Function Approximation

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ENCE 201, Fall Semester 2023

September 25, 2023

Overview

Part 2

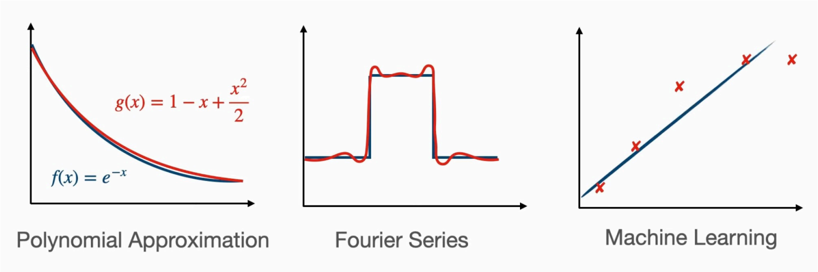
Function Approximation

Motivating Ideas

Function Approximation

A function approximation asks us to select a function, g(x), among a well-defined set of options that approximates – closely matches – a second function, f(x), in a task-specific way.

Approximation Examples: Many approaches ...



Strategy 1: Polynomial Approximation

Polynomial Approximation

Replace function f(x) by a simplier polynomial approximation g(x). Then, use g(x) in computations instead of f(x).

Example 1: Replace $y = f(x) = e^{-x}$ by a quadratic approximation:

$$f(x) = e^{-x} \quad \longrightarrow \quad g(x) = 1 - x + \frac{x^2}{2}. \tag{1}$$

Example 2: Replace $y = sin(x)$ on $x \in [0, \pi]$ by a quadratic approximation:

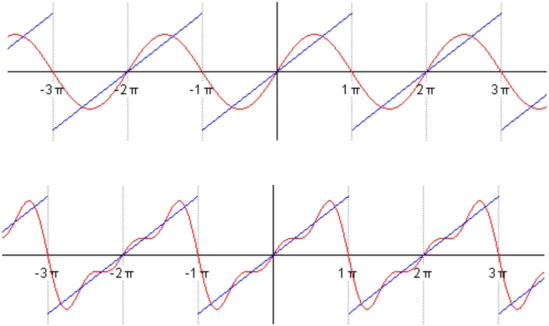
$$f(x) = sin(x) \quad \longrightarrow \quad g(x) = \frac{4x}{\pi^2}\left[\pi - x\right]. \tag{2}$$

Strategy 2: Fourier Series

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of trigonometric (i.e., sines and cosines) and/or exponential functions.

Example 1: Progressive refinement of sawtooth function:



Fourier Series

and

Fourier Integral

Fourier Series (Motivating Idea)

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of trigonometric (i.e., sines and cosines) and/or exponential functions.

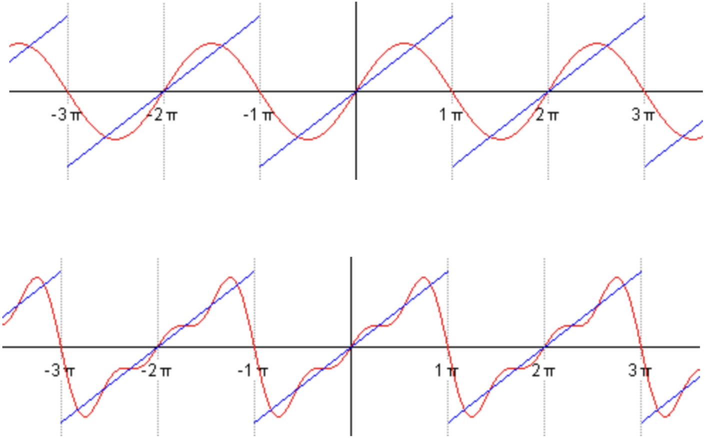
Periodic Function

A function f(x) is periodic if and only if there exists a positive number 2p such that for every x in the domain of f, $f(x+2p) = f(x)$. The number 2p is called the period of f.

Applications: Modeling of waveforms (e.g., surface waves on ocean; ocean tides; acoustics, musical tones; weather phenomena), analysis of resonant frequencies in a structure.

Fourier Series

Example 1: Progressive refinement of sawtooth function:



Fourier Series (Mathemtics)

Mathematics: (triginometry version) For x $\in [0, 2P]$:

$$f(x) \approx A_o + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi x}{P}) + B_n \sin(\frac{n\pi x}{P}) \right]. \tag{19}$$

The coefficients are given by:

$$A_o = \frac{1}{P} \int_0^{2P} f(x) dx, \tag{20}$$

and $$A_n = \frac{1}{P} \int_0^{2P} f(x) \cos(\frac{n\pi x}{L}) dx, \quad \text{for n} \geq 1, \tag{21}$$

$$B_n = \frac{1}{P} \int_0^{2P} f(x) \sin(\frac{nx}{L}) dx, \quad \text{for n} \geq 1. \tag{22}$$

Fourier Series (Mathematics)

Example 1: Fourier Series for Sawtooth Function ...

The sawtooth shape can be written:

$$f(x) = \begin{cases} x & 0.0 \leq x \leq \pi, \\ x - 2\pi & \pi < x \leq 2\pi \end{cases} \tag{23}$$

Substituting 23 into 20 – 22 gives:

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) cos(nx) dx = 0.0. \tag{24}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) sin(nx) dx = -\frac{2}{n}(-1)^n. \tag{25}$$

Fourier Series (Mathematics)

Example 1: Sawtooth function continued ...

The Fourier exapansion is:

$$f(x) = 2\left[sin(x) - \frac{sin(2x)}{2} + \frac{sin(3x)}{3} - \cdots + (-1)^{n+1}\frac{sin(nx)}{n}\cdots\right]$$
$$= 2\sum(-1)^{n+1}\frac{sin(nx)}{n}, \quad \text{for} \quad -\pi \leq x \leq \pi. \tag{26}$$

Note 1: The Fourier series coverges only conditionally because of the discontinuity of f(x) at x $\pm$ π.

Note 2: At the discontinuity itself, the Fourier series will coverge to the arithmetic mean of the end values.

Fourier Series (Mathemtics)

Mathematics: (exponential series). Recall Euler's formula:

$$e^{inx} = \cos(nx) + i\sin(nx). \tag{27}$$

Rearranging equation 27 gives:

$$\cos(nx) = \left[\frac{e^{inx} + e^{-inx}}{2}\right], \quad \sin(nx) = \left[\frac{e^{inx} - e^{-inx}}{2i}\right]. \tag{28}$$

Substituting 27 into 19, and rearranging terms gives:

$$f(x) = \frac{A_o}{2} + \sum_{n=1}^{\infty}\left[A_n\left[\frac{e^{inx} + e^{-inx}}{2}\right] + B_n\left[\frac{e^{inx} - e^{-inx}}{2i}\right]\right]. \tag{29}$$

Fourier Series

Simplifying:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\pi nx/L} \tag{30}$$

where:

$$c_n = \frac{1}{2}\left[A_n - iB_n\right], c_{-n} = \frac{1}{2}\left[A_n + iB_n\right], c_o = \frac{A_o}{2}. \tag{31}$$

Fourier Integral Analysis

Motivation:

- In many practical problems, the function involved is non-periodic (i.e., Fourier series are not possible).

Solution:

- Consider limiting form of Fourier Series when $p \longrightarrow \infty$.
- Fourier Series becomes Fourier Integral Analysis

Fourier Transform Pair: Waveforms in time/frequency domains:

$$f(t) = \int_{-\infty}^{\infty} g(w) e^{iwt} dw.$$
$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) e^{iw\tau} d\tau. \quad (32)$$

Discrete Fourier Transform

Discrete Fourier Transform (DFT)

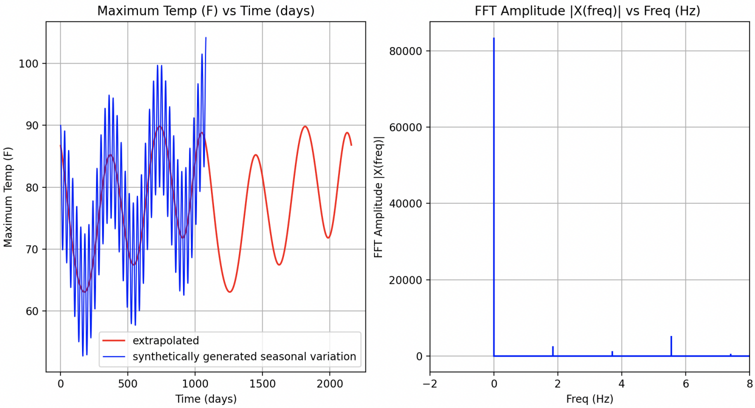
- Represents waveforms in both the time and frequency domains.
- Standard implementation of DFT requires $O(n^2)$ computational work, where n is the size of the data.

Fast Fourier Transform (FFT)

- An algorithm that computes the DFT of a sequence, most often in the time domain to a representation in the frequency domain.
- The inverse FFT transforms the frequency domain representation back into the time domain.
- FFT requires $O(nlog(n))$ computational work, so is very efficient.

Fast Fourier Transform Analysis

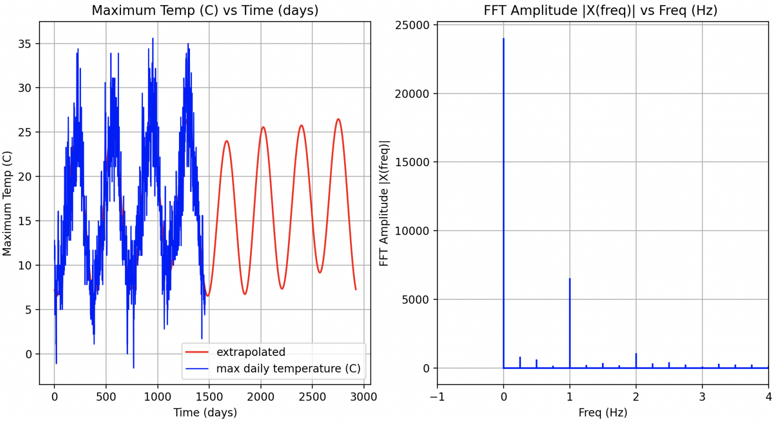
Example 1: FFT analysis of synthetically generated time series ...



Source Code: See python-code.d/math/

Fast Fourier Transform Analysis

Example 2: FFT analysis of max daily temperature in Seattle.



Source Code: See python-code.d/math/

Python Code Listings

Code 2: FFT of Max Temperature in Seattle

```
# ==========================================================================
# TestFastFourierTransform03.py: Compute FFT extrapolation and dominant
# frequencies in Temperature Measurements, Seattle, WA.
#
# Modified by: Mark Austin                                    September 2023
# ==========================================================================

from math import cos,pi
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# ==========================================================================
# main method ...
# ==========================================================================

def main():
    print("--- Enter TestFastFourierTransform03.main()      ...");
    print("--- ============================================ ...");

    # Part 1: Set parameters ...

    print("--- Part 1: Set parameters ...");

    sr = 365 # <-- sampling rate (no samples per year) ...

    # Part 2: Read Seattle weather ...
```

Code 2: FFT of Max Temperature in Seattle

```
    print("--- Part 2: Read datafile for weather in Seattle ...");

    rainfall = pd.read_csv('../data/seattle-weather.csv')
    x = np.array( rainfall['temp_max'].values)

    # Part 3: Compute fast fourier transform for maximum temperature ...

    print("--- Part 3: Compute FFT for max daily temperature in Seattle ...");

    X = np.fft.fft(x)
    N = len(X)
    n = np.arange(N)
    T = N/sr
    freq = n/T

    print("---          No samples per year = {:10.2f} ...".format(sr) );
    print("---          No data points: N = {:10.2f} ...".format(N) );
    print("---          No periods:     T = {:10.2f} ...".format(T) );
    print(n)
    print("--- Fourier transform result ...");

    # print(X)

    # Part 4: Plot data + extrapolated curve ...

    print("--- Part 4: Plot data + extrapolated curve ...");

    plt.figure(figsize = (12, 6))
    plt.subplot(121)
```

Code 2: FFT of Max Temperature in Seattle

```
for num_ in [6]:
   fft_list             = np.copy(X)
   fft_list[num_:-num_] = 0

   # Inverse Fast Fourier transform

   t = np.fft.ifft(fft_list)

   # Plot general trend ...

   plt.plot(np.concatenate([t,t]), color = 'red', label='extrapolated' )

plt.plot( np.arange(0, x.size), x, 'b', label = 'max daily temperature (C)', linewid
plt.title("Maximum Temp (C) vs Time (days)")
plt.ylabel("Maximum Temp (C)", fontsize=10, rotation=90)
plt.xlabel("Time (days)", fontsize=10, rotation=0)
plt.grid()
plt.legend()

# Part 5: Plot data in frequency domain ...

print("--- Part 5: Plot data in frequency domain ...");

plt.subplot(122)
plt.stem( freq, np.abs(X), 'b', markerfmt=" ", basefmt="-b")
plt.title("FFT Amplitude |X(freq)| vs Freq (Hz) ")
plt.xlabel('Freq (Hz)')
```

Code 2: FFT of Max Temperature in Seattle

```
    plt.ylabel('FFT Amplitude |X(freq)|')
    plt.grid()
    plt.xlim(-1, 4)

    plt.show()

    print("--- ========================================== ... ");
    print("--- Leave TestFastFourierTransform03.main()  ...");

# call the main method ...

main()
```