University of Maryland, Dec 2016

Distributed Optimization of Continuous-time Multi-agent Networks

Yiguang Hong

Academy of Mathematics & Systems Science
Chinese Academy of Sciences

Outline

1. Background
2. Formulation
3. Continuous-time optimization
   1. Fundamental problem
   2. Optimization for physical systems
4. Conclusions

1. Background

Convex optimization: min $f(y)$, $y \in R^m$

where f is convex

If $f(y)$ is differentiable, the discrete-time dynamics:

$$y(t+1) = y(t) - k\,\nabla f(y(t)), \quad k>0$$

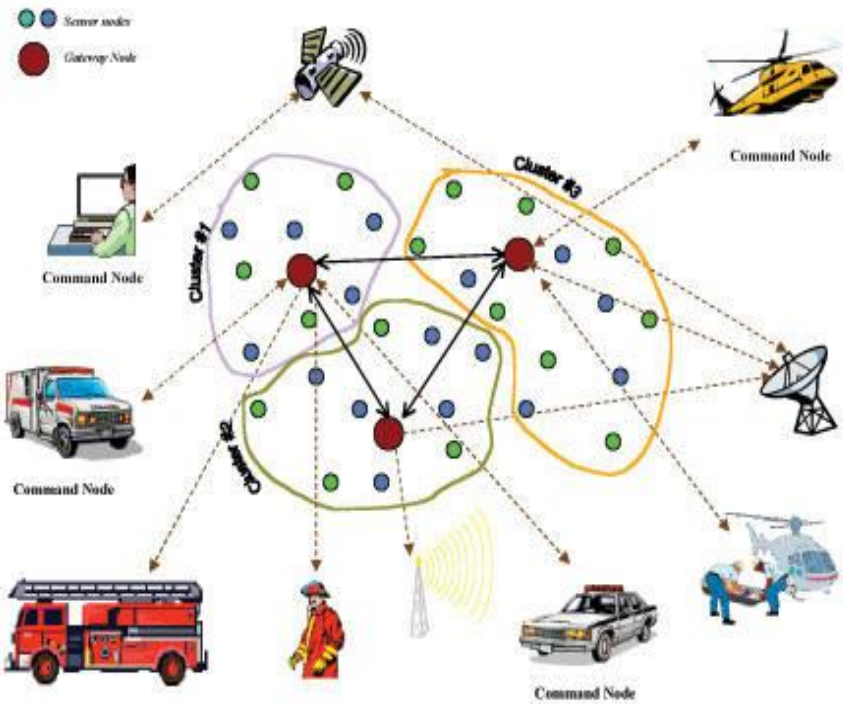
or continuous-time dynamics:

$$\dot{y}(t) = -k\,\nabla f(y(t)), \quad k>0$$

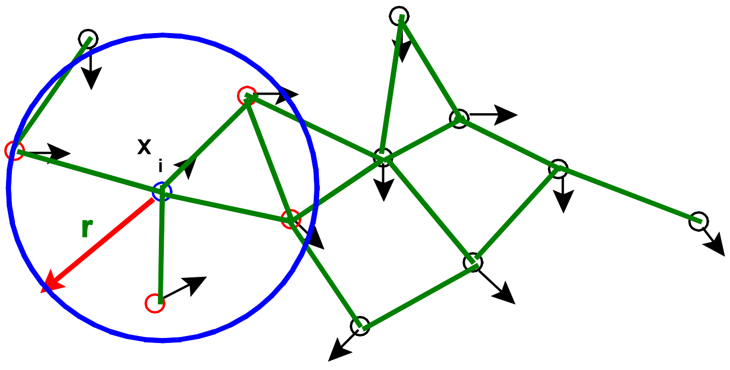
Distributed design

Agent → Multi-agent systems (MAS): distributed design in a network without a center

- Network topology and Information flow
- Design of protocols and algorithms
- Complexity (unbalanced, uncertain, asynchronous, heterogeneous, …)



Consensus: the basic problem



Neighbor Graph

Agent dynamics: $dx_i/dt = u_i \;\; i = 1,\ldots 2$
Leader (or desired position): x_0

Neighbor-based communication (N_i : the neighbor set of agent i)

Distributed control: $u_i = \Sigma_j (x_j - x_i),\; j \in N_i$

Multi-agent consensus (agreement, synchronization):
•**Leader-following:** $x_i - x_0 \rightarrow 0$
•**Leaderless:** $x_i - x_j \rightarrow 0$

Distributed optimization

- Distributed optimization: Optimization (task) + distributed design (consensus)
- Distributed convex optimization → distributed matrix optimization, distributed MPC and dynamic programming, …

- Applications: industry and energy (smart grids, sensor network, manufacture), economics and society (social networks, marketing, traffic), biology and ecology ……

2. Formulation

Convex optimization: $\min_{z \in R^m} \; f(z)$

with $f(z)$ convex

→Distributed version: $f(z) = \sum_{i=1}^{n} f_i(z)$

- each agent i knows its own cost function f_i or its gradient ∇f_i
- Local cost function f_i may not have the same optimal solution of f

Distribution formulation

Convex optimization: $\min_{z \in R^m} \quad f(z)$

Constraints: $g(z) \le 0$; $z \in \Omega$, with $g(z)$, Ω: convex

→ Distributed version: $f(z) = \sum_{i=1}^{n} f_i(z)$

Constraints:

- Global: known by every agent ➔ conventional one
- Local: for agent i: $g_i(z_i) \le 0$ and/or $z_i \in \Omega_i$ (with Ω= nonempty intersection of all local constraint sets Ω_i)
- Coupled: $g\,(z_1, z_2 \ldots z_n) \le 0$

Preliminaries: convex analysis

- A function $f(x)$ is **convex** if

$$f(cx+(1-c)y) \leqslant cf(x)+(1-c)f(y)$$

  for any x, y and $0<c<1$.
- It is **strictly convex** if it is convex and "=" holds iff $x=y$.
- It is **strongly convex** if it is strictly convex and there is σ such that

$$f(cx+(1-c)y) \leqslant cf(x)+(1-c)f(y)$$

$$\underline{-\tfrac{1}{2}\sigma c(1-c)||x-y||_2^2}$$

Convex set

- K is a convex set if, for $0 < \lambda < 1$

$$(1-\lambda)x+\lambda y \in K \qquad x \in K, y \in K$$

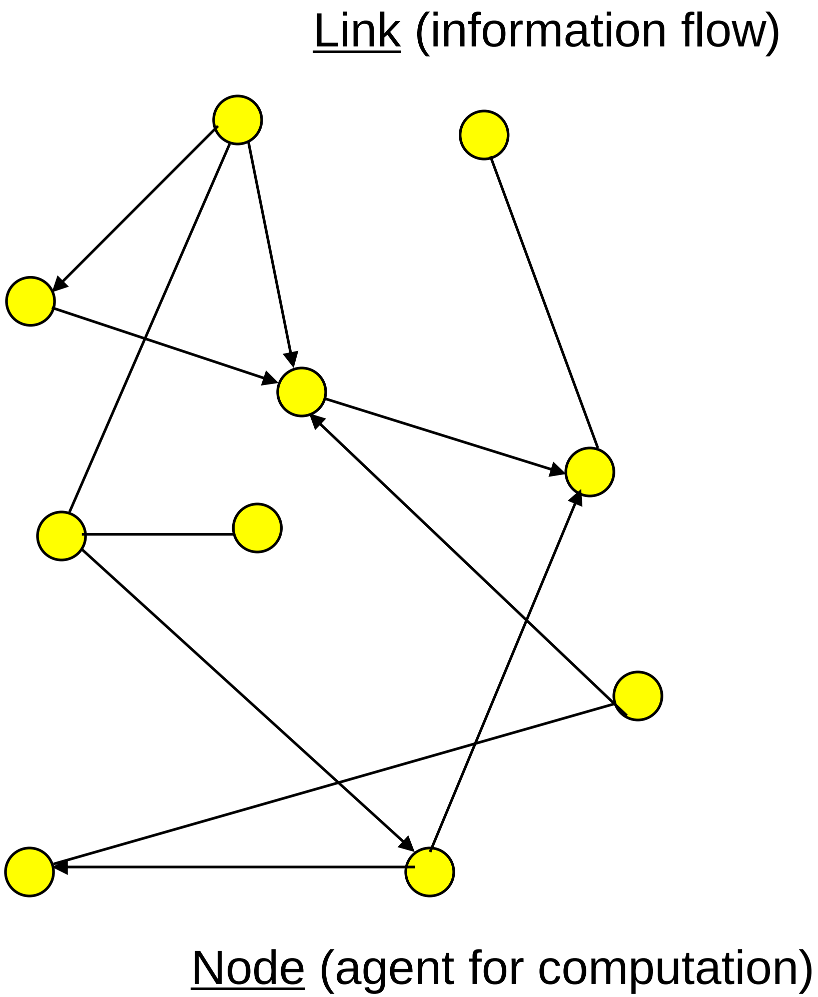
- $d(x,K)$: distance between set K and x

$$\|x\|_K \triangleq \inf\{\|x-y\| | y \in K\}$$

Preliminaries: graph

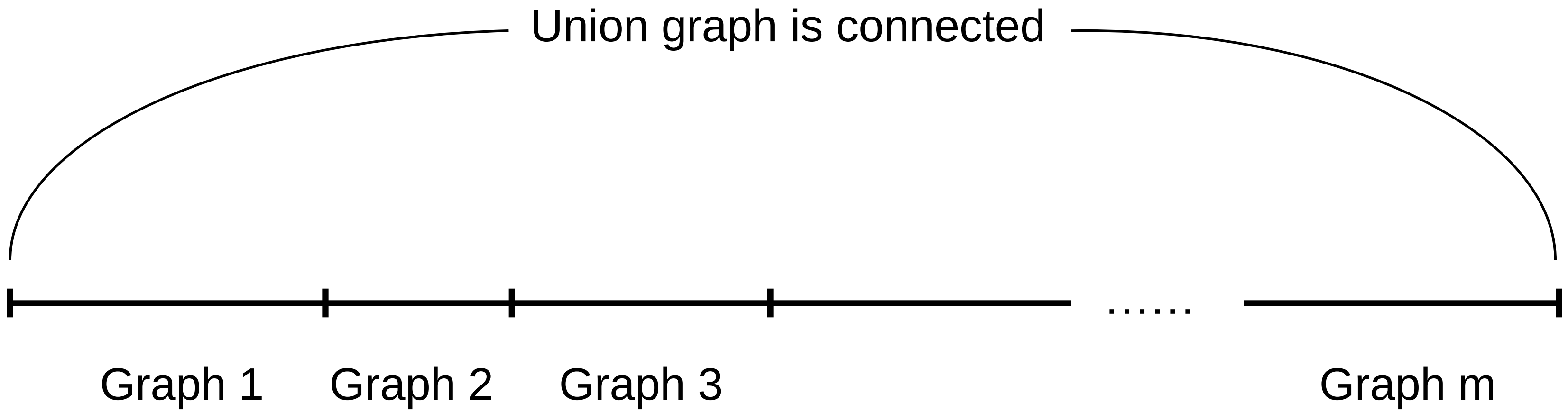
Graph for the interaction between agents → Laplacian or stochastic matrices

- Undirected or directed graph (balanced)
- Fixed or switched graph



Switching → joint connection

- Joint connection: union graph in $[t, \infty)$ is connected for any t: a necessary condition
- Uniform joint connection: $\exists T$, union graph on $[t, t+T]$ is connected



➔ Many extensions …

- Non-convex optimization → constrained convex problem …
- Online or robust optimization: regret analysis …
- Zero-sum game (saddle point): $\min\max f(x,y)$
- Aggregative game
- Coverage: search/rescue, evasion/pursuit
- Machine learning ……

Discrete-time optimization

Joint work with students (Y. Lou, G. Shi, Y. Zhang, P. Yi) and professors (Profs. Xie, Jonhasson, and Liu, et al)

- Convex intersection computation with approximate projection (IEEE TAC 2014, full paper): accurate projection → approximate projection set; the critical approximate angle.

- Non-convex intersection computation (IEEE Trans Wireless Communications 2015, full paper): ring intersection with application to localization even when the intersection set is empty

- Random sleep algorithms (SCL 2013, CTT 2015): update with random sleep procedure, due to random failure, or sleep to save energy, or stochastic disturbance, etc

Discrete time optimization

- Zero-sum game (IEEE TAC 2016, full paper): the parties against each other to solve the saddle point problem; adaptive heterogeneous stepsizes for unbalanced graphs
- Optimization with quantization (IEEE TCNS 2014, full paper): exact optimization can be achieved with one bit when the graph is fixed, with at most 3 bits when it is switching
- Optimization with constraints (SIAM Control & Optimization, 2016, IEEE TAC, under review): convergence rate for stochastic algorithm, nonmsooth optimization with equality constraints

Recent Attention: continuous-time

Conventional optimization algorithm: discrete time

Recent years: analysis and design of continuous-time algorithm

- Few works done for continuous-time approximation or constrained optimization in the past: Arrow et al (1958), Ljung (1977), Brockett (1988) , …
- A way to connect discrete-time decision and continuous-time control

3. Continuous-time optimization

Fundamental Problems:

- Connectivity: time-varying graphs, balanced weights
- Uncertainty: communication, measurement, environment
- Constraints: local, coupled, …

Cyber-physical (hybrid) problems:

- Communication cost: random sleep, event-based, quantization …
- Disturbance rejection hybrid/hierarchical computation
- Complicated dynamics: nonlinear physical agents

Why continuous-time model?

New era → new problems:

- Optimization solved not with digital computers, but by physical systems
- Cyber-physical system: hybrid model with discrete-time communication and continuous-time physical systems
- New design viewpoint from continuous-time dynamics
- Maybe quantum computation?

Comparison

	Discrete-time	Continuous-time
Tools	Variational inequality, monotone property, fixed point …	Lyapunov function, passivity, input-output stability …
Design	Time-varying stepsize, ADMM, dual variable, …	Dynamic compensation, autonomous equation, singular perturbation …
Theory	Convex optimization, saddle-point dynamics, …	Nonlinear control, differential inclusion, robust control …

Continuous-time optimization

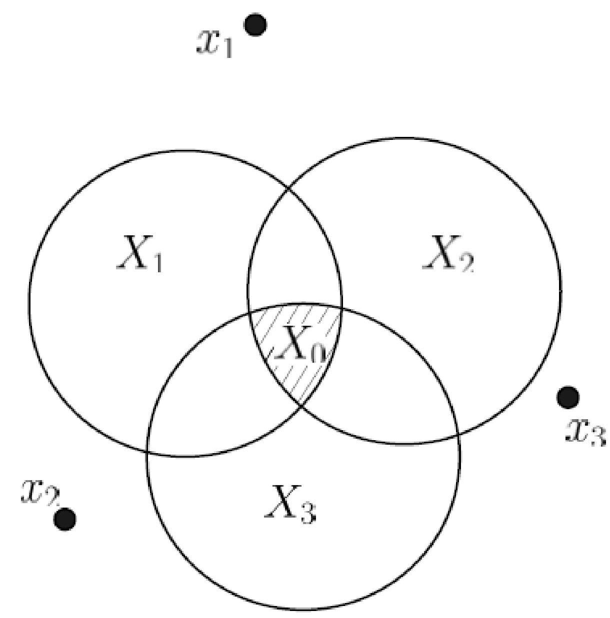
Joint work with G. Shi, Y. Lou, P. Yi, X. Wang, Z. Deng, Y. Zhang, X. Zeng, et al

1. Convex intersection computation: IEEE TAC 2013; and approximate projection: Automatica 2016
2. Optimization with constraints: SCL 2014, Automatica 2016, IEEE TAC 2017
3. Optimization with disturbance rejection: IEEE T-Cybernetics 2015, CTT 2014, IET CTA 2017
4. High order/nonlinear agent dynamics: Unmanned Systems 2016, Automatica 2017

3.1 Distributed convex intersection

- Basic formulation: Agent $dx_i/dt = u_i$ only knows the information of its own closed convex set X_i and its neighbor x_j → the agents achieve consensus within X_0 $(=\cap X_i)$, which is not empty

- Aim: distributed algorithm with switching interaction topologies



Formulation

Find a point in the intersection set of a group of convex set

- The problem originally studied by Aronszajn 1950, Gubin, et al 1967, Deutsch, 1983: alternating projection algorithm (APA: a centralized solution)

- Projected consensus algorithm (PCA: a decentralized version of APC) with time-varying directed interconnection, or its randomized version; Nedic et al 2010, Shi et al, 2012…

Projected consensus algorithm (PCA)

PCA for continuous-time system: accurate projection for optimization + neighbor-based rule for consensus

- Centralized design → Distributed design: neighbor-based rule, not completely connected
- Conventional analysis → set analysis (non-smoothness)
- Switching interaction topology (non-smoothness): common Lyapunov function

Main Results (TAC 2013)

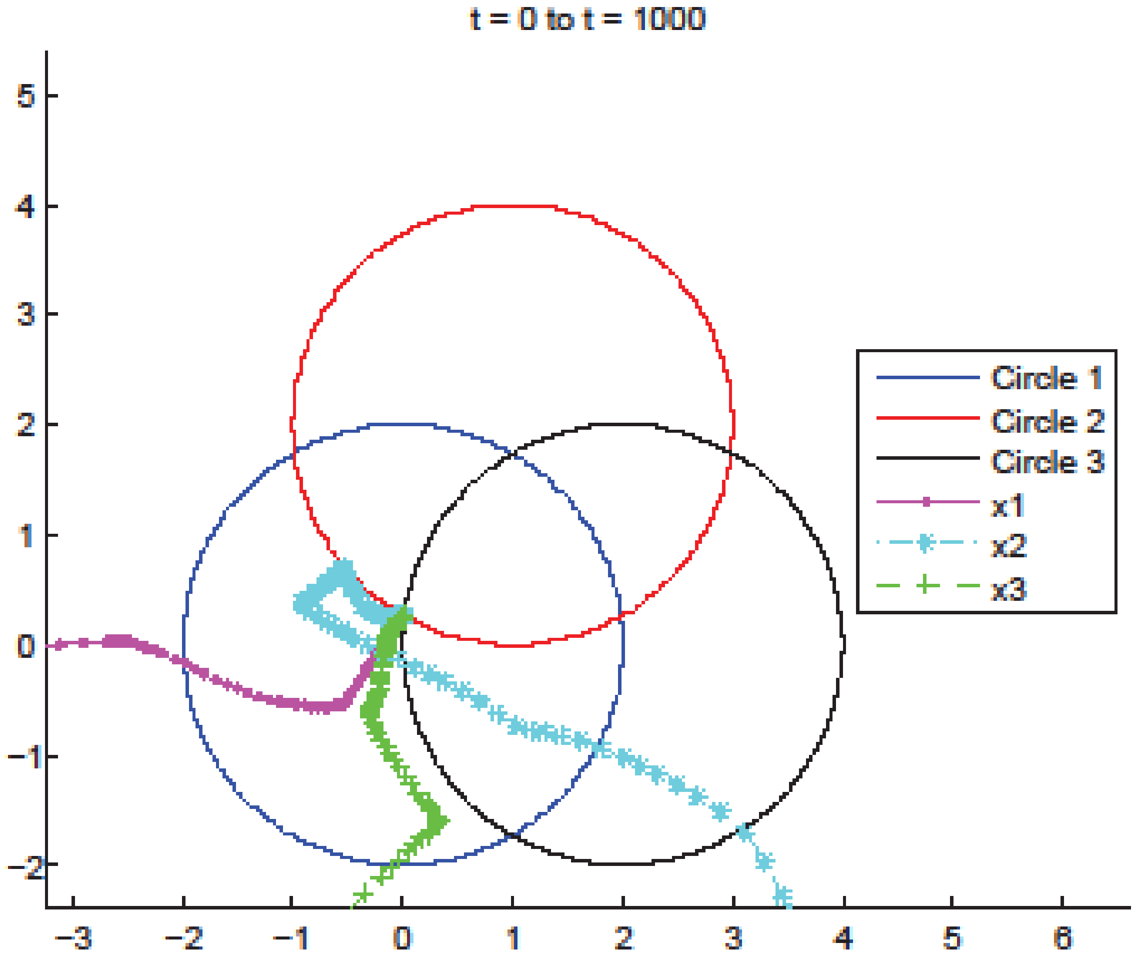
PCA: local projection for intersection
+ neighbor-based rule for consensus

Result 1: Global convex intersection of MAS
← uniformly jointly strongly connected

Consistent with the existing discrete time results

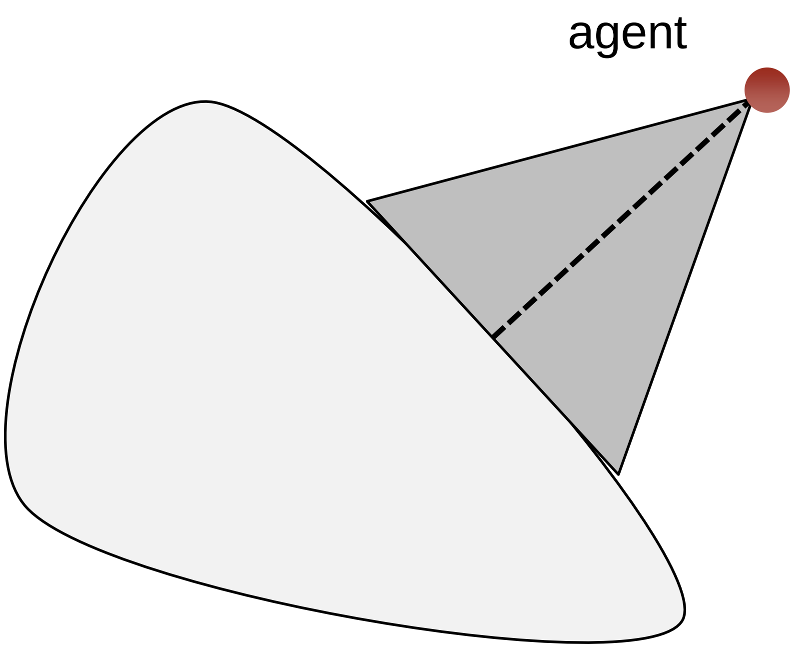
Result 2: In the bidirectional case, MAS achieves global convex intersection = $[t, \infty)$ joint connection

Numerical simulation



Approximate PCA

PCA → APCA: approximate projection with $0 \leq \theta \leq \theta^* < \pi/2$ (APCA)



In practice, it is hard or expensive to get accurate projection →Approximately projected consensus algorithm (APCA) for unknown projection: IEEE TAC 2014

Critical approximate angle

Suppose $\alpha_{ik} = 1,\ \theta_k = \theta\ \forall i, k$

$n \geq 1$ nodes

- $0 < \theta < \pi/4$ implies

$$\sup_{x(0)} \limsup_{k\to\infty} |x_i(k)|_{X_0} < \infty,\ \ i = 1, ..., n$$

$n = 1$ node

- $\theta = \pi/4$ implies

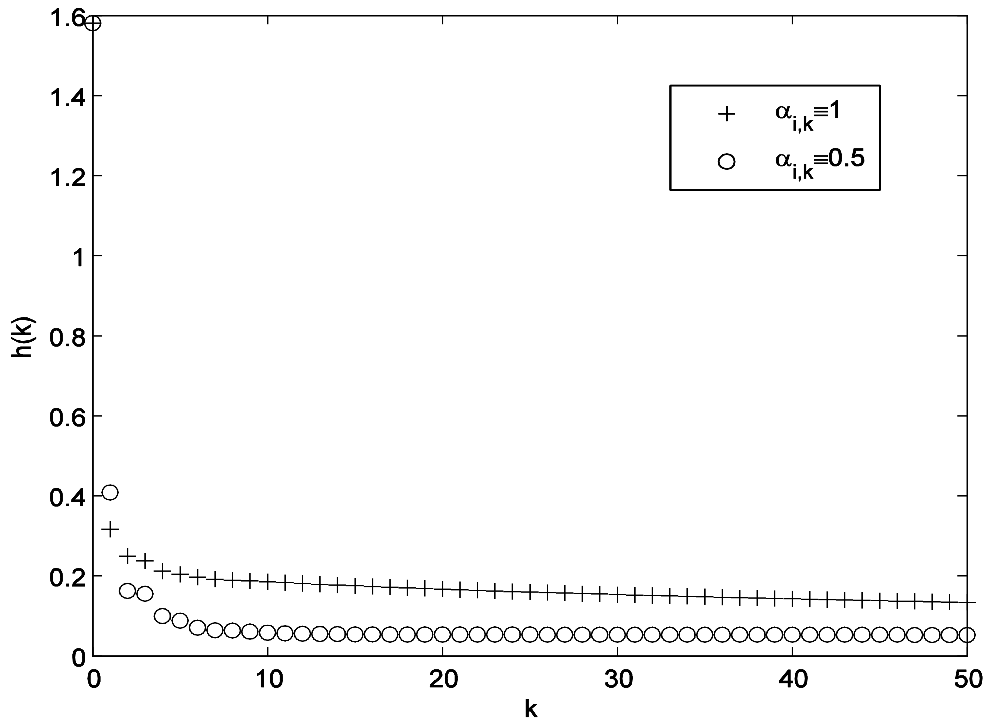
$$\limsup_{k\to\infty} |x_*(k)|_{X_*} < \infty$$

- $\pi/4 < \theta < \pi/2$ implies

$$\limsup_{k\to\infty} |x_*(k)|_{X_*} = \infty,$$

$$|x_*(0)|_{X_*} > \sup_{y_1, y_2 \in X_*} |y_1 - y_2|/(\tan\theta - 1)$$

Approximate projection performs better than the accurate one! (IEEE TAC 2014)

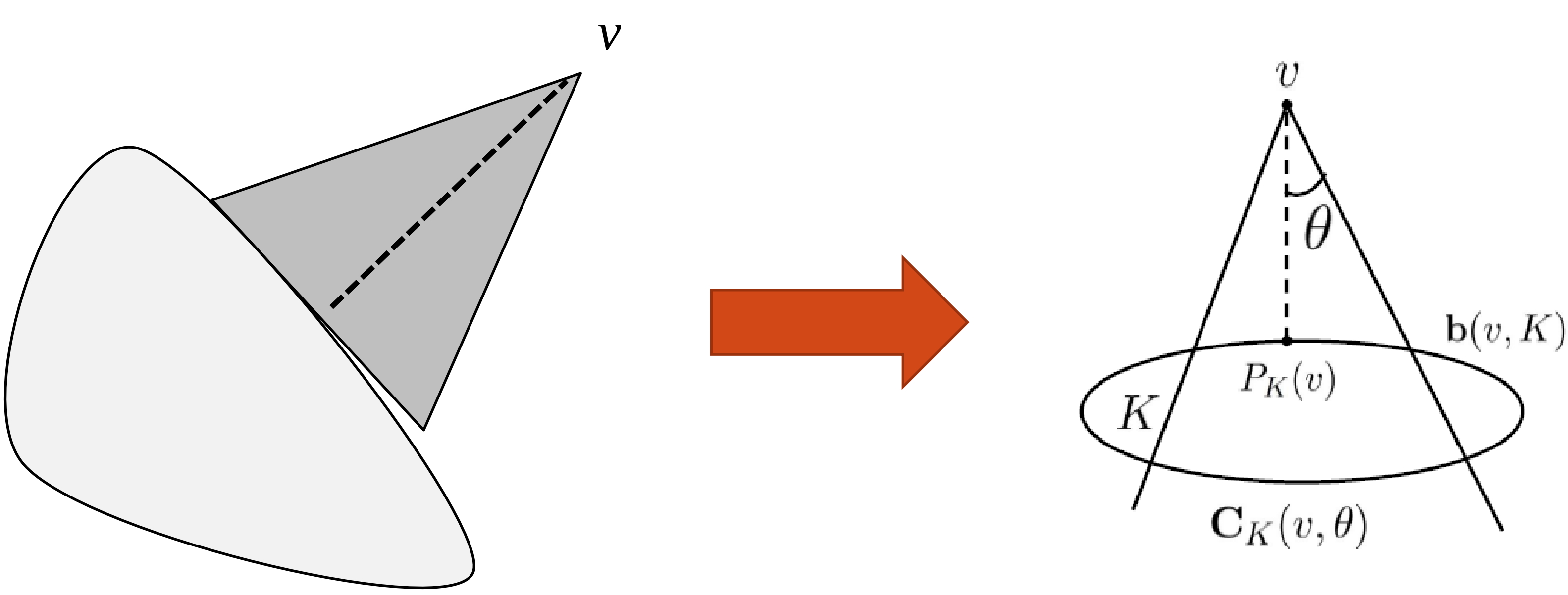


$$h(k) = \max_{1 \leq i \leq 3} |x_i(k)|_{X_0}$$

Continuous-time case

Accurate projection: hard to obtain in practice
Approximate projection is a cheap choice

Approximate angle: $0 \le \theta \le \theta^* < \pi/2$; modified projection sets



Main Results

- Connectivity: uniformly jointly-connected for balanced graph in continuous-time case
- Approximate projection + consensus rule + stepsize condition → optimal consensus (Automatica 2016)
- Difference between continuous and discrete time cases:
  - Definition of approximate projection → virtual stepsize based on finite curvature,
  - The intersection set $X_0 = \cap X_i$ may be empty but the aim can be achieved by one algorithm
  - No critical approximate angle (essential difference between continuous-time and discrete-time cases)

3.2 Optimization with constraints

- **Constraints: from multiple objectives and condition limitations → analysis and design of distributed optimization algorithms**
- **Application: smart grids, sensor network, social systems, wireless communciation, …**

Given constraints: global, local, coupled …

Active constraints: invariance, bounds …

Well known constraints

Constraints for $$\min_{x\in\Omega} f(x), \quad f(x) = \sum_{i=1}^{n} f^i(x_i)$$

1. Local inequality constraint: $g_j^i(x_i) \le 0$
2. Resource allocation: $\sum_{i=1}^{n} x_i = d_0$
3. Local equality constraints: $A^i x = b^i$
4. Constraint sets: $x_i \in \Omega_i$

- **Various combinations of constraints: 2+4 & 3+4 with some conditions …**
- **Applications to sensor networks or smart grids**

Remarks

Start with simple cases: Lipschitz of the gradient + undirected graph +

- Strict convexity → asymptotical stability
- Strong convexity → exponential stability

Extensions with many challenges:

- Convexity → non-unique solution, multiple equilibria
- Nonsmooth functions → nonsmooth analysis
- Directed time-varying graph → auxillary? dynamics to estimate unbalanced weights, analysis based on common bound
- Communication cost → quantization, random sleep, event-based

Local inequality constraints

Problem: $\min f(x),\ f(x) = \sum_{i=1}^{N} f_i(x)$

$$g_j^i(x) \leq 0,\ j = 1, ..., J^i, i = 1, ..., N$$

- Distributed control:

$$\begin{aligned}
\dot{x}_i &= -\nabla f_i(x_i) - \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) \\
&\quad - \sum_{j \in \mathcal{N}_i} a_{ij}(v_i - v_j) - \sum_{j=1}^{J^i} \lambda_{ij} \nabla g_j^i(x_i) \\
\dot{v}_i &= \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j); \\
\dot{\lambda}_{ij} &= [g_j^i(x_i)]^+_{\lambda_{ij}}, j = 1, ..., J^i.
\end{aligned}$$

- Convergence based on a hybrid LaSalle invariance principle (SCL 2015).

Resource allocation

Problem:

$$\min_{x_i \in \mathbf{R}^m, i \in \mathcal{N}} \quad \sum_{i \in \mathcal{N}} f_i(x_i),$$

$$subject\ to \ \sum_{i \in \mathcal{N}} x_i = \sum_{i \in \mathcal{N}} d_i.$$

- Distributed control:

$$\dot{x}_i = -\nabla f_i(x_i) + \lambda_i$$
$$\dot{\lambda}_i = -\sum_{j \in \mathcal{N}_i} (\lambda_i - \lambda_j) - \sum_{j \in \mathcal{N}_i} (z_i - z_j) + (d_i - x_i)$$
$$\dot{z}_i = \sum_{j \in \mathcal{N}_i} (\lambda_i - \lambda_j)$$

- Results: convergence; exponential convergence; additional constraint sets (CCC 2015， Automatica 2016)

3.3 Optimization with disturbance

- Stochastic disturbance (noise, package loss …) discussed in some optimization results
- Modeled deterministic disturbance may be considered when the agents are physical (UAV, robots) and moving in practical environment
- Exact optimization with disturbance rejection: agent dynamics + optimization goal + exogenous disturbance

Basic Distributed Algorithm

Design： optimization + consensus + internal-model-based disturbance rejection

$$\dot{v}_i = \alpha\beta \sum_{j=1}^{N} a_{ij}(x_i - x_j)$$

$$\dot{\eta}_i = (I_n \otimes F)\eta_i + (I_n \otimes G)u_i$$

$$u_i = \underbrace{-\alpha\nabla f_i(x_i) - v_i}_{\text{optimal term}} \underbrace{-\beta \sum_{j=1}^{N} a_{ij}(x_i - x_j)}_{\text{consensus term}}$$

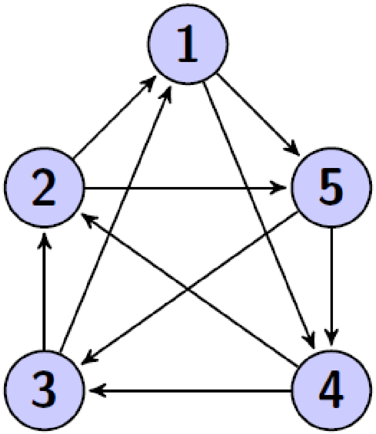
$$\underbrace{-(I_n \otimes \Psi)\eta_i}_{\text{internal model term}}.$$

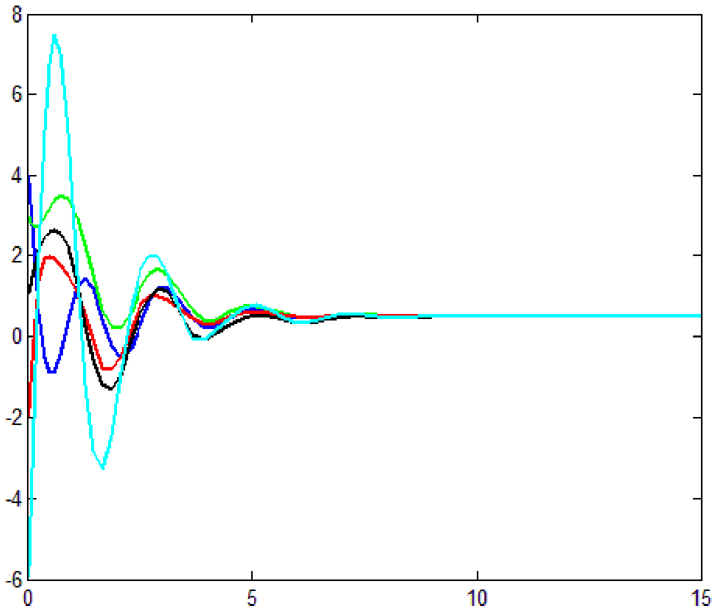
Main Results

- The exact optimization can be achieved with known disturbance frequency by internal model (Control Theory & Technology, 2014)
- It is also achieved semi-globally with unknown frequency by adaptive internal model (CCC 2014)
- The agent dynamics can be extended to a nonlinear case （IEEE T-Cybernetics, 2015）
- Event-triggered design for both communication and gradient measurement (IET CTA, 2016)

Simulation（5 agents）

Topology and error trajectories





$$f_1(x) = (x+2)^2, \quad f_2(x) = (x-5)^2$$

$$f_3(x) = x^2 \ln(1+x^2) + x^2$$

$$f_4(x) = \frac{x^2}{\sqrt{x^2+1}} + x^2, \quad f_5(x) = \frac{x^2}{\ln(2+x^2)}$$

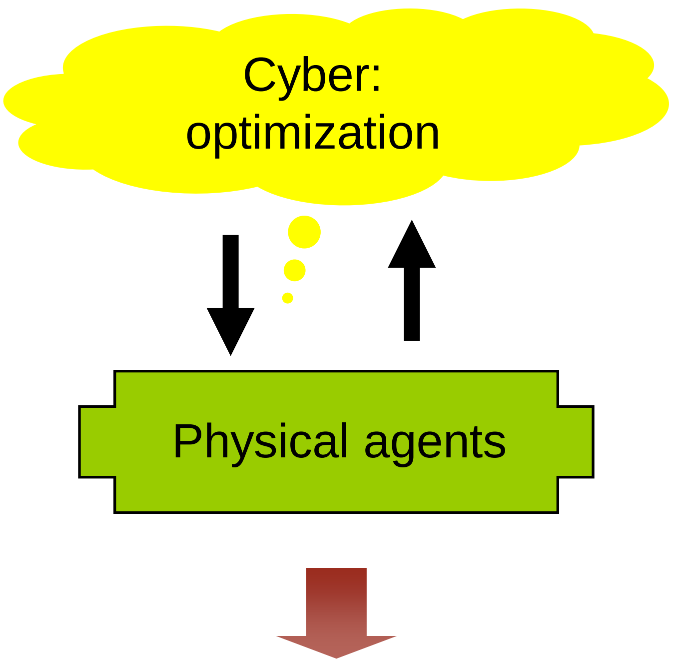
3.4 Nonlinear/High-order Agent

Motivation:

- Integration of control and optimization
- Cyber optimization solved by physical systems

Results:

1. Euler–Lagrangian (EL) systems: nonlinear second order systems
2. High order linear systems → special nonlinear systems



Distributed optimization as cyber-physical systems …

Optimization of EL systems

Mechanical systems in the Euler-Lagrange form：

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i = \tau_i.$$

N heterogeneous agents with uncertain parameters

Distributed optimization control design for EL systems：

$$\tau_i = -k\dot{q}_i - \alpha\nabla f_i(q_i) - k\sum_{j\in N_i} a_{ij}(q_i - q_j) - kv_i$$

$$\dot{v}_i = \sum_{j\in N_i} a_{ij}(q_i - q_j)$$

Task: tracking, formation, coverage

Constraint: obstacle, energy, resource …

Results for EL systems

- Basic assumptions: Strong convexity & Undirected graph
- Result 1 (Unmanned Systems 2016): Lipschitz of gradient → semi global convergence (exponential).
- Result 2 (Automatica 2017): Global Lipschitz of gradient → global convergence (exponential) (optimization of double integrator + tracking control of EL systems
- Result 3 (Kybernetika 2017): Event-triggered optimization design for EL systems
- Result 4 (IFAC conference 2016): Optimization design with kinematic constraints (saturation of velocity and acceleration)

High order system

For agents in the form of n-th integrator: $x_i^{(n-1)} = v_i$;

The algorithm for each agent:

$$v_i = -\sum_{\iota=1}^{n-1} k_{n-\iota} x_i^{(\iota)} - \beta \sum_{j \in N_i} a_{ij}(x_i - x_j) - \alpha \nabla f_i(x_i) - w_i$$

$$\dot{w}_i = \alpha\beta \sum_{j \in N_i} a_{ij}\Big((x_i - x_j) + \sum_{\iota=1}^{n-1}(x_i^{(\iota)} - x_j^{(\iota)})\Big),$$

Results: strong convexity + undirected graph + global Lipschitz → exponential convergence

→ minimum phase nonlinear systems and observer-based output feedback design

4. Conclusions

- Distributed optimization: optimization algorithms based on local information → scalability, reliability, and maybe security …
- Challenges: operations research + control systems +complex network +computational complexity + …
- Applications: estimation (sensor), simultaneous routing & resource allocation (wireless communication), opinion dynamics (social networks), intersection computation (computer), ……

Research framework

		constraint	uncertainty	dynamics
Phy sical	Control	Non-holonomic, saturation, event-based …	Identification, adaptive con., robust con.	Stochastic, time-varying, nonlinear …
	Optimiz.	Inequality, bounded set, equality …	Data-based, online regret, robust opti.	High order, multi-scale, …
Cyber	Network	Communication, environment, energy …	Survivability, security, failure …	Link dynamics, split/merge, switching …

Some recent results for MAS

- Distributed optimization (IEEE TAC 2013, 2014, 2016; SCL 2013, 2015; IEEE T-Cybernetics 2015; SIAM Con. & Opti. 2016; Automatica 2016a, 2016b, 2017)
- Containment control & multiple leaders (Automatica 2014)
- Distributed output regulation (IEEE TAC 2013, 2014, 2016; Automatica 2015; IJRNC 2013)： Internal model based design
- Attitude synchronization and formation (Automatica 2014)
- Coverage: cooperative sweeping (Automatica 2013)
- Distributed Kalman filter (IEEE TAC 2013)
- Quantization in control and optimization (IEEE TCNS 2014, IEEE TAC 2016)
- Target surrounding (IEEE TAC 2015)
- Opinion dynamics (Physica A 2013, Automatica 2016)
- … …

Thank you！