Matrices and Vectors: Basic Introduction

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Overview

Part 2

Definition of Matrices

Definition of a Matrix

Definition. A matrix (or array) of order m by n is simply a set of numbers arranged in a rectangular block of m horizontal rows and n vertical columns. We say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

is a matrix of size (or dimension) $(\mathtt{m} \times \mathtt{n})$.

In the double subscript notation a_{ij} for matrix element a(i,j), the first subscript i denotes the row number, and the second subscript j denotes the column number.

Definition of Vectors

Definition of Row and Column Vectors

Definition. A row vector is simply a (1×n) matrix, i.e.,

$$V = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_n \end{bmatrix} \quad (17)$$

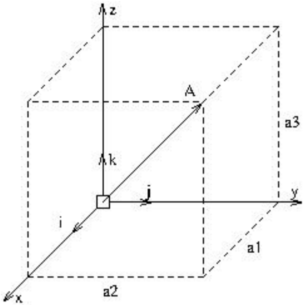
Definition. A column vector is a (m × 1) matrix, e.g.,

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \cdots \\ v_m \end{bmatrix} \quad (18)$$

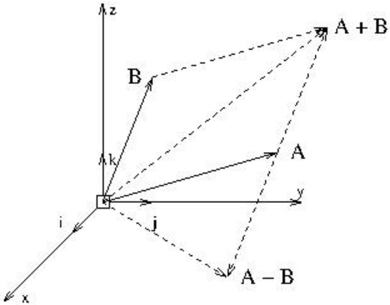
In both cases, the i-th element of the column vector is denoted v_i.

Vector Properties

Properties of Vector Arithmetic



Components of Three–Dimensional Vector



Vector Addition and Subtraction

- a + b = b + a
- a + 0 = a
- c(a + b) = ca + cb
- (a + b) + c = a + (b + c)
- a + (-a) = 0
- 1 a = a.

Dot Product

Definition. The dot product of two vectors $a = [a_1, a_2, a_3, \cdots, a_n]$ and $b = [b_1, b_2, b_3, \cdots, b_n]$ is:

$$a.b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_4 + \cdots + a_n b_n. \quad (19)$$

Note: a.b = b.a. If a and b are perpendicular then a.b = 0.

Engineering Applications

- Mechanical work is the dot product of force and displacement vectors (Jou).
- Power is the dot product of force and velocity vectors (W).
- Fluid Mechanics.

Dot Product

Example 1. Let $a = [1, 2, 3]$ and $b = [0, -1, 2]$. The dot product:

$$a.b = \sum_{i=1}^{n} a_i b_i = 1 \times 0 + 2 \times -1 + 3 \times 2 = 4. \quad (20)$$

A dot product can also be written as a row vector multiplied by a column vector, e.g.,

$$\begin{bmatrix} 1, 2, 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = 4. \quad (21)$$

The vector dimensions are: $(1 \times 3)\ (3 \times 1) \rightarrow (1 \times 1)$.

Dot Product

Properties. Let $a = [a_1, a_2, a_3, a_4]$, $b = [b_1, b_2, b_3, b_4]$ and $c = [c_1, c_2, c_3, c_4]$. And let d be a non-zero constant.

The dot product:

$$a.b = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \tag{22}$$

obeys the properties:

- a.a $= \|a\|^2$.
- a.(b + c) = a.b + a.c
- a.b = b.a
- $a.b = 0 \iff a = 0$ or $b = 0$ or $a \perp b$.
- 0.a = 0
- (da).b = d(a.b)
- a.b $= |a|.|b| \cos(\theta)$.

Cross Product

Definition. Consider two vectors A and B in three dimensions:

$$A = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$
$$B = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

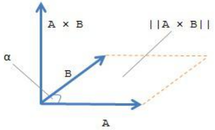
The cross product of A and B is:

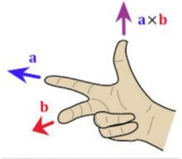
$$C = A \times B = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$
$$= (a_2 b_3 - a_3 b_2)\,\hat{i} + (a_3 b_1 - a_1 b_3)\,\hat{j} + (a_1 b_2 - a_2 b_1)\,\hat{k}.$$

Cross Product

Geometric Interpretation

$A \times B$ is a vector that is perpendicular to both A and B.





- The magnitude of $\|A \times B\|$ is equal to the area of the parallelogram formed using A and B as the sides.
- The angle between A and B is: $\|A \times B\| = \|A\| \; \|B\| \; sin(\alpha)$.
- The cross product is zero when the A and B are parallel.

Linear Independence of Vectors

Linear Independence

A set of vectors $(v_1, v_2, v_3, \cdots, v_n)$ is said to be linearly independent if the equation

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0. \tag{23}$$

can only be satisfied by $a_i = 0$ for i = 1, ... n.

Put another way: no vector in the sequence can be written as a linear combination of the other vectors.

Linear Independence of Vectors

Example 1. Consider three vectors $v_1 = (1, 1)$, $v_2 = (-3, 2)$, and $v_3 = (2, 4)$ in two-dimensional space.

The vectors will be linearly independent if the only solutions to

$$a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{24}$$

are $a_1 = a_2 = a_3 = 0$. Writing these equations in matrix form:

$$\begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{25}$$

Linear Independence of Vectors

Apply row operations (details to follow):

$$\begin{bmatrix} 1 & 0 & 16/5 \\ 0 & 1 & 2/5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (26)$$

which can be rearranged:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + a_3 \begin{bmatrix} 16/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (27)$$

We conclude that since a_1 and a_2 can be written in terms of a_3, the equations are linearly dependent.

Linear Independence of Vectors

A Few Observations

- Vectors v_1 through v_3 are two dimensional.
- Can show that three (or more) vectors in two-dimensional space will always be linearly dependent.
- Can show that four (or more) vectors in three-dimensional space will always be linearly dependent.
- This is why a stool with three legs (vectors) will always be steady (linearly independent), but one with four legs (vectors) will sometimes rock (linearly dependent).