Department of Civil and Environmental Engineering, Spring Semester, 2024

ENCE 201 Midterm 2, Open Notes and Open Book

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Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are two questions. Partial credit will be given for partially correct answers, so please show all your working.

Note: Please see the **class web page for instructions on how to submit your exam paper**.

Question	Points	Score
1	20	
2	20	
Total	40	

Question 1: 20 points.

This question covers solution of matrix equations using Gauss Elimination. Consider the matrix equations Ax = b, where:

$$\begin{bmatrix} 0 & 7 & 3 \\ 3 & 0 & 7 \\ 7 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 17 \end{bmatrix}. \tag{1}$$

[1a] (4 pts). Compute the det(A)?

$$\det(A) = 0 \det\begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix} - 7 \det\begin{bmatrix} 3 & 7 \\ 7 & 0 \end{bmatrix} + 3 \det\begin{bmatrix} 3 & 0 \\ 7 & 3 \end{bmatrix}$$

$$= 0(0-21) - 7(0-49) + 3(9-0)$$

$$= 370.$$

[1b] (4 pts). Based on your solution to part `1a`, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

1. $\det(A) \neq 0 \rightarrow \text{rank}(A) = 3$ = no rows in A.
2. Matrix inverse A^{-1} exists, $Ax = b \rightarrow x = A^{-1}.b$.
3. Number of sol'ns $Ax = b \Rightarrow 1$.

[1c] (4 pts). In terms of "complexity of matrix structure" and "row operations," what are the goals of the method of Gauss Elimination? Why does the method work?

Goal: Design a sequence of elementary row ops to simplify the Matrix structure without changing the solution.

[1d] (8 pts). Use the method of Gauss Elimination with pivoting to compute the solution to equation 1. This is a hand calculation, so show all of your working.

Augmented matrix (A|b):

$$\left[\begin{array}{ccc|c} 0 & 7 & 3 & 10 \\ 3 & 0 & 7 & 13 \\ 7 & 3 & 0 & 17 \end{array}\right] \xrightarrow{\text{Swap: } R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 7 & 3 & 0 & 17 \\ 3 & 0 & 7 & 13 \\ 0 & 7 & 3 & 10 \end{array}\right] \xrightarrow[R_2 \to R_2 - 3R_1]{R_1/7}$$

$$\left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & -9/7 & 7 & 40/7 \\ 0 & 7 & 3 & 10 \end{array}\right] \xrightarrow[R_2/7]{\text{Swap: } R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & 1 & 3/7 & 10/7 \\ 0 & -9/7 & 7 & 40/7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \frac{9}{7}R_2} \left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & 1 & 3/7 & 10/7 \\ 0 & 0 & 370/49 & 370/49 \end{array}\right]$$

Back Substitution:

$$\frac{370}{49} x_3 = \frac{370}{49} \longrightarrow x_3 = 1.0$$

$$x_2 + 3/7\, x_3 = \frac{10}{7} \longrightarrow x_2 = 1.0$$

$$x_1 + 3/7\, x_2 = \frac{17}{7} \longrightarrow x_1 = 2.0,$$

Check Solution:

$$\begin{bmatrix} 0 & 7 & 3 \\ 3 & 0 & 7 \\ 7 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 17 \end{bmatrix} \checkmark$$

Question 2: 20 points

[2a] (5 pts). Derive the Newton-Raphson formula:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right] \tag{2}$$

Hint : Start off by writing down a Taylors series expansion for $f(x+h)$. State all of your assumptions.

$$f(x+h) = f(x) + h f'(x) + \underbrace{\frac{h^2}{2!} f''(x) + \dots}_{\text{higher order terms (HOT).}} = 0$$

Ignoring HOT:

$$f(x+h) = f(x) + h f'(x) = 0$$

$$\Rightarrow x_{n+1} = x_n + h = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right]. \quad \text{— } A$$

[2b] (5 pts). Consider the function

$$y(x) = \sin^2(x). \tag{3}$$

Show that the Newton-Raphson update formula can be written as:

$$x_{n+1} = x_n - \frac{1}{2}\tan(x_n). \tag{4}$$

Be sure to show all of your working.

$$y(x) = \sin^2(x) \longrightarrow \frac{dy}{dx} = 2\sin(x)\cos(x) = \sin(2x).$$

$$\Rightarrow x_{n+1} = x_n - \frac{\sin^2(x_n)}{2\sin(x_n)\cos(x_n)}$$

$$= x_n - \frac{1}{2}\tan(x_n) \quad \text{— } B.$$

[2c] (5 pts). If $\dot{g}(x)$ and $\ddot{g}(x)$ are the first and second derivatives of $g(x)$, respectively, and

$$f(x) = \left[\frac{g(x)}{\dot{g}(x)}\right] \tag{5}$$

show that the modified Newton-Raphson formula is given by:

$$x_{n+1} = x_n - \left[\frac{g(x_n)\dot{g}(x_n)}{\dot{g}(x_n)\dot{g}(x_n) - g(x_n)\ddot{g}(x_n)}\right] \tag{6}$$

$$f(x) = \left[\frac{g(x)}{\dot{g}(x)}\right] \longrightarrow f'(x) = \frac{g'(x)^2 - g(x) \cdot g''(x)}{g'(x)^2} \quad \text{—— } (C)$$

Plug (C) into (A) & rearranging terms:

$$x_{n+1} = x_n - \left[\frac{g(x_n) \cdot g'(x_n)}{\dot{g}(x_n) \cdot \dot{g}(x_n') - g(x_n) \cdot g''(x_n)}\right].$$

[2d] (5 pts). Use a starting value $x_o = \pi/4$ and the modified Newton Raphson Formula to find an improved estimate of the root of the polynomial:

$$y(x) = \sin^2(x). \tag{7}$$

Note: Do no more than 1 iteration !!.

$$f(x) = \sin^2(x) \longrightarrow f'(x) = 2\cos(x)\sin(x) = \sin(2x)$$

$$f''(x) = 2\cos(2x).$$

Let $x_0 = \pi/4$. $f(x_0) = \sin^2(\pi/4) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$

$$f'(x_0) = \sin(\pi/2) = 1$$

$$f''(x_0) = 2\cos(\pi/2) = 0.$$

$$x_1 = x_0 - \left[\frac{\sin^2(x_0)\sin(2x_0)}{\sin^2(2x) - \sin^2(x) \cdot 2 \cdot \cos(2x)}\right] = \frac{\pi}{4} - \frac{(1/\sqrt{2})^2 \cdot 1}{1^2 - (1/\sqrt{2}) \cdot 0}$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$