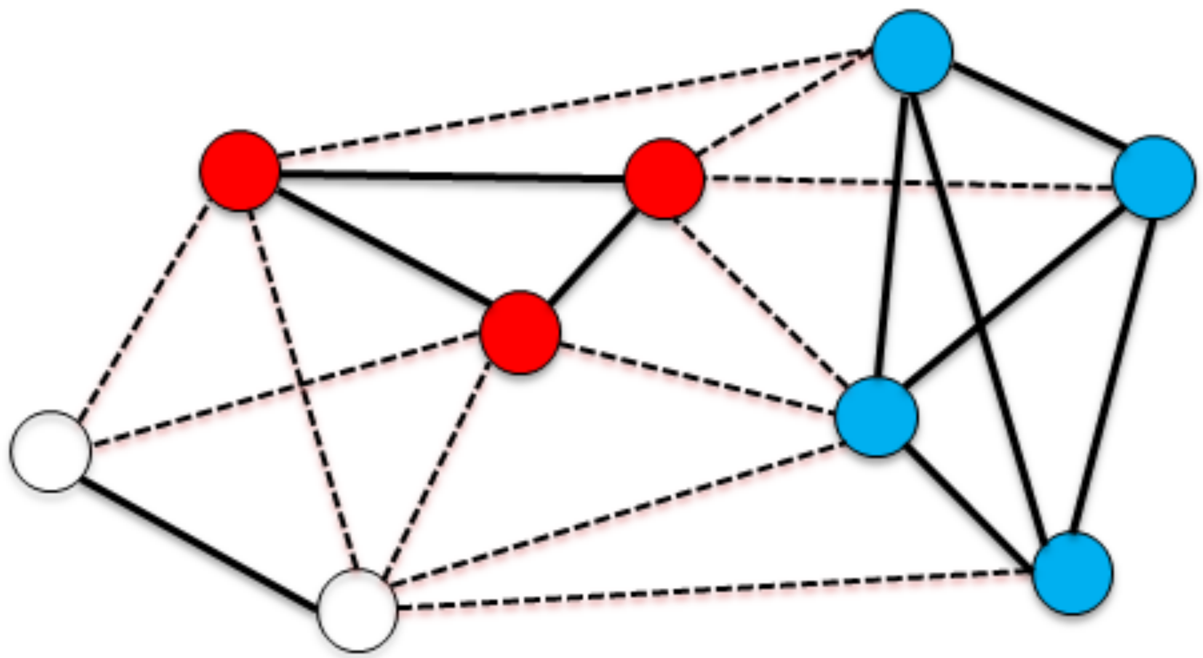
$$\mathbb{P}\Big(\lim_{t\to\infty} x_i(t) = l_j(\mathbf{x}(0)),\ i \in \mathrm{V}_j,\ j = 1, \ldots, m\Big) = 1.$$

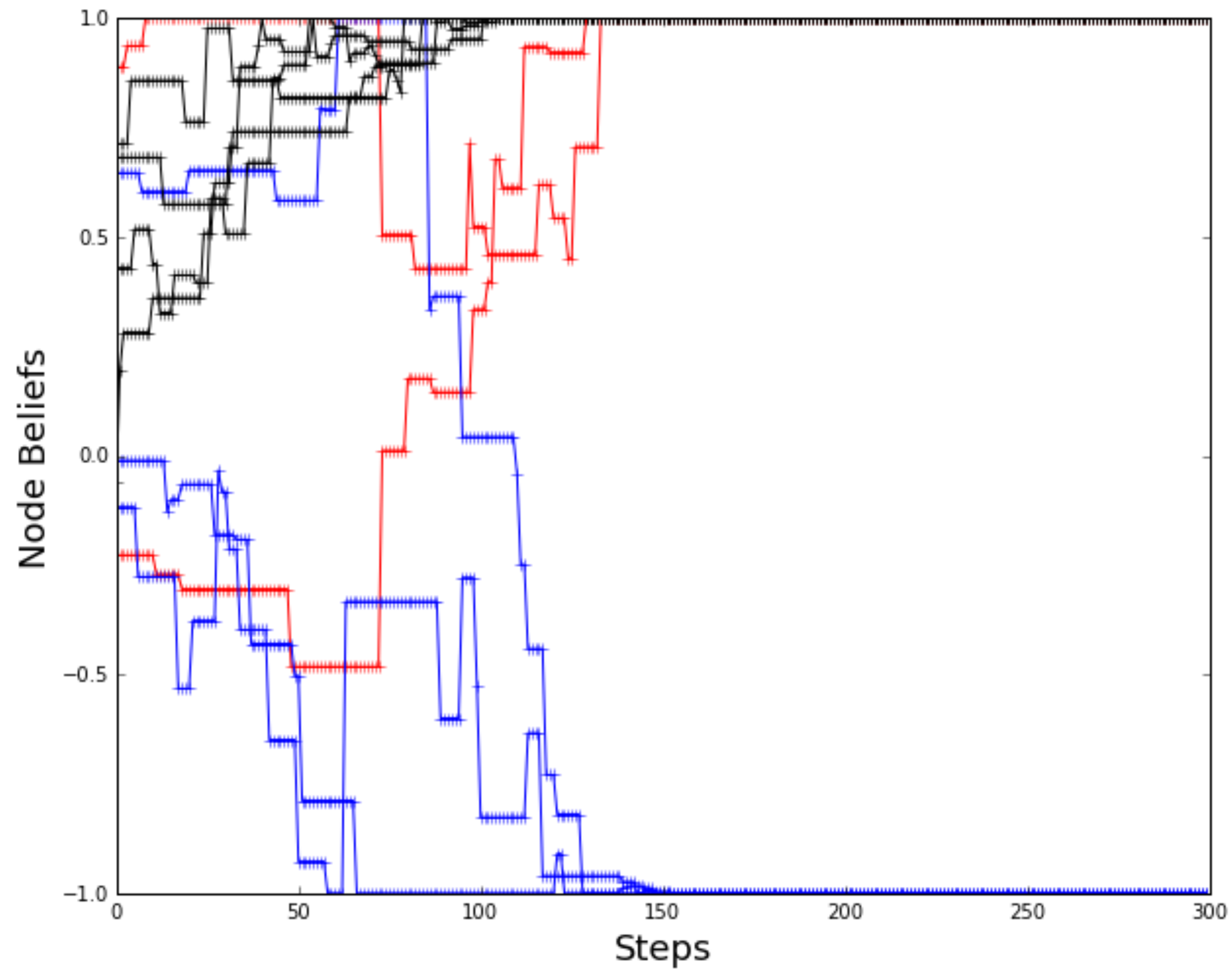
Separation Events



- **The power of minority groups.**

Numerical Example



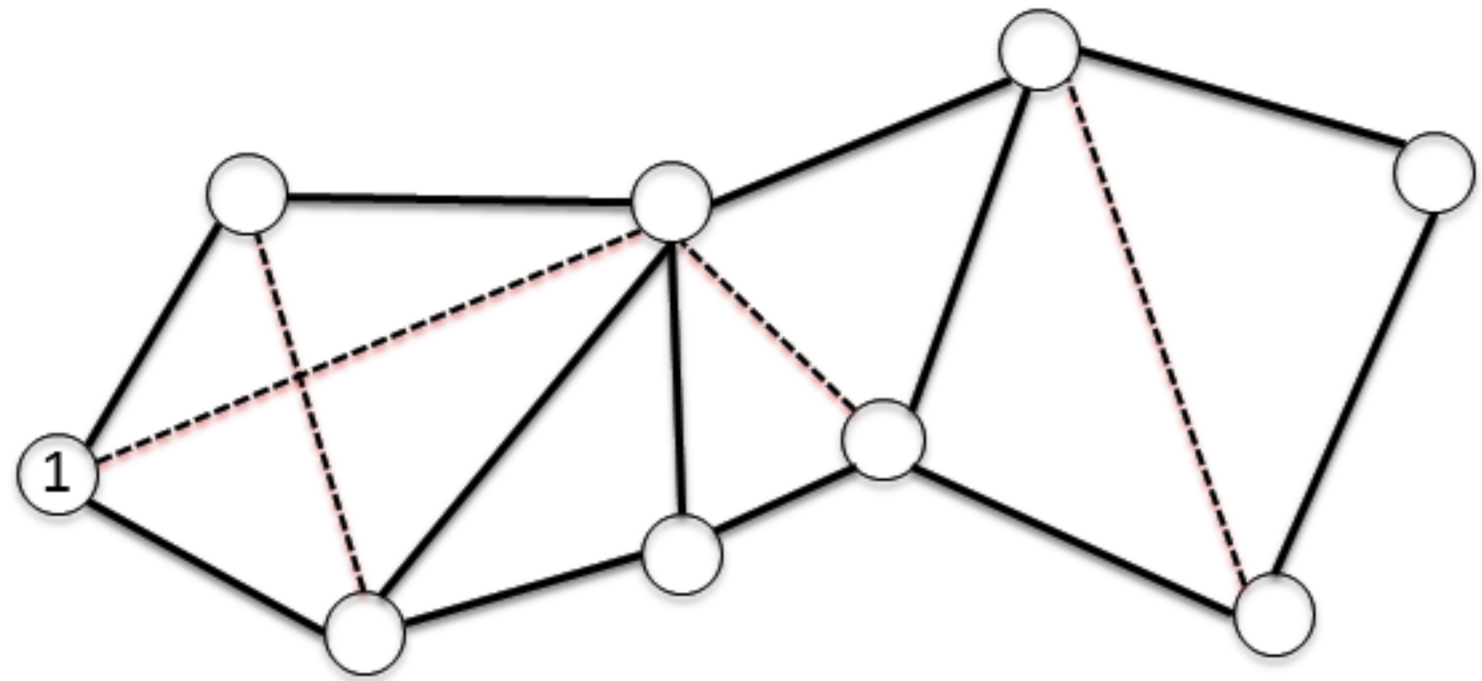


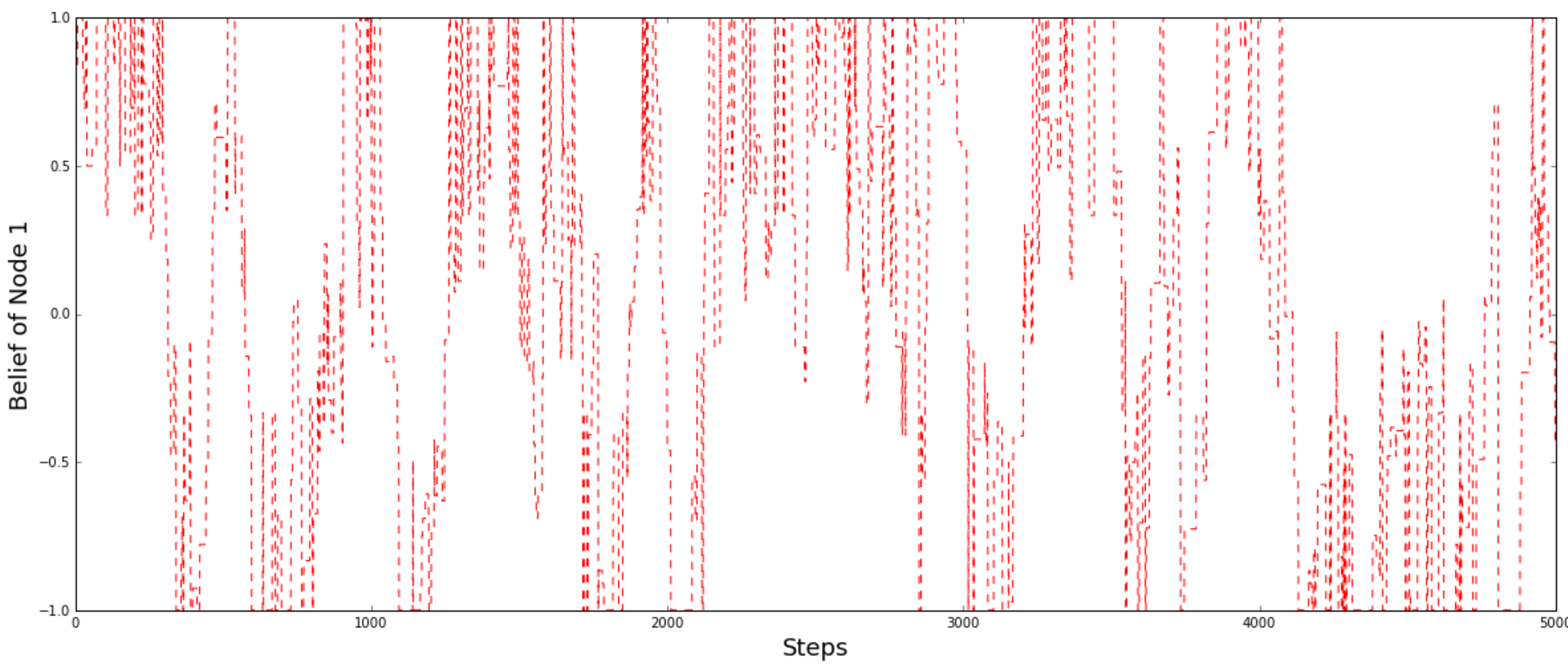
Oscillation of Opinions

Theorem. Let $\alpha \in (0, 1/2)$. Assume that G is a complete graph and the positive graph G^+ is connected. When β is sufficiently large, for almost all initial value $\mathbf{x}(0)$ w.r.t. Lebesgue measure, there holds for all $i \in \mathrm{V}$ that

$$\mathbb{P}\Big(\liminf_{t\to\infty} x_i(t) = -A,\ \limsup_{t\to\infty} x_i(t) = A\Big) = 1.$$

Numerical Example





Related Publications

Shi et al. 2013 *IEEE Journal on Selected Areas in Communications;*
Shi et al. 2015, 2016 *IEEE Transactions on Control of Network Systems;*
Shi et al. 2016 *Operations Research.*

Thank you!