Department of Civil and Environmental Engineering, Fall Semester, 2023

ENCE 201 Midterm 2, Open Notes and Open Book

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Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are two questions. Partial credit will be given for partially correct answers, so please show all your working.

Please see the **class web page for instructions on how to submit your exam paper**.

Question	Points	Score
1	20	
2	20	
Total	40	

Question 1: 20 points.

Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e.,

$$[A] = [L]\,[U] \tag{1}$$

where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our examples in class assumed that upper diagonal elements would be unity (i.e., $U_{ii} = 1$). Python sets the lower diagonal elements to unity (i.e., $L_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $(n^2 + n)$ to n^2 might work.

[1a] (3 pts). Draw and label a diagram that illustrates: (1) the number of equations, and (2) the layout of unknowns in the LU decomposition problem. Clearly indicate on your diagram why these assumptions are needed.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & & \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix}}_{n^2 \text{ elements}} = \underbrace{\begin{bmatrix} L_{11} & 0 & \\ L_{21} & L_{22} & \\ \vdots & & \ddots \\ L_{n1} & & L_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{1n} \\ 0 & u_{22} & u_{2n} \\ & \ddots & \vdots \\ & & u_{nn} \end{bmatrix}}_{n^2 + n \text{ elements}}$$

← need to impose 'n' constraints to reduce number of unknowns to n^2.

[1b] (5 pts). Calculate the matrix reduction of the matrix

$$[A] = \begin{bmatrix} 4.0 & 2.0 & 2.0 \\ 2.0 & 10.0 & 4.0 \\ 12.0 & 12.0 & 24.0 \end{bmatrix}. \tag{2}$$

by assuming that $U_{ii} = L_{ii}$. Show all of your working ...

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 12 & 12 & 24 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Apply constraint: $L_{ii} = u_{ii}$.

First row $\quad u_{11} \cdot L_{11} = 4 \implies u_{11} = L_{11} = 2.$

[1b] Continued ...

2nd row: $L_{21} \cdot U_{11} = 2 \longrightarrow L_{21} = 1$

$L_{21} \cdot U_{12} + L_{22} \cdot U_{22} = 10 \longrightarrow L_{22} = U_{22} = 3.$

$L_{21} \cdot U_{13} + L_{22} \cdot U_{23} = 4 \longrightarrow U_{23} = 1.$

Third row

$L_{31} \cdot U_{11} = 12 \longrightarrow L_{31} = 6.$

$L_{31} \cdot U_{12} + L_{32} \cdot U_{22} = 12 \longrightarrow L_{32} = 2.$

$L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + L_{33} \cdot U_{33} = 24$

$\Rightarrow 6.1. + 2.1 + L_{33} = 24$

$\Rightarrow L_{33} = U_{33} = 4.$

Hence:

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 12 & 12 & 24 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

[1c] (4 pts). Use forward and backward substitution to solve $[A]\,[x] = [10, 26, 60]^T$.

Step 1: Solve $L \cdot z = b$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 60 \end{bmatrix} \Rightarrow \begin{matrix} z_1 = 5 \\ z_2 = 7 \\ z_3 = 4. \end{matrix}$$

Step 2: Solve. $U x = z$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 2 \\ x_3 = 1 \end{matrix}$$

[1d] (4 pts). Check your solution.

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 12 & 12 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 10 \\ 26 \\ 60 \end{bmatrix} \checkmark$$

[1e] (4 pts). Hence, write down the det[A]?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

$$\det(A) = \det(L).\det(U) = (2.3.4)(2.3.4)$$
$$= 24^2 = 576.$$

Note: Using co-factors method (covered in class)

$$\det(A) = \det \begin{bmatrix} 4 & 2 & 2 \\ 2 & 10 & 4 \\ 12 & 12 & 24 \end{bmatrix}$$

$$= 4 \det \begin{bmatrix} 10 & 4 \\ 12 & 24 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 4 \\ 12 & 24 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 10 \\ 12 & 12 \end{bmatrix}$$

$$= 4(240-48) - 2(48-48) + 2(24-120)$$

$$= 576 \checkmark$$

Question 2: 20 points

This question covers function interpolation with the methods of divided differences and lagrange interpolation, and integration of a function using a variety of integration rules. The whole question is motivated by the small dataset:

x	0	1	2
f(x)	1	2	5

[2a] (5 pts). Use the method of **divided differences** to show that the data set is interpolated by the quadratic function:

$$p(x) = 1 + x^2 \tag{3}$$

Be sure to show all of your working.

x	$f(x)$		
0	1		
1	2	1	
2	5	3	1

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2-1}{1-0} = 1$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{5-2}{2-1} = 3$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3-1}{2} = 1$$

$$p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 1 + 1(x-0) + 1(x-0)(x-1)$$

$$= 1 + x^2.$$

[2b] (5 pts). Check your answer in part 2a by computing the functional form via the method of **Lagrange Interpolation.**

Be sure to show all of your working.

x	$f(x)$
0	1
1	2
2	5

Format:

$$p(x) = f_0 p_0(x) + f_1 p_1(x) + f_2 p_2(x)$$

where,

$$p_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2).$$

$$p_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2).$$

$$p_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}x(x-1).$$

Hence,

$$p(x) = 1 \cdot \frac{1}{2}(x-1)(x-2) + -2x(x-2) + \frac{5}{2}x(x-1).$$

$$= \left(\tfrac{1}{2} - 2 + \tfrac{5}{2}\right)x^2 + \left(-\tfrac{3}{2} + 4 + \tfrac{-5}{2}\right)x + 1$$

$$= 1 + x^2 \checkmark$$

Now suppose that we want to compute the integral:

$$\int_0^2 p(x)dx \tag{4}$$

where p(x) is as defined in equation 3.

[2c] (2 pts). What is the analytic solution to this problem?

$$I = \int_0^2 p(x)\,dx = \left[x + \frac{x^3}{3} \right]_0^2 = 2 + 8/3 - 0 - 0 = 14/3$$

(A)

[2d] (4 pts). Suppose that a numerical approximation to equation 4. is computed using the Trapezoid Rule and h = 1. What can you say about the magnitude of the expected error?

$$I_1 = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2) \right]$$

$$= \frac{1}{2}\left[1 + 4 + 5 \right] = 10/2 = 5. \quad \text{—— (B)}$$

Magnitude of error: $\leq \frac{f''(\varepsilon)}{12} h^2 (b-a)$. $\quad 0 \leq \varepsilon \leq 2$.

With 2 intervals $h = 1$, $b - a = 2$, $f''(\varepsilon) = 2$.

$\Rightarrow$ Expected error $\leq \frac{2}{12} \cdot 1 \cdot 2 = 1/3$.

Actual error = (A) − (B) = 1/3. ✓ It works ✓✓.

[2e] (4 pts). If the exercise is repeated using Simpson's rule, will the numerical approximation be more accurate? Why?

Using Simpson's Rule.

$$\int_0^2 p(x)\,dx \approx \frac{h}{3}\left(f(x_0) + 4 f(x_1) + f(x_2) \right]$$

$$= \frac{1}{3}\left[1 + 4 \times 2 + 5 \right]$$

$$= \frac{14}{3} \longleftarrow \text{Simpson's Rule is exact!}$$