Linear Matrix Equations – Part 1

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Overview

Part 2

Linear

Matrix Equations

Linear Matrix Equations

Definition. A system of m linear equations with n unknowns may be written

$$\begin{array}{ccccccccc} a_{11}x_1 & + a_{12}x_2 & + & a_{13}x_3 + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + a_{22}x_2 & + & a_{23}x_3 + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & \vdots & & & \ldots & & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + & a_{m3}x_3 + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1}$$

Points to note:

- The constants a_{11}, a_{21}, a_{31}, $\cdots$ a_{mn} and b_1, b_2, $\cdots$ b_m are called the equation coefficients.
- The variables x_1, x_2 $\cdots$ x_n are the unknowns in the system of equations.

Linear Matrix Equations

Matrix Form. The matrix counterpart of 1 is $[A] \cdot [X] = [B]$, where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad (2)$$

Points to note:

- Matrices A and X have dimensions $(m \times n)$ and $(n \times 1)$, respectively.
- Column vector B has dimensions $(n \times 1)$.

Analysis of Solutions to Matrix Equations

Key Observations

- For two- and three-dimensions, graphical methods and intuition work well.
- For problems beyond three dimensions, much more difficult to understand the nature of solutions to linear matrix equations.
- We need to rely on mathematical analysis instead.

Basic Questions

- How many solutions will a set of equations will have?
- How to determine when no solutions exist?
- If there is more than one solution, how many solutions exist?

Fortunately, hand calculations on very small systems can provide hints on a pathway forward.

Matrix Determinant

Preample to Matrix Determinant

Strategy. Understand this problem by premultiplying the equations by constants in such a way that when they are combined variables will be eliminated.

Hand Calculation 1: Multiply equation 7 by a_{21} and equation 8 by a_{11}. This gives:

$$a_{21} \cdot a_{11} \cdot x_1 + a_{21} \cdot a_{12} \cdot x_2 = a_{21} \cdot b_1 \quad (10)$$

$$a_{11} \cdot a_{21} \cdot x_1 + a_{11} \cdot a_{22} \cdot x_2 = a_{11} \cdot b_2 \quad (11)$$

Next, subtract equation 10 from equation 11 and rearrange:

Preample to Matrix Determinant

$$x_2 = \left[\frac{a_{11} \cdot b_2 - a_{21} \cdot b_1}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \right]. \tag{12}$$

Finally, get x_1 by back-substituting x_2 into either equation 7 or 8.

Turns out there is more than one way to compute a solution ...

Hand Calculation 2: Multiply equation 7 by a_{22} and equation 8 by a_{12}, then subtract and rearrange:

$$x_1 = \left[\frac{a_{22} \cdot b_1 - a_{12} \cdot b_2}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \right] \tag{13}$$

Compute x_2 by back-substituting x_1 into either equation 7 or 8.

Preample to Matrix Determinant

Key Point. The denominators of equations 12 and 13 are the same.

They correspond to the **determinant** of a (2×2) matrix, namely:

$$det(A) = det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}. \quad (14)$$

Note. The family of equations will have a unique solution when $\det(A) \neq 0$.

Preample to Matrix Determinant

Equations in Three Dimensions. (i.e., $m = n = 3$),

$$det(A) = det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}. \quad (15)$$

where,

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{23} \end{bmatrix}, M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}. \quad (16)$$

Again, a unique solution exists when $\det(A) \neq 0$. $\det(A)$ will be zero when two or more planes are parallel.

Matrix Determinant

General Formula. Let A be a $(n \times n)$ matrix.

For each a_{ij} there is a sub-matrix A'_{ij} obtained by deleting the i-th row and j-th column of A.

Let $M_{ij} = \det\,(A'_{ij})$.

i-th row expansion

$$\det(A) = \sum_{j=1}^{n} (-1)^{(i+j)} M_{ij}.$$

j-th row expansion

$$\det(A) = \sum_{i=1}^{n} (-1)^{(i+j)} M_{ij}.$$

$(-1)^{(i+j)}$					
	+	−	+	−	+
	−	+	−	+	−
	+	−	+	−	+
	−	+	−	+	−

Matrix Determinant

Example 1. The most straight forward way of computing the determinant of:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & -2 \end{bmatrix} \quad (17)$$

is to expand terms about the row or column having the most zero elements – in this case, the first row. This gives:

$$\det(A) = 2\det \begin{bmatrix} -1 & 1 \\ 6 & -2 \end{bmatrix} = 2(2-6) = -8. \quad (18)$$

Because det(A) evaluates to a non-zero number, we expect that the inverse of A will exist, and as such, the rank(A) = 3.

Matrix Determinant

Example 2. Compute the determinant of:

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 1 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix} \tag{19}$$

To minimize computation we expand terms about the row or column having the most zero elements – in this case, the third column. This gives:

$$\det(A) = -5\det \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{bmatrix} = -5M_{34}. \tag{20}$$

Matrix Determinant

Expanding the second determinant about the first row gives:

$$\begin{aligned} M_{34} &= 2\det\begin{bmatrix} -2 & 7 \\ -6 & 8 \end{bmatrix} + \det\begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} + 3\det\begin{bmatrix} 4 & -2 \\ 6 & -6 \end{bmatrix}, \\ &= 2(-16+42) + (32-42) + 3(-24+12) \\ &= 6. \end{aligned} \quad (21)$$

Equation 20 evaluates to -5(6) = -30.

Matrix Determinant

Matrix Determinant Properties:

- $\det(AB) = \det(A)\det(B) = \det(BA)$.
- If $\det(A) \neq 0$, then A is invertable (non-singular).
- $\det(A^T) = \det(A)$.
- $\det(A)$ is multiplied by -1 when two rows of A are swapped.
- If two rows of A are identical, then $\det(A) = 0$.