Cover's Open Problem: "The Capacity of the Relay Channel"

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Joint work with Xiugang Wu and Leighton Pate Barnes.

Father of the Information Age

Claude Shannon (1916-2001)

The Bell System Technical Journal

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A Mathematical Theory of Communication

By C. E. SHANNON

Introduction

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a

Communication in the Presence of Noise*

CLAUDE E. SHANNON†, MEMBER, IRE

Summary—**A method is developed for representing any communication system geometrically. Messages and the corresponding signals are points in two "function spaces," and the modulation process is a mapping of one space into the other. Using this representation, a number of results in communication theory are deduced concerning expansion and compression of bandwidth and the threshold effect. Formulas are found for the maximum rate of transmission of binary digits over a system when the signal is perturbed by various types of noise. Some of the properties of "ideal" systems which transmit at this maximum rate are discussed. The equivalent number of binary digits per second for certain information sources is calculated.**

* Decimal classification: 621.38. Original manuscript received by the Institute, July 23, 1940. Presented, 1948 IRE National Convention, New York, N. Y., March 24, 1948; and IRE New York Section, New York, N. Y., November 12, 1947.
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I. Introduction

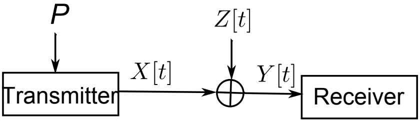
A GENERAL COMMUNICATIONS system is shown schematically in Fig. 1. It consists essentially of five elements.

1. *An information source.* The source selects one message from a set of possible messages to be transmitted to the receiving terminal. The message may be of various types; for example, a sequence of letters or numbers, as in telegraphy or teletype, or a continuous function of time $f(t)$, as in radio or telephony.

2. *The transmitter.* This operates on the message in some way and produces a signal suitable for transmission to the receiving point over the channel. In teleph-

"A method is developed for representing any communication system geometrically..."

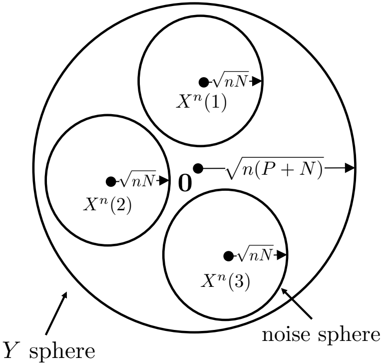
AWGN Channel



Capacity

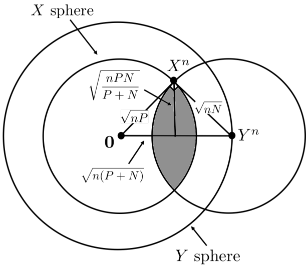
$$C = \log\left(1 + \frac{P}{N}\right)$$

Converse: Sphere Packing



$$\#\text{ of } X^n \leq \frac{\left|\text{Sphere}\left(\sqrt{n(P+N)}\right)\right|}{\left|\text{Sphere}\left(\sqrt{nN}\right)\right|} \doteq \frac{2^{\frac{n}{2}\log 2\pi e(P+N)}}{2^{\frac{n}{2}\log 2\pi eN}} = 2^{\frac{n}{2}\log\left(1+\frac{P}{N}\right)}$$

Achievability: Geometric Random Coding

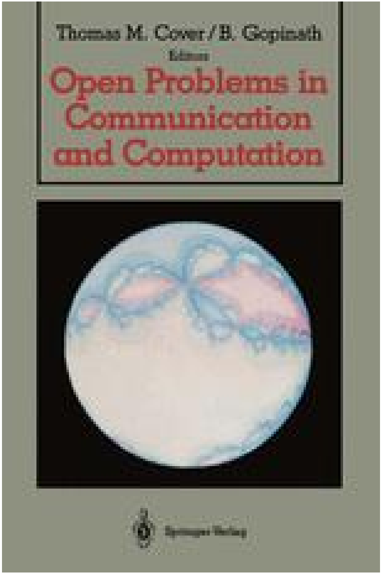


$$\Pr(\exists \text{ false } X^n) \leq \frac{|\text{Lens}|}{\left|\text{Sphere}\left(\sqrt{nP}\right)\right|} \times 2^{nR}$$

$$\doteq \frac{2^{\frac{n}{2}\log 2\pi e \frac{PN}{P+N}}}{2^{\frac{n}{2}\log 2\pi e P}} \times 2^{nR}$$

$$= 2^{-\frac{n}{2}\log\left(1+\frac{P}{N}\right)} \times 2^{nR}$$

The story goes...



CHAPTER I.

INTRODUCTION

Thomas M. Cover and B. Gopinath

The papers in this volume are the contributions to a special workshop on problems in communication and computation conducted in the summers of 1984 and 1985 in Morristown, New Jersey, and the summer of 1986 in Palo Alto, California. The structure of this workshop was unique: no recent results, no surveys. Instead, we asked for outstanding open problems in the field. There are many famous open problems, including the question

$$P = NP?,$$

the simplex conjecture in communication theory, the capacity region of the broadcast channel, and the two-helper problem in information theory.

Beyond these well-defined problems are certain grand research goals. What is the general theory of information flow in stochastic networks? What is a comprehensive theory of computational complexity? What about a unification of algorithmic complexity and computational complexity? Is there a notion of energy-free computation? And if so, where do information theory, communication theory, computer science, and physics meet at the atomic level? Is there a duality between computation and communication? Finally, what is the ultimate impact of algorithmic complexity on probability theory? And what is its relationship to information theory?

The idea was to present problems on the first day, try to solve them on the second day, and present the solutions on the third day. In actual fact, only one problem was solved during the meeting -- El Gamal's problem on noisy communication over a common line. This was solved by Gallager. Shortly thereafter, however, Hajek solved two of Cover's prob-

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Cover's Open Problem

3.15 THE CAPACITY OF THE RELAY CHANNEL

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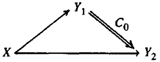
Consider the following seemingly simple discrete memoryless relay channel:



Here Y_1, Y_2 are conditionally independent and conditionally identically distributed given X, that is, $p(y_1, y_2 \mid x) = p(y_1 \mid x)\, p(y_2 \mid x)$. Also, the channel from Y_1 to Y_2 does not interfere with Y_2. A $(2^{nR}, n)$ code for this channel is a map $x : 2^{nR} \to X^n$, a relay function $r : Y_1^n \to 2^{nC_0}$, and a decoding function $g : 2^{nC_0} \times Y_2^n \to 2^{nR}$. The probability of error is given by

$$P_e^{(n)} = P\{\, g(r(y_1), y_2) \neq W \,\},$$

where W is uniformly distributed over 2^{nR} and

$$p(w, y_1, y_2) = 2^{-nR} \prod_{i=1}^{n} p(y_{1i} \mid x_i(w)) \prod_{i=1}^{n} p(y_{2i} \mid x_i(w)).$$

Let $C(C_0)$ be the supremum of the achievable rates R for a given C_0, that is, the supremum of the rates R for which $P_e^{(n)}$ can be made to tend to zero.

We note the following facts:

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1. $C(0) = \sup_{p(x)} I(X; Y_2)$.
2. $C(\infty) = \sup_{p(x)} I(X; Y_1, Y_2)$.
3. $C(C_0)$ is a nondecreasing function of C_0.

What is the critical value of C_0 such that $C(C_0)$ first equals $C(\infty)$?

REFERENCES

[1] T. Cover and A. El Gamal, "Capacity Theorems for the Relay Channel," *IEEE Trans. Inf. Theory*, IT-25, No. 5, pp. 572-584 (Sept. 1979).

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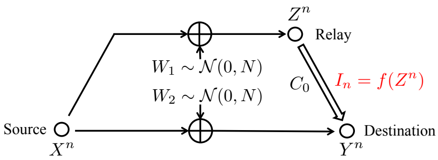
Gaussian case



$$C(\infty) = \frac{1}{2} \log \left(1 + \frac{2P}{N}\right)$$

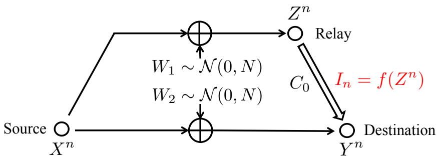
Achievability: $C_0^* = \infty$.

Cutset-bound (Cover and El Gamal'79):

$$C_0^* \geq \frac{1}{2} \log \left(1 + \frac{2P}{N}\right) - \frac{1}{2} \log \left(1 + \frac{P}{N}\right).$$

Potentially, $C_0^* \to 0$ as $P/N \to 0$.

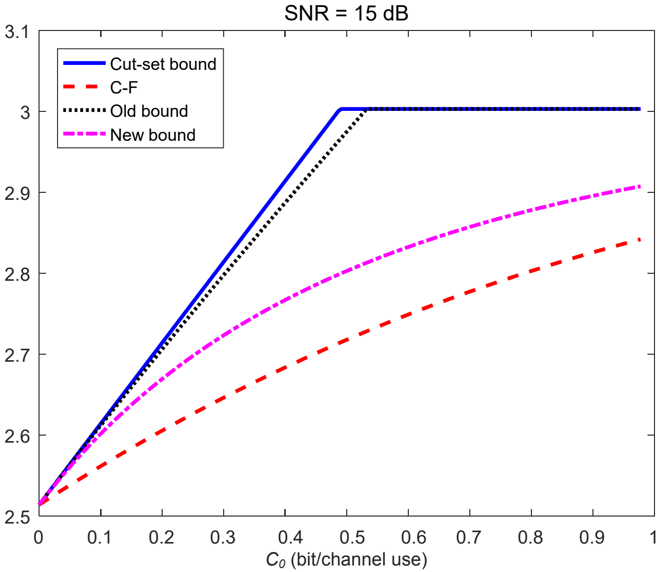
Main Result



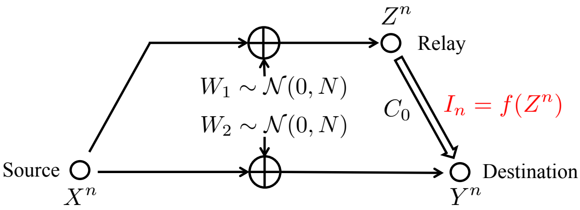
Theorem

$$C_0^* = \infty$$

Upper Bound on the Capacity

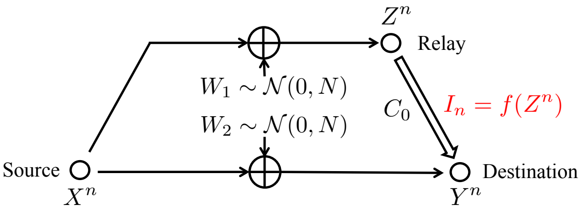


Cutset Bound



$$
\begin{aligned}
nR &\leq I(X^n; Y^n, I_n) + n\epsilon_n \\
&= I(X^n; Y^n) + I(X^n; I_n | Y^n) + n\epsilon_n \\
&= I(X^n; Y^n) + \underbrace{H(I_n | Y^n)}_{\leq nC_0} - \underbrace{H(I_n | Y^n, X^n)}_{\geq 0} + n\epsilon_n \\
&\leq n(I(X; Y) + C_0 + \epsilon_n)
\end{aligned}
$$

Cutset Bound



$$
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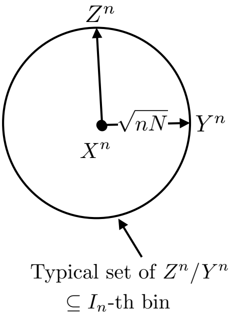
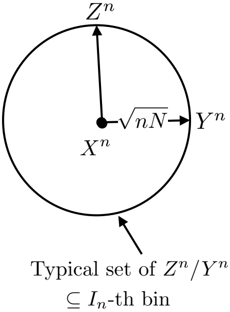
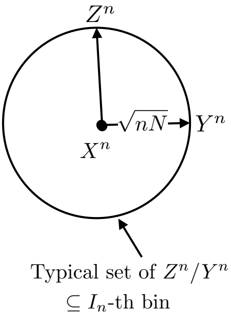
&= I(X^n; Y^n) + \underbrace{H(I_n | Y^n)}_{\le nC_0} - \underbrace{H(I_n | X^n)}_{\ge 0} + n\epsilon_n \\
&\le n(I(X; Y) + C_0 + \epsilon_n)
\end{aligned}
$$

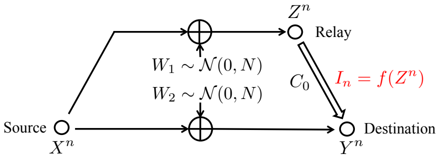

Z^n
$\sqrt{nN}$
Y^n
X^n
Typical set of Z^n/Y^n
$\subseteq I_n$-th bin



If $H(I_n|X^n) = 0$,



If $H(I_n|X^n) = 0$, then $H(I_n|Y^n) = 0$.

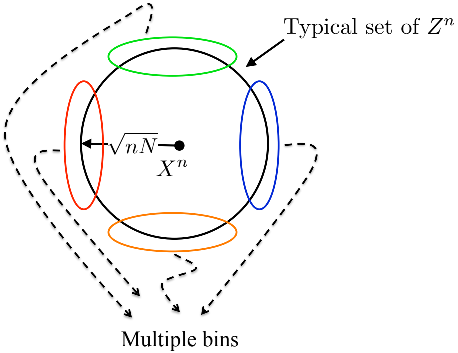


$$R \leq I(X^n; Y^n) + \underbrace{H(I_n|Y^n)}_{\leq ?} - \underbrace{H(I_n|X^n)}_{= -n \log \sin \theta_n} + n\epsilon_n$$

Goal:

$$\underbrace{\underbrace{I_n = f(Z^n) - Z^n - X^n}_{H(I_n|X^n) = -n \log \sin \theta_n} - Y^n}_{H(I_n|Y^n) \leq ?}$$

$H(I_n|X^n) \neq 0$



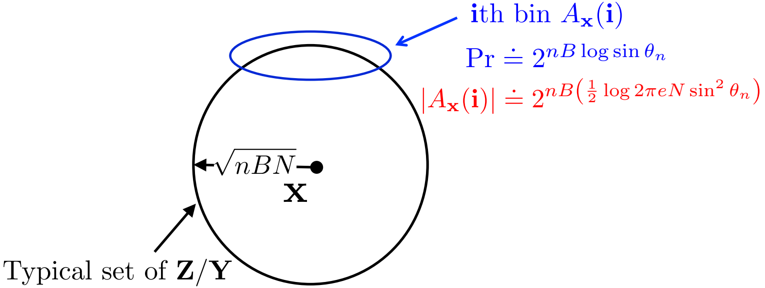
$\#$ of bins $=?$ $\mathbb{P}$(each bin) $=?$

From n- to nB- Dimensional Space

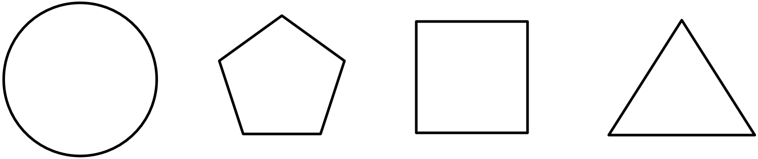
- $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{I}$: B-length i.i.d. from $\{(X^n(b), Y^n(b), Z^n(b), I_n(b))\}_{b=1}^{B}$.
- If $H(I_n|X^n) = -n \log \sin\theta_n$, then for any typical $(\mathbf{x}, \mathbf{i})$

$$p(\mathbf{i}|\mathbf{x}) \doteq 2^{nB \log \sin\theta_n},$$

$$\mathbb{P}(\mathbf{Z} \in A(\mathbf{i})|\mathbf{x}) \doteq 2^{nB \log \sin\theta_n}$$



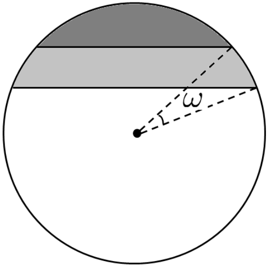
Isoperimetric Inequalities



Isoperimetric Inequality in the Plane (Steiner 1838)

Among all closed curves in the plane with a given enclosed area, the circle has the smallest perimeter.

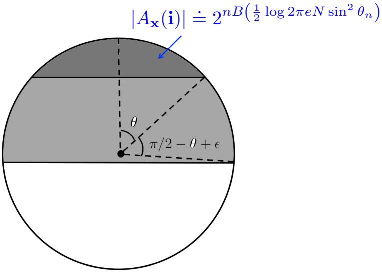
Isoperimetric Inequalities



Isoperimetric Inequality on the Sphere (Levy 1919)

Among all sets on the sphere with a given volume, the spherical cap has the smallest boundary, or the smallest volume of ω-neighborhood for any $\omega > 0$.

Blowing-up Lemma



Isoperimetric Inequality on the Sphere + Measure Concentration:

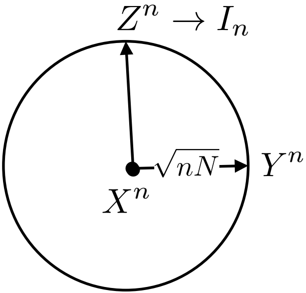
$$\mathbb{P}(\mathbf{Z} \in \text{blow-up of } A_{\mathbf{x}}(\mathbf{i})|\mathbf{x}) \approx 1.$$

$$\Downarrow$$

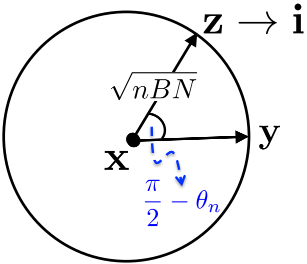
$$\mathbb{P}(\mathbf{Y} \in \text{blow-up of } A_{\mathbf{x}}(\mathbf{i})|\mathbf{x}) \approx 1.$$

Geometry of Typical Sets

- *n*-dimensional space:
  Almost all (X^n, Y^n, Z^n, I_n)



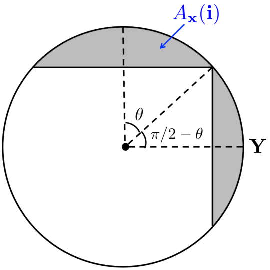
- *nB*-dimensional space:
  Almost all $(\mathbf{x}, \mathbf{y}, \mathbf{i})$

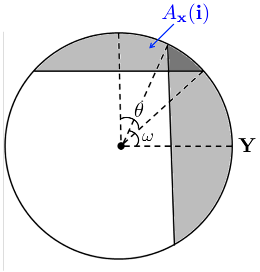


Information Inequality: (Wu and Ozgur, 2015)

$$H(I_n|Y^n) \leq n(2\log\sin\theta_n + \sqrt{2\log\sin\theta_n \ln 2}\log e).$$

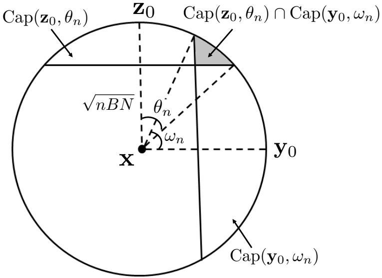
A new approach





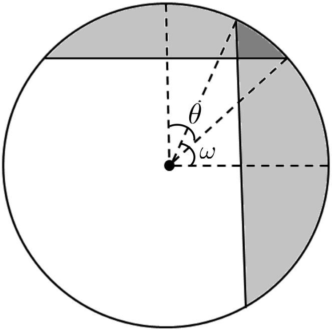
Control the intersection of a sphere drawn around a randomly chosen $\mathbf{Y}$ and $A_{\mathbf{x}}(\mathbf{i})$.

Easy if $A_{\mathbf{x}}(\mathbf{i})$ is a spherical cap



$$|\text{Cap}(\mathbf{z}_0, \theta_n) \cap \text{Cap}(\mathbf{Y}, \omega_n)| \doteq 2^{nB\left(\frac{1}{2}\log 2\pi e N(\sin^2\theta_n - \cos^2\omega_n)\right)}$$

Strengthening of the Isoperimetric Inequality

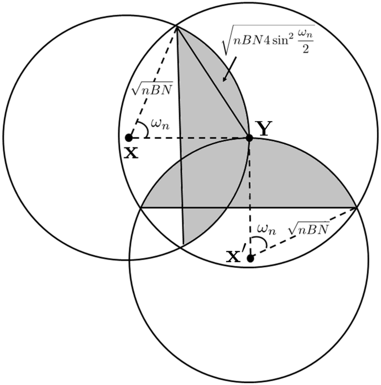


Strengthening of the Isoperimetric Inequality:

Among all sets on the sphere with a given volume, the spherical cap has minimal intersection volume at distance ω for almost all points on the sphere for any $\omega > \pi/2 - \theta$.

Proof: builds on the Riesz rearrangement inequality.

A Packing Argument



$$\text{Given } \mathbf{Y}, \# \text{ of } \mathbf{I} \leq \frac{\left|\text{Sphere}\left(\mathbf{Y}, \sqrt{nBN4\sin^2\frac{\omega_n}{2}}\right)\right|}{2^{nB\left(\frac{1}{2}\log 2\pi e N(\sin^2\theta_n - \cos^2\omega_n)\right)}}$$

Summary

- Solved an problem posed by Cover and named "The Capacity of Relay Channel" in *Open Problems in Communication and Computation*, Springer-Verlag, 1987.
- Developed a converse technique that significantly deviates from standard converse techniques based on single-letterization and has some new ingredients:
  - Typicality
  - Measure Concentration
  - Isoperimetric Inequality