# Robust Model Predictive Control

## (A Story of Tube Model Predictive Control)

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ISR @ UMD College Park, February 22, 2016

# Part

Model Predictive Control (MPC)

Robust Model Predictive Control (RMPC)

Tube Model Predictive Control (TMPC)

Trends & Directions

Closing

# MPC Analogy

## Jean Piaget (1896 – 1980)

Cognitive Psychology

Children learning and environment controlling

**1. Image**

**2. Aim**

**3. Action**

**4. Collation**

# MPC Analogy

## Jacques Richalet (1936 – )

Predictive Functional Control

Credits for Brilliant Analogy

**1. Image**

**1. Model**

**2. Aim**

**2. Reference**

**3. Action**

**3. Control**

**4. Collation**

**4. Feedback**

# MPC Paradigm

Goals:

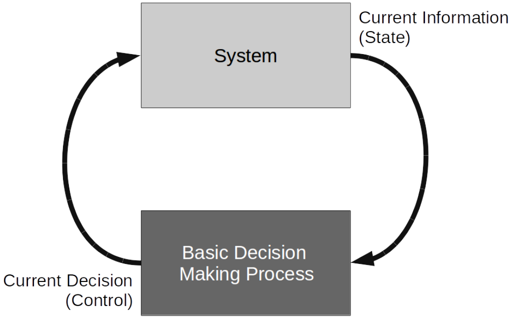
Constraint satisfaction,

Stability, and

Optimized performance.

Tool:

Model predictive control.




Model predictive control (MPC):

Repetitive decision making process (DMP).

Basic DMP is finite horizon optimal control.

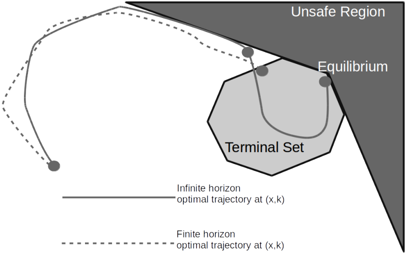
# Basic DMP (Finite Horizon Optimal Control)

Given an integer $N \in \mathfrak{N}$ and a state $x \in \mathbb{X}$ select predicted sequences of

control actions $\mathbf{u}_{N-1} := \{u_0, u_1, \ldots, u_{N-1}\}$, and

controlled states $\mathbf{x}_N := \{x_0, x_1, \ldots, x_{N-1}, x_N\}$,

which, for each $k \in \{0, 1, \ldots, N-1\}$, satisfy




$x_{k+1} = f(x_k, u_k)$ with $x_0 = x$,

$x_k \in \mathbb{X}$,

$u_k \in \mathbb{U}$, and

$x_N \in \mathbb{X}_f$ ,

and which minimize $V_N(\mathbf{x}_N, \mathbf{u}_{N-1}) := \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N)$ .

# Key Facts

Main properties:

- MPC law $u_0^0(\cdot)$ is feedback implicitly evaluated at current state.
- Predictions and optimized predictions are, however, open–loop.
- Consistently improving and stabilizing (under mild assumptions).

Theoretical implementation:

- Mathematical (nonlinear) programming in general case.
- Strictly convex programming in most frequent cases.

Practical implementation:

- Online optimization.
- Offline parameteric optimization and online look–up tables.
- Combinations of the online and offline parameteric optimization.

# Part

Model Predictive Control (MPC)

Robust Model Predictive Control (RMPC)

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# RMPC Paradigm

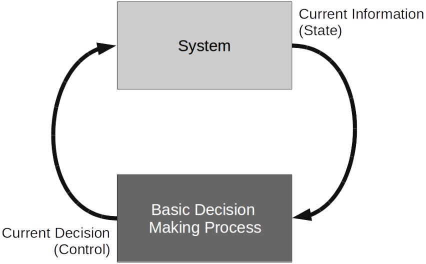
Goals:

Robust constraint satisfaction,

Robust stability,

Optimized robust performance, and

Computational practicability.




Tool:

Robust model predictive control.

Robust model predictive control (RMPC):

Repetitive decision making process (DMP).

Basic DMP is finite horizon robust optimal control.

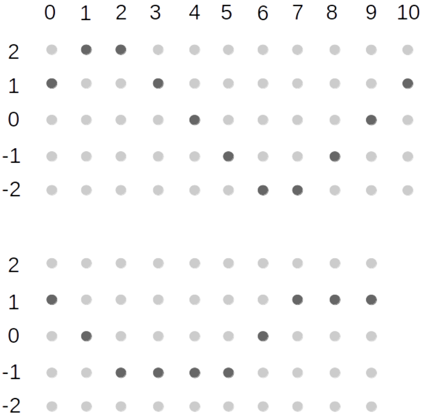
# Pivotal Concerns

- Intricate interaction of uncertainty with:
  - System evolution,
  - Constraints, and
  - Performance.

- Fragility (non–robustness) of conventional MPC.

- Convoluted interplay between:
  - Quality of guaranteed structural properties and
  - Complexity of associated computational methods.

# Predicting Without Uncertainty ($x^+ = x + u$)




States $x_k$ depend on:

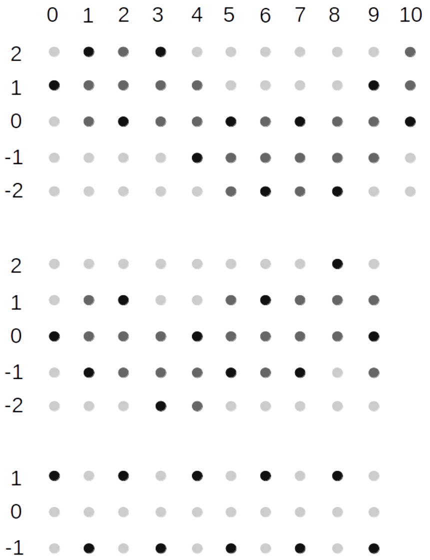
initial state $x_0$, and

controls $u_0, u_1, \ldots, u_{k-1}$.

Controls $u_k$ depend on:

initial state $x_0$.

# Predicting Under Uncertainty ($x^+ = x + u + w$)




States $x_k$ depend on:

initial state $x_0$,

controls $u_0, u_1, \ldots, u_{k-1}$, and

disturbances $w_0, w_1 \ldots, w_{k-1}$.

Controls $u_k$ depend on:

current state $x_k$.

Disturbances $w_k$ are independent.

# Fragility of Conventional MPC

Key issues (mostly due to state constraints):

- Asymptotic stability needs not be a robust property (Teel),
- Optimal control of a continuous control system might induce a discontinuous controlled dynamics, and
- Optimal control might be a fragile process itself (Raković).

Message:

- There's no thing such as a free lunch.
- Ensure robustness by design rather than hoping to get it for free.

# Dynamic Programming Based RMPC

## Richard E. Bellman (1920 – 1984)

Dynamic Programming

Curse of Dimensionality

## Minimax DP Recursion (with boundary conditions $V_0(\cdot) := V_f(\cdot)$ and $\mathbb{X}_0 := \mathbb{X}_f$)

Max value functions $J_k(\cdot)$:
$\forall (x,u) \in \mathbb{X}_k \times \mathbb{U},\ J_k(x,u) = \max_w \{\ell(x,u,w) + V_{k-1}(f(x,u,w))\ :\ w \in \mathbb{W}\}.$

Minimax value functions $V_k(\cdot)$:
$\forall x \in \mathbb{X}_k,\ V_k(x) = \min_u \{J_k(x,u)\ :\ u \in \mathbb{U}\ \wedge\ \forall w \in \mathbb{W},\ f(x,u,w) \in \mathbb{X}_{k-1}\}.$

Minimax optimal control laws $u_k(\cdot)$ are the optimizers of the minimax value functions:
$\forall x \in \mathbb{X}_k,\ u_k(x) = \arg\min_u \{J_k(x,u)\ :\ u \in \mathbb{U}\ \wedge\ \forall w \in \mathbb{W},\ f(x,u,w) \in \mathbb{X}_{k-1}\}.$

Domains of the minimax value functions $\mathbb{X}_k$ are the minimax controllability sets:
$\mathbb{X}_k = \mathcal{F}^{-1}(\mathbb{X}_{k-1})$ *where* $\mathcal{F}^{-1}(X) = \{x \in \mathbb{X}\ :\ \exists u \in \mathbb{U},\ \forall w \in \mathbb{W},\ f(x,u,w) \in X\}.$

For DP based RMPC, one makes (repetitive) use of (a selection of) $u_N(\cdot)$ and $V_N(\cdot)$ over $\mathbb{X}_N$.

# Closed–Loop RMPC

System:

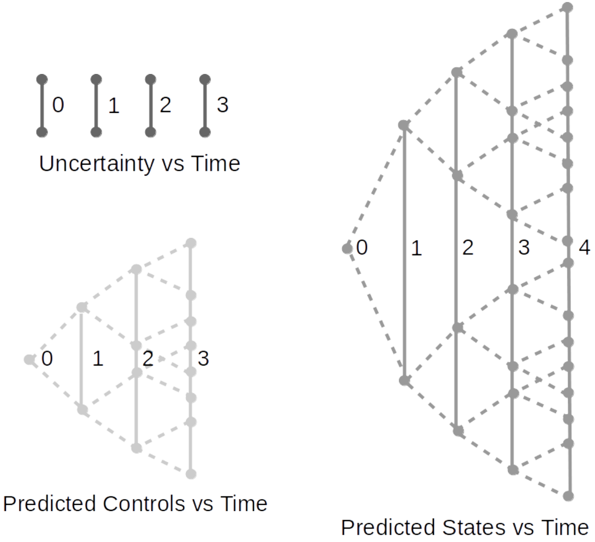
$x^+ = x + u + w$

Uncertainty:

$w \in [-1, 1]$

Prediction horizon:

$N = 4$




Closed–Loop (or brute force scenarios based) RMPC is clearly intractable!

System:

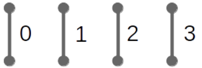
$x^+ = x + u + w$

Uncertainty:

$w \in [-1, 1]$

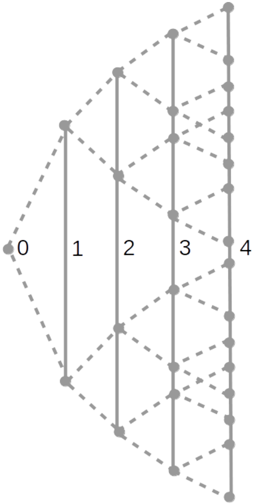
Prediction horizon:

$N = 4$




Uncertainty vs Time


Predicted Controls vs Time




Predicted States vs Time

Open–Loop (or a careless man crossing street) RMPC is clearly insensitive!

# Breaking Down Complexity

- What RMPC a rational and intelligent man should be happy with?
  - Improved computability w.r.t. DP based and closed–loop RPMC.
  - Improved sensibility w.r.t. conventional and open–loop RMPC.
  - And anything else on top of that as a bonus.

- Reconsider the whole approach to MPC under uncertainty.
  (In the spirit of "design the whole and then its parts".)

- Two key steps for simplifying complexity:
  - Utilization of parameterized predictions under uncertainty.
  - Acceptance of generalized notions and natural performance criteria.

# Separable RMPC (Linear–Polytopic Setting)

Parameterization via partial

states

$$x_k = \sum_{j=0}^{k} x_{(j,k)}$$

controls

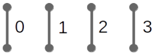
$$u_k = \sum_{j=0}^{k} u_{(j,k)}$$

dynamics

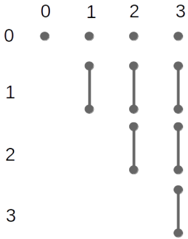
$$x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)}$$

initial conditions

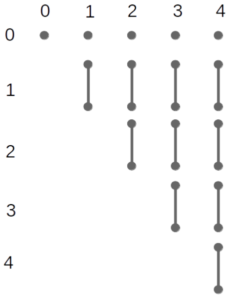
$$x_{(0,0)} = x \ \wedge \ x_{(k,k)} = w_{k-1}$$




Uncertainty vs Time




Predicted Controls vs Time




Predicted States vs Time

$$x^+ = x + u + w,$$
$$w \in [-1, 1],$$
$$N = 4.$$

Employ separable RMPC as compatible with the superposition principle!

# Separable prediction structure

| $(x, u)$–part $\diamond$ time | 0 | 1 | 2 | ... | $k$ | ... | $N-1$ | $N$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $x_{(0,0)} = x$ | $x_{(0,1)}$ | $x_{(0,2)}$ | $\cdots$ | $x_{(0,k)}$ | $\cdots$ | $x_{(0,N-1)}$ | $x_{(0,N)}$ |
| 0 | $u_{(0,0)}$ | $u_{(0,1)}$ | $u_{(0,2)}$ | $\cdots$ | $u_{(0,k)}$ | $\cdots$ | $u_{(0,N-1)}$ | |
| 1 | | $x_{(1,1)} = w_0$ | $x_{(1,2)}$ | $\cdots$ | $x_{(1,k)}$ | $\cdots$ | $x_{(1,N-1)}$ | $x_{(1,N)}$ |
| 1 | | $u_{(1,1)}$ | $u_{(1,2)}$ | $\cdots$ | $u_{(1,k)}$ | $\cdots$ | $u_{(1,N-1)}$ | |
| 2 | | | $x_{(2,2)} = w_1$ | $\cdots$ | $x_{(2,k)}$ | $\cdots$ | $x_{(2,N-1)}$ | $x_{(2,N)}$ |
| 2 | | | $u_{(2,2)}$ | $\cdots$ | $u_{(2,k)}$ | $\cdots$ | $u_{(2,N-1)}$ | |
| | | | | $\cdots$ | | | | |
| | | | | $\cdots$ | | | | |
| $k$ | | | | | $x_{(k,k)} = w_{k-1}$ | $\cdots$ | $x_{(k,N-1)}$ | $x_{(k,N)}$ |
| $k$ | | | | | $u_{(k,k)}$ | $\cdots$ | $u_{(k,N-1)}$ | |
| | | | | | | $\cdots$ | | |
| | | | | | | $\cdots$ | | |
| $N-1$ | | | | | | | $x_{(N-1,N-1)} = w_{N-2}$ | $x_{(N-1,N)}$ |
| $N-1$ | | | | | | | $u_{(N-1,N-1)}$ | |
| $N$ | | | | | | | | $x_{(N,N)} = w_{N-1}$ |
| $N$ | | | | | | | | |
| total | $x_{(0,0)}$ | $\sum_{j=0}^{1} x_{(j,1)}$ | $\sum_{j=0}^{2} x_{(j,2)}$ | $\cdots$ | $\sum_{j=0}^{k} x_{(j,k)}$ | $\cdots$ | $\sum_{j=0}^{N-1} x_{(j,N-1)}$ | $\sum_{j=0}^{N} x_{(j,N)}$ |
| total | $u_{(0,0)}$ | $\sum_{j=0}^{1} u_{(j,1)}$ | $\sum_{j=0}^{2} u_{(j,2)}$ | $\cdots$ | $\sum_{j=0}^{k} u_{(j,k)}$ | $\cdots$ | $\sum_{j=0}^{N-1} u_{(j,N-1)}$ | |

The x–rows dynamics $x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)}$ are deterministic.

For worst case cost use column–wise the $k^{th}$–(x,u)–rows.

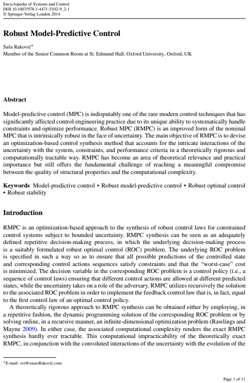
For worst case constraints use row–wise the $k^{th}$–(x,u)–columns.

# Main Existing Parameterizations in RMPC

| RMPC Method | Parameterized States | Parameterized Controls | Prediction Structure |
|---|---|---|---|
| NO–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = K \sum_{j=0}^{k} x_{(j,k)}$ | Separable |
| OL–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = u_{(0,k)}$ | Separable |
| TIASF–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = u_{(0,k)} + K \sum_{j=1}^{k} x_{(j,k)}$ | Separable |
| TVASF–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = u_{(0,k)} + \sum_{j=1}^{k} K_{(j,k)} x_{(j,k)}$ | Separable |
| APDF–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = u_{(0,k)} + \sum_{j=1}^{k} M_{(j,k)} x_{(j,j)}$ | Separable |
| SSF–RMPC | $x_k = \sum_{j=0}^{k} x_{(j,k)}$ | $u_k(x_k, x) = u_{(0,k)} + \sum_{j=1}^{k} u_{(j,k)}(x_{(j,k)})$ | Separable |
| CL–RMPC | $x_k$ | $u_k(x_k, x) = u_k(x_k)$ | Aggregated |
| DP–Based–RMPC | $x_k$ | $u_k(x_k, x) = u_k(x_k)$ | Aggregated ($\infty$) |

- OL–RMPC (Blanchini; Lee and Yu;...),
- TIASF–RMPC (Chisci and Zappa; Kouvaritakis and Cannon;...),
- TVASF–RMPC (...;Löfberg;...),
- APDF–RMPC (van Hessem and Bosgra; Löfberg; Kerrigan;...),
- SSF–RMPC (Raković; Raković, Kouvaritakis, Cannon and Panos),
- CL–RMPC (Bertsekas; Scoekert and Mayne; ...), and
- DP–Based–RMPC (Bellman; Bertsekas; Mayne;...).

# RMPC Personal References



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**Robust Model-Predictive Control**

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**Abstract**

Model-predictive control (MPC) is indisputably one of the rare modern control techniques that has significantly affected control engineering practice due to its unique ability to systematically handle constraints and optimize performance. Robust MPC (RMPC) is an improved form of the nominal MPC that is intrinsically robust in the face of uncertainty. The main objective of RMPC is to devise an optimization-based control synthesis method that accounts for the intricate interactions of the uncertainty with the system, constraints, and performance criteria in a theoretically rigorous and computationally tractable way. RMPC has become an area of theoretical relevance and practical importance but still offers the fundamental challenge of reaching a meaningful compromise between the quality of structural properties and the computational complexity.

**Keywords** Model-predictive control • Robust model-predictive control • Robust optimal control • Robust stability

**Introduction**

RMPC is an optimization-based approach to the synthesis of robust control laws for constrained control systems subject to bounded uncertainty. RMPC synthesis can be seen as an adequately defined repetitive decision-making process, in which the underlying decision-making process is a suitably formulated robust optimal control (ROC) problem. The underlying ROC problem is specified in such a way so as to ensure that all possible predictions of the controlled state and corresponding control actions sequences satisfy constraints and that the "worst-case" cost is minimized. The decision variable in the corresponding ROC problem is a control policy (i.e., a sequence of control laws) ensuring that different control actions are allowed at different predicted states, while the uncertainty takes on a role of the adversary. RMPC utilizes recursively the solution to the associated ROC problem in order to implement the feedback control law that is, in fact, equal to the first control law of an optimal control policy.

A theoretically rigorous approach to RMPC synthesis can be obtained either by employing, in a repetitive fashion, the dynamic programming solution of the corresponding ROC problem or by solving online, in a recursive manner, an infinite-dimensional optimization problem (Rawlings and Mayne 2009). In either case, the associated computational complexity renders the exact RMPC synthesis hardly ever tractable. This computational impracticability of the theoretically exact RMPC, in conjunction with the convoluted interactions of the uncertainty with the evolution of the

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**Invention of Prediction Structures and Categorization of Robust MPC Syntheses** ⋆

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**Abstract:** This plenary paper is concerned with robust model predictive control (MPC) synthesis. In particular, the novel notion of prediction structures is introduced, and then utilized to derive a precise and compact overview of the existing robust MPC (RMPC) syntheses as well as to indicate direct improvements of their system theoretic properties. The prediction structures paradigm allows for a systematic, implementation and examples independent, comparison and classification of the existing RMPC syntheses. The corresponding categorization of the currently available RMPC syntheses is derived by: (*i*) introducing the adequate indices as measures of structural (including topological and system theoretic) properties and computational complexity, and (*ii*) analysing the trade–off between the guaranteed structural properties and the necessary computational complexity. The associated analysis is based on the classical game and utility theory notions; for simplicity, it is delivered by deploying the aggregated structural property and computational complexity indices; furthermore, it is complemented with an unambiguous analysis of the "holy–grail" trade–off between the quality of structural properties and the degree of computational complexity.

*Keywords:* Robust Model Predictive Control, Prediction Structures, Parameterized Tubes.

1. INTRODUCTION

Robust model predictive control is an area of theoretical relevance and practical importance (Rawlings and Mayne, 2009). The field has attracted significant research attention over the last few decades. Nevertheless, the area still offers the fundamental challenge of reaching a reasonable compromise between computational practicability and quality of both topological and system theoretic properties. From a theoretical point of view, RMPC synthesis is an adequately defined repetitive decision making process, in which the underlying decision making process is a suitably formulated robust optimal control (ROC) problem. Thus, theoretically exact RMPC synthesis can be attained by employing, in a repetitive fashion, the dynamic programming (Bellman, 1957) solution of the underlying ROC problem (Mayne et al., 2000; Bertsekas, 2007). Unfortunately, the associated computational complexity is, in general, impracticable. It is, hence, rather tempting to aim to develop the methods which yield directly approximate solutions (Jones et al., 2007) to the underlying exact ROC problem. However, this approach is unwieldy, since the inherited complexity of the traditionally employed minimax optimization (Scokaert and Mayne, 1998; Kerrigan and Maciejowski, 2003) is bound to induce a reasonably high degree of computational complexity to any approximation based methodology that would guarantee reasonable system theoretic properties. All in all, the prohibitive computational complexity of the exact RMPC synthesis (or its direct minimax optimization based approximations) has motivated the development of alternative approaches.

The existing approaches to RMPC can be divided broadly into two categories depending on the treatment of the uncertainty and its interactions with evolution of the system, constraints and performance criteria. The first category of alternatives includes the methods that aim to employ the inherent robustness, when plausible, of deterministic MPC. These methods employ essentially deterministic MPC synthesis, albeit applied to a suitably modified description of the system, constraints and performance criteria, and utilize the notion of input–to–state stability. This category of the alternative approaches to RMPC results potentially in computationally practicable methods. However, the presence of uncertainty is taken care of in an indirect way, and furthermore such approaches seem to be limited since the deterministic MPC is itself an inherently fragile (non–robust) process. In this sense, it is known that the stability property of deterministic MPC is non–robust (Grimm et al., 2004), but the situation is, in fact, even worse since the optimal control of constrained discrete time systems is a fragile process itself (Raković, 2009). A more detailed overview of this category of RMPC synthesis methods can be found in (Raimondo et al., 2009). The second category of alternative approaches to RMPC includes the methods that account for the effects of the uncertainty directly, but aim to exploit suitable parameterization of the underlying control policy in order to reduce the corresponding computational complexity (Blanchini, 1990; Chisci et al., 2001; Lõfberg, 2003b,a; Langson et al., 2004; Mayne et al., 2005; S. V. Raković, 2005; Goulart et al., 2006; Raković et al., 2012d). This category of approaches has been developed compatibly with a commonly accepted consensus: *there is a need for simplifying approximations of the underlying control policy and for cost functions that*

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What delayed the use of the superposition principle for design of RMPC?

# Part

Model Predictive Control (MPC)

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Trends & Directions

Closing

# TMPC Paradigm

- The state tubes are sequences of sets of possible states.
- The control tubes are sequences of sets of possible controls.
- State and control tubes play role of state and control sequences.
- Tubes are induced from the dynamics, uncertainty and control policy.
- Optimal tubes are obtained via tube optimal control.
- TMPC is repetitive online utilization of related tube optimal control.
- All RMPC methods result in tubes.

Parameterization of tubes and control policy is of major importance.

# TMPC Methods

- Rigid TMPC (2002 – 2006)
  with D. Q. Mayne and involving some collaborations.

- Homothetic TMPC (2007 – 2009)
  involving some collaborations.

- Parameterized TMPC (2007 – 2010)
  involving some collaborations.

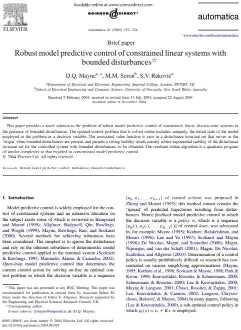
- Elastic TMPC (2015 – 2016)
  with W. S. Levine and B. Açikmeșe.

# Rigid TMPC Key Features

- Parameterizations
  - States $x_k = z_k + s_k$.
  - Controls $u_k = v_k + Ks_k$.
  - "Nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - "Local" dynamics $s_{k+1} = (A + BK)s_k + w_k$.
- Tubes (with $S(1) := \{x \ : \ Cx \leq 1\}$)
  - State cross–sections $X_k = z_k \oplus S(1)$.
  - Control cross–sections $U_k = v_k \oplus KS(1)$.
  - "Nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - "Local" dynamics $(A + BK)S(1) \oplus \mathbb{W} \subseteq S(1)$.

# Rigid TMPC References

State and, observer based, output feedback rigid TMPC.


Available online at www.sciencedirect.com

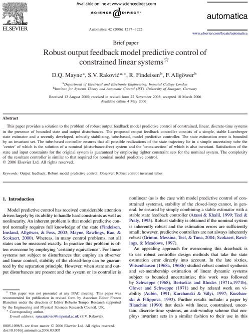
automatica

Automatica 41 (2005) 219–224

Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances

D.Q. Mayne, M.M. Seron, S.V. Raković


Available online at www.sciencedirect.com

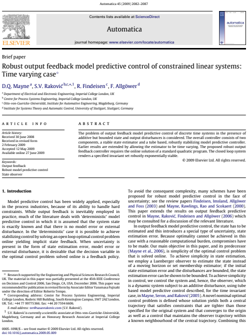
automatica

Automatica 42 (2006) 1217–1222

Brief paper

Robust output feedback model predictive control of constrained linear systems

D.Q. Mayne, S.V. Raković, R. Findeisen, F. Allgöwer


Automatica 45 (2009) 2082–2087

Automatica

Brief paper

Robust output feedback model predictive control of constrained linear systems: Time varying case

D.Q. Mayne, S.V. Raković, R. Findeisen, F. Allgöwer

A few other papers supporting strongly the methodology.

# Homothetic TMPC Key Features

- Parameterizations
  - States $x_k = z_k + s_k$.
  - Controls $u_k = v_k + Ks_k$.
  - Decoupled "nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - Decoupled "local" dynamics $s_{k+1} = (A + BK)s_k + w_k$.
- Tubes (with $S(\alpha) := \{x \ : \ Cx \leq \alpha 1\}$ for $\alpha \in \mathfrak{R}_{\geq 0}$)
  - State cross–sections $X_k = z_k \oplus S(\alpha_k)$.
  - Control cross–sections $U_k = v_k \oplus KS(\alpha_k)$.
  - Decoupled "nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - Decoupled "local" dynamics $(A + BK)S(\alpha_k) \oplus \mathbb{W} \subseteq S(\alpha_{k+1})$.
  - Coupled dynamics $Az_k + Bv_k \oplus (A + BK)S(\alpha_k) \oplus \mathbb{W} \subseteq z_{k+1} \oplus S(\alpha_{k+1})$.

# Homothetic TMPC References

Approximate reachability and state feedback homothetic TMPC.

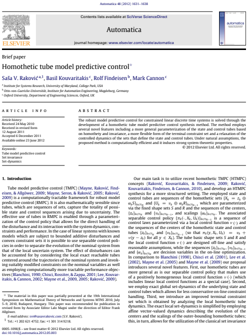

Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Approximate Reachability Analysis for Linear Discrete Time Systems Using Homothety and Invariance

Saša V. Raković, Mirko Fiacchini

Keywords: Reachability Analysis, Approximations, Homothety, Set Invariance.

1. INTRODUCTION


Automatica 48 (2012) 1631–1638

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Automatica

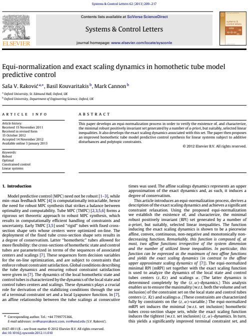
journal homepage: www.elsevier.com/locate/automatica

Brief paper

Homothetic tube model predictive control

Saša V. Raković, Basil Kouvaritakis, Rolf Findeisen, Mark Cannon

1. Introduction


Systems & Control Letters 62 (2013) 209–217

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Equi-normalization and exact scaling dynamics in homothetic tube model predictive control

Saša V. Raković, Basil Kouvaritakis, Mark Cannon

1. Introduction

# Parameterized TMPC Key Features

- Parameterizations
  - States $x_k = \sum_{j=0}^{k} x_{(j,k)}$.
  - Controls $u_k = \sum_{j=0}^{k} u_{(j,k)}$.
  - Partial deterministic dynamics $x_{(j,k+1)} = Ax_{(j,k)} + Bu_{(j,k)}$.
  - Partial initial conditions $x_{(0,0)} = x \ \wedge \ x_{(k,k)} = w_{k-1}$.
- Minkowski decomposable tubes (free and optimized online)
  - State cross–sections $X_k = x_{(0,k)} \oplus \bigoplus_{j=1}^{k} X_{(j,k)}$.
  - Partial state cross–sections $X_{(j,k)} = \text{convh}(\{x_{(i,j,k)} \ : \ i \in \mathcal{I}\})$.
  - Control cross–sections $U_k = u_{(0,k)} \oplus \bigoplus_{j=1}^{k} U_{(j,k)}$.
  - Partial control cross–sections $U_{(j,k)} = \text{convh}(\{u_{(i,j,k)} \ : \ i \in \mathcal{I}\})$.
  - Partial extreme deterministic dynamics $x_{(i,j,k+1)} = Ax_{(i,j,k)} + Bu_{(i,j,k)}$.

# Parameterized TMPC References

Robust control invariance and state feedback parameterized TMPC.


IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 55, NO. 7, JULY 2010

## Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks

Saša V. Raković and Miroslav Barić


IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 57, NO. 11, NOVEMBER 2012

## Parameterized Tube Model Predictive Control

Saša V. Raković, Basil Kouvaritakis, Mark Cannon, Christos Panos, and Rolf Findeisen


INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL

## Fully parameterized tube model predictive control

Saša V. Raković, Basil Kouvaritakis, Mark Cannon and Christos Panos

# Elastic TMPC Key Features

- Parameterizations
  - States $x_k = z_k + s_k$.
  - Controls $u_k = v_k + K(a_k)s_k$.
  - Decoupled "nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - Decoupled "local" dynamics $s_{k+1} = (A + BK(a_k))s_k + w_k$.
- Tubes (with $S(a) := \{x \ : \ Cx \leq a\}$ for $a \in \mathfrak{R}^q_{\geq 0}$)
  - State cross–sections $X_k = z_k \oplus S(a_k)$.
  - Control cross–sections $U_k = v_k \oplus K(a_k)S(a_k)$.
  - Decoupled "nominal" dynamics $z_{k+1} = Az_k + Bv_k$.
  - Decoupled "local" dynamics $(A + BK(a_k))S(a_k) \oplus \mathbb{W} \subseteq S(a_{k+1})$.
  - Coupled dynamics
    $Az_k + Bv_k \oplus (A + BK(a_k))S(a_k) \oplus \mathbb{W} \subseteq z_{k+1} \oplus S(a_{k+1})$.

State feedback elastic TMPC.

## Elastic Tube Model Predictive Control

Saša V. Raković, William S. Levine and Behçet Açıkmeşe

***Abstract*— This paper introduces elastic tube model predictive control (MPC) synthesis. The proposed framework is a natural generalization of the rigid and homothetic tube MPC design methods. The cross–sections of the employed state and control tubes are allowed to change more elastically, while the local component of the tubes control policy is permitted to take a more general form. The related stabilizing terminal conditions are also adequately generalized in order to take advantage of more flexible tubes and tubes control policy parameterizations. These novel features result in an improved tube MPC at a cost of manageable increase in computational complexity.**

### I. Introduction

Due to its unique ability to systematically handle constraints and optimize performance, MPC has become an elemental and contemporary research field that has seen important advances in underlying theory [1]–[3] as well as numerous industrial applications [4]. Robust MPC (RMPC) is an improved MPC variant providing structural properties (e.g. performance, invariance and stability) that are robust in face of the bounded uncertainty. A real intricacy in RMPC arises due to the facts that the closed loop RMPC [5] provides strong structural properties but it is computationally unwieldy, while the conventional MPC is not necessarily robust [6] even though it is computationally convenient. Being one of MPC's fundamental subfields, RMPC has been the subject of intensive research investigations [5]–[17]. See also comprehensive monographs [2], [18], main survey papers [1], [3] and encyclopedic and plenary articles [19], [20] for an in–depth overview of the current state of affairs in RMPC and for a number of additional relevant references.

Tube MPC (TMPC) has emerged as a dominant design framework for RMPC, since it addresses effectively the fundamental challenge of reaching a meaningful compromise between the quality of guaranteed structural properties and the associated computational complexity. TMPC considers predicted behaviour in terms of the sets of possible states and controls due to the spread of trajectories caused by the uncertainty. This results in state and control tubes that represent either the exact or outer bounding sequences of the sets of possible states and associated controls. The sensible parameterizations of the state and control tubes and related tubes control policy lead to computationally highly attractive TMPC that induces strong structural properties. TMPC is particularly effective for constrained linear systems subject to additive uncertainty [10], [11], [14]–[16].

Rigid TMPC (RTMPC) [11] is the first generation of TMPC. The cross–sections of the state and control tubes in RTMPC are simple translations of the fixed cross–section basic shape sets $S(1)$ and $K(1)S(1)$ so that $X_k = z_k \oplus S(1)$ and $U_k = v_k \oplus K(1)S(1)$ where $z_k$ and $v_k$ are the centers of the state and control tubes. The related control laws take form $\pi_k(x_k, z_k, v_k) = v_k + K(1)(x_k - z_k)$. Homothetic TMPC (HTMPC) [14], [15] is the second generation of TMPC. The cross–sections of the state and control tubes in HTMPC are homothetic copies of the fixed cross–section basic shape sets $S(1)$ and $K(1)S(1)$ so that $X_k = z_k \oplus \alpha_k S(1)$ and $U_k = v_k \oplus \alpha_k K(1)S(1)$ where scalar $\alpha_k$ is the scaling factor of the state and control tubes. Parameterized TMPC (PTMPC) [16], [19] is the third generation of TMPC. The cross–sections of the state and control tubes in PTMPC are expressed in terms of partial cross–sections so that $X_k = X_{(0,k)} \oplus X_{(1,k)} \oplus \ldots \oplus X_{(k,k)}$ and $U_k = U_{(0,k)} \oplus U_{(1,k)} \oplus \ldots \oplus U_{(k,k)}$ where sets $X_{(j,k)}$ and $U_{(j,k)}$ are paramaterized via finitely many points. The related control laws $\pi_k(x_k, X_k, U_k)$ are separable and nonlinear functions composed of partial control rules $\pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)})$ so that $\pi_k(x_k, X_k, U_k) = \sum_{j=0}^{k} \pi_{(j,k)}(x_{(j,k)}, X_{(j,k)}, U_{(j,k)})$ for $x_k = \sum_{j=0}^{k} x_{(j,k)}$ with $x_{(j,k)} \in X_{(j,k)}$. The online implementations of RTMPC, HTMPC and PTMPC reduce to convex optimization for which the number of decision variables and constraints scales linearly w.r.t. the prediction horizon in cases of RTMPC and HTMPC, and quadratically in case of PTMPC. RTMPC [11] induces strong system theoretic properties that are, however, weaker than those guaranteed by HTMPC [14], [15]. Both RTMPC and HTMPC are considerably outperformed by PTMPC [16], [19]. As a matter of fact, PTMPC outperforms all major methods for RMPC such as those based on the constraint tightening with or without prestabilization [7], [8] or on the use of affine–in–the–predicted–states and affine–in–the–past–disturbances control policies [9], [12].

The main aim of our proposal is to improve considerably RTMPC and HTMPC design methods at a cost of controlled increase in computational complexity. This goal is achieved by introducing more flexible parameterizations of the state and control tubes and related tubes control policy. The cross–sections $X_k$ of the state tubes are parameterized in terms of the centers $z_k$ and vector–valued elasticity parameters $a_k$ as $X_k = z_k \oplus S(a_k)$. Likewise, the cross–sections $U_k$ of the control tubes are parameterized in terms of the centers $v_k$ and elasticity parameters $a_k$ as $U_k = v_k \oplus K(a_k)S(a_k)$. The related tubes control policy is formed from the control laws $\pi_k(x_k, z_k, v_k, a_k) = v_k + K(a_k)(x_k - z_k)$. In the spirit of RTMPC and HTMPC, the suitable set–valued function $S(\cdot)$, with values $S(a)$, and matrix–valued function $K(\cdot)$,

Saša V. Raković and Behçet Açıkmeşe are with the University of Texas at Austin, USA. William S. Levine is with the University of Maryland at College Park, USA.



**Elastic Tube Model Predictive Control**

Saša V. Raković (1), William S. Levine (2), Behçet Açıkmeşe (3),

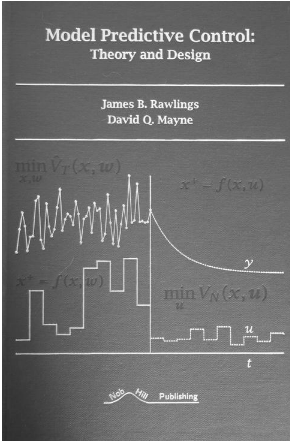
(1) The University of Texas, Austin, USA (2) The University of Maryland, College Park, USA (3) The University of Washington, Seattle, USA

This paper introduces elastic tube model predictive control (MPC) synthesis. The proposed framework is a natural generalization of the rigid and homothetic tube MPC design methods. The cross–sections of the employed state and control tubes are allowed to change more elastically, while the local component of the tubes control policy is permitted to take a more general form. The related stabilizing terminal conditions are also adequately generalized in order to take advantage of more flexible tubes and tubes control policy parameterizations. These novel features result in an improved tube MPC at a cost of manageable increase in computational complexity.



Conference version out. Journal version in progress.

# TMPC Methods in Books

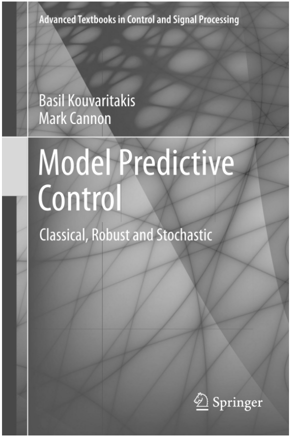



Includes my work with D. Q. Mayne on

- Rigid state feedback TMPC.
- Rigid, observer based, output feedback TMPC.
- Time–varying, observer based, output feedback TMPC.

And a few other results that I developed.

# TMPC Methods in Books




Includes my work on

- Rigid TMPC (with D. Q. Mayne).
- Homothetic TMPC.
- Parameterized TMPC.

And a few other results that I developed.

# Part

# Present Trends

- Scenarios based "quasi–robust" MPC.
- Real–time RMPC (Real–time TMPC).
- Stochastic MPC.
- Decentralized MPC under uncertainty.
- Networked MPC under uncertainty.
- Economic MPC under uncertainty.

And many more, MPC has seen tremendous expansion.

# Potential Directions

- Modelling uncertainty for MPC.
- Contemporary uncertainty in MPC.
- Resilient MPC.
- Fault–tolerant MPC.
- MPC for autonomous systems.
- Collaboratively adaptive MPC.

And many more classical and contemporary directions, since MPC is applicable to a wide range of conventional and modern areas.

# A big picture in MPC

- Integration of identification and MPC
  (e.g., Adaptive MPC).
- Integration of uncertainty modelling and MPC
  (e.g., Flexible MPC under uncertainty).
- Integration of estimation and MPC
  (e.g., Output feedback MPC).
- Integration of fault tolerance and MPC
  (e.g., Reconfigurable and actively fault tolerant MPC).
- Integration of MPC's general components and optimization
  (i.e., Integrated MPC synthesis).

Make sure that the sum of parts is equal to the whole!

# Part

Model Predictive Control (MPC)

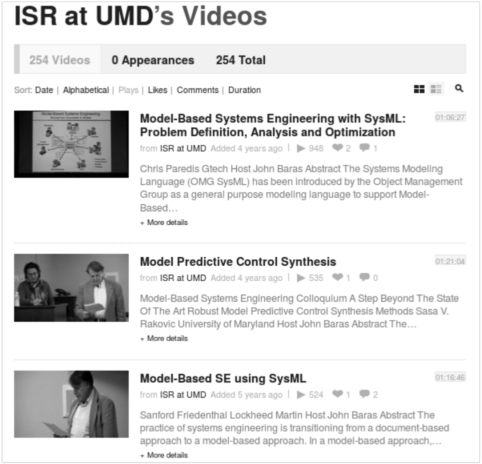
Robust Model Predictive Control (RMPC)

Tube Model Predictive Control (TMPC)

Trends & Directions

Closing

# Last Time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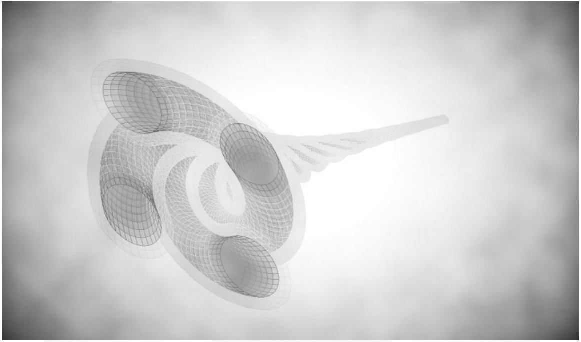
Model-Based Systems Engineering with SysML: Problem Definition, Analysis and Optimization
01:06:27
from ISR at UMD Added 4 years ago | 948 2 1
Chris Paredis Gtech Host John Baras Abstract The Systems Modeling Language (OMG SysML) has been introduced by the Object Management Group as a general purpose modeling language to support Model-Based…
+ More details
Model Predictive Control Synthesis
01:21:04
from ISR at UMD Added 4 years ago | 535 1 0
Model-Based Systems Engineering Colloquium A Step Beyond The State Of The Art Robust Model Predictive Control Synthesis Methods Sasa V. Rakovic University of Maryland Host John Baras Abstract The…
+ More details
Model-Based SE using SysML
01:16:46
from ISR at UMD Added 5 years ago | 524 1 2
Sanford Friedenthal Lockheed Martin Host John Baras Abstract The practice of systems engineering is transitioning from a document-based approach to a model-based approach. In a model-based approach,…
+ More details

# This Time

More of a story telling and a slightly different angle!



Questions are, as always, welcome!

# ACC 2016 Events

- Double Invited Session
  "MPC, Quo Vadis?".
  - with W. S. Levine, B. Açikmeșe and I. V. Kolmanovsky
  - 12 papers by well–known contributors in MPC.
- Workshop
  "MPC Under Uncertainty: Theory, Computations and Applications".
  - with W. S. Levine, B. Açikmeșe and I. V. Kolmanovsky
  - Concise and unifying tutorial to MPC under uncertainty.