ON THE CONTROLLABILITY OF BILINEAR SYSTEMS WITH DELAY

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ABSTRACT

Title of Thesis: On the Controllability of Bilinear Systems with Delay

Luther Pearson Hampton III, Master of Science, 1975

Thesis directed by: Professor John S. Baras

In this thesis we investigate systems of the type

$$\frac{dx}{dt} = \left(A + \sum_{i=1}^{p} u_i(t) B_i \right) x \quad (t) + Cx(t-\tau)$$

where x(t) R^n, u_i = 1,...p are scalar functions, measurable and bounded on finite intervals, and A, B_i, C, = 1...p are nxn matrices. In particular we devise criteria for local accessibility, controllability of more general nonlinear systems with delays and a "Bang-Bang" theory for these systems. These results generalize those existing for bilinear systems without delays and for linear delay differential systems.

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CHAPTER 1

INTRODUCTION

I.1 Motivation and Generalities

Recently bilinear systems have received a great amount of attention in the literature. This is mainly attributable to two factors. First, there is hope that this type of variable structure systems will prove to be more adequate in handling certain phenomena in the field of non linear systems. Second, this class lies in between linear and non linear cases and it is therefore believed that through its study some light can be shed on the theory of non linear systems.

Bilinear systems are systems which are linear in the control and linear in the state but not linear in the control and state jointly, a typical example would be of the form

$$x(t) = \left[A(t) + \sum_{1}^{P} B_i(t)u_i(t) \right] x(t) + Cu(t) \qquad (1)$$

$$x(0) = x_o$$

where $x(t) \in R^n, u_i(t)$ scalars, and the matrices A, B_i, C of the appropriate dimensions.

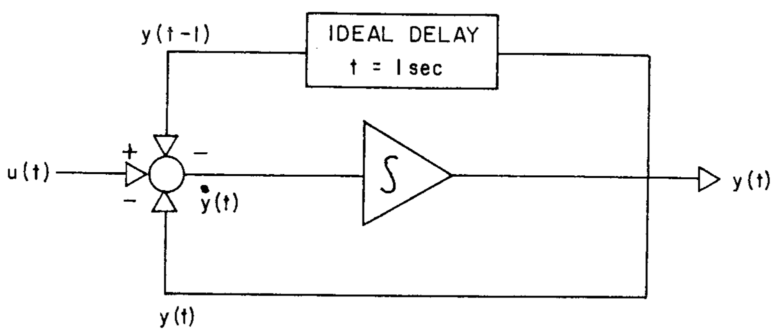
ON THE CONTROLLABILITY OF BILINEAR SYSTEMS WITH DELAY

Another class of systems, that of delay systems, has attracted the interest of researchers from the early days of system theory. This has been due to a variety of examples involving hereditary behavior. Most of this work has been done on linear systems with delays, that is systems of the form

$$\dot{x}(t) = A(t)\, x\,(t) + B(t)\, x\,(t-\tau) + C(t)\, u(t) \tag{2}$$

$$x(\theta) = \varphi(\theta); \theta \in [\tau- \;,0]$$

where $x(t) \in R^n$, $A(t)$, $B(t)$, $C(t)$ matrices of the appropriate dimension, $u(t) \in R^p$ and $\varphi(t) \in C\left\{\left[-\tau,0\right], R^n\right\}$, the continuous functions mapping $\left[-\tau,0\right]$ into R^n. As an elementary example of this form of control system consider the following network

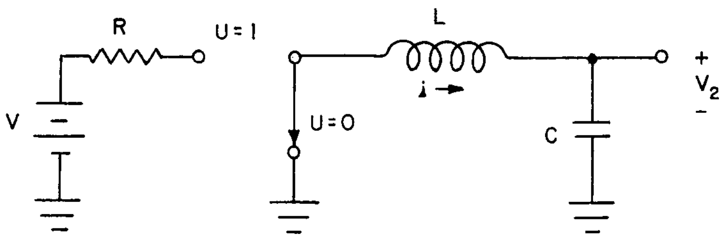


the relationship between $u(t)$ and $y(t)$ is given by

$$\dot{y}(t) = -y(t) - y(t-1) + u(t)$$

and clearly in order to predict the behavior of this system from some time $t = t_0$ onwards it is necessary to know the output $y(t)$ for $t \in \left[t_0-1, t_0\right]$ hence this is a system of the form (2).

As an example of this form of system consider the following voltage regulator



Choosing as states the voltage across the capacitor $x_2 = v_2\sqrt{C}$ and the current through the inductor $x_1 = i\sqrt{L}$, and letting $\omega = \frac{1}{\sqrt{LC}}$ the equations become

$$L\dot{i} = -v_2 + u(V - Ri)$$

$$C\dot{v}_2 = i$$

or

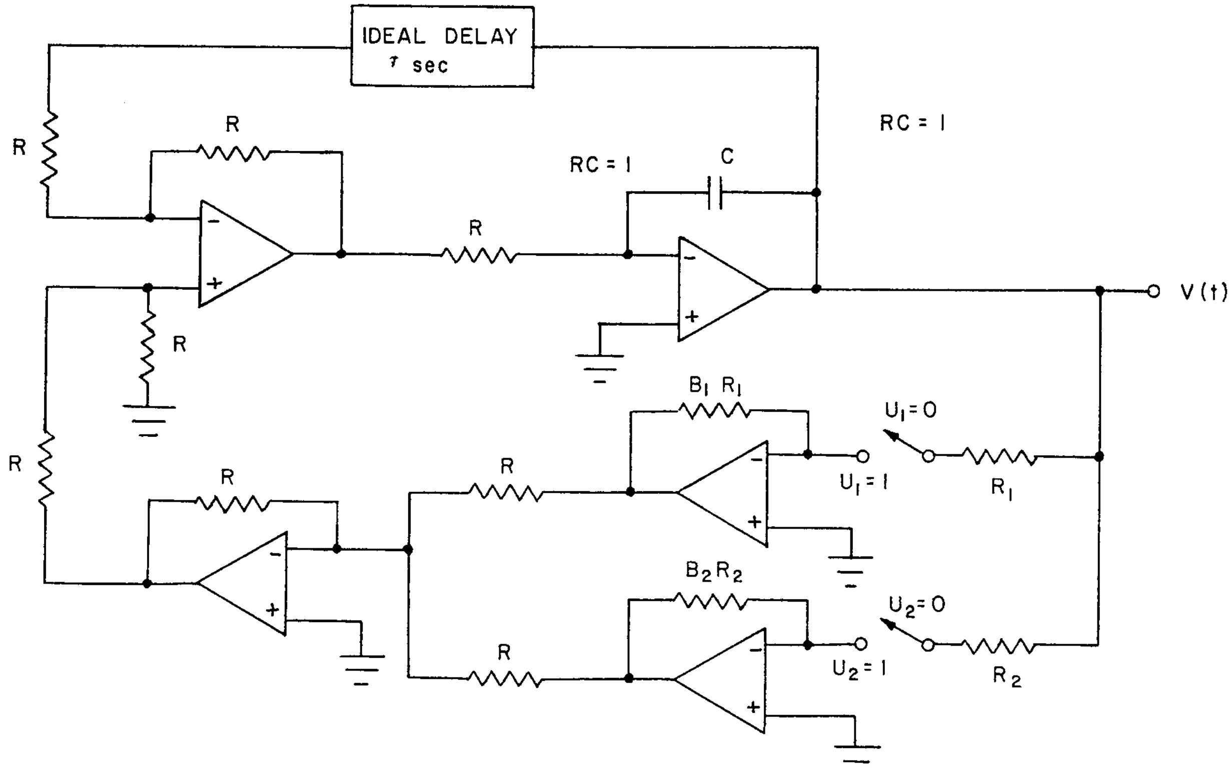
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + u\begin{pmatrix} \frac{R}{L} & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{pmatrix} \frac{V}{\sqrt{L}} \\ 0 \end{pmatrix} u$$

which is in the form of the equation (1).

A great amount of literature is available on bilinear systems, see for example Brockett [1], Hermes [2], Jurdjevic and Sussman [4] - [6] and Bruni et al [7].

As has been previously mentioned the literature in this field is varied and rich, see for example Weiss [8], Popov [9], Banks [11], Hale [13], and Delfour and Mitter [14].

There are however a variety of practical problems where the dynamics display both hereditary behavior and bilinearity. Here is a representative example.



Consider the problem of controlling the output v(t) of the operational amplifier by manipulating the switches $u_1(t)$, $u_2(t)$. Is this possible and if so how would this be done? The equation for this system is:

$$\dot{v}(t) = Av(t) + Cv(t-\tau) + \sum_{1}^{2} B_i u_i(t)\ v(t)$$

I.2 Objectives

In this thesis systems of the form

$$\dot{x}(t) = Ax(t) + \sum_{1}^{p} B_i \; u_i(t) \; x \; (t) + Cx(t-\tau)$$

$$x(\theta) = \varphi(\theta); \theta \in \left[\tau - \tau, 0\right]$$

will be studied

where $x(t) \in R^n$, $u_i(\cdot)$ bounded and measurable on any finite interval and A, B_i, C matrices of the appropriate dimensions.

The objectives of this thesis are:

(1) to reach a definition of the state consistent with accepted definitions,

(2) with respect to this idea present sufficient conditions for complete controllability of systems of this form

(3) to examine various properties of the trajectories of these systems.

(4) Also, certain properties of the sets attainable from any particular initial function will be studied and a theorem on compactness of the attainable set will be presented. This will lead to a "bang-bang". Theorem for these systems. These will hopefully pave the way in later papers for solutions to optimal control problems in systems of this form.

I.3 Previous Work and Mathematical Background

In order to summarize previous material several notions are needed. First notice that given the system

$$\dot{x}(t) = \left[A + \sum_{1}^{p} B_i u_i \right] x(t) + Cx(t-\tau) \tag{1}$$

it is impossible to uniquely specify a solution from any particular starting time t_o without first specifying an initial function on $\left[t_o - \tau, t_o\right]$. Were this not given the value of $x(t-\tau)$ would be unknown on $\left[t_o, t_o + \tau\right]$. It would be impossible to proceed. Thus in keeping with the accepted definition of the state as the minimum amount of information necessary to predict the behavior of the system from any time forward we consider, as usual [13], the state as a function which is defined on $\left[t_o - \tau, t_o\right]$ and consider the system in this manner. As will be shown later, in order to guarantee existence and uniqueness the initial state or function will be assumed continuous. Thus the state space can be $C\left\{\left[-\tau, 0\right]; R^n\right\}$ the Banach space of continuous functions mapping $\left[-\tau, 0\right]$ into R^n and the state at time t_1 will be the trajectory of the system on $\left[t_1 - \tau, t_1\right]$.

These same arguments naturally apply for the linear system with delays

$$\dot{x}(t) = A(t)\, x(t) + C(t)\, u(t) + B(t)\, x(t-\tau) \tag{2}$$

$$x(\theta) = \varphi(\theta); \theta \in \left[\tau - \tau, 0\right].$$

In some problems it is only desired to know certain properties of the set of attainable points in R^n. This leads to the following set of definitions.

Let $x(t;t_o,\varphi,u)$ be the solution of system (1) or (2) starting at t_o with initial condition $\varphi \in C\left\{\left[t_o-\tau,t_o\right]; R^n\right\}$ and using control u. For a fixed t define

$$x_t(\delta) = x(t+\delta;\ t_o,\varphi,u)$$

then $x_t(\delta)$ is an element of $C\left\{\left[\tau- ,0\right];R^n\right\}$ for $\delta \in \left[-\tau,0\right]$ and in keeping with the above arguments can be defined as the state of the system (1) or (2), see Hale [13].

<u>Definition I.3.1:</u> The linear system (2) is said to be completely controllable to $H \subset C^1\left\{\left[-\tau,0\right];R^n\right\}$ (where C^1 is the subspace of continuously differentiable functions) at time t_1 if given any function $\psi \in H$ and any initial condition φ there exists an admissible control u(t) such that

$$x_{t_1}(\delta) = \psi(\delta);\delta \in \left[-\tau,0\right].$$

<u>Definition I.3.2</u>: The linear system (2) is completely euclidean controllable at time t_1 if given any $y \in R^n$ and any $\varphi \in C$ then there exists an admissible control u such that

$$x(t_1;t_o,\varphi,u) = y$$

For convenience we let $C = C\left\{\left[-\tau,o\right];R^n\right\}$

<u>Definition I.3.3</u>: The linear system (2) is pointwise complete if the range of the map

$$g: C\left\{\left[-\tau,0\right];R^n\right\} \longrightarrow R^n$$

$$\varphi \longmapsto x(t;t_o,\varphi,o)$$

is R^n for every $t \geq t_o$.

For $t \geq t_o$ define

$$W(t_o,t_1) = \int_{t_o}^{t_1} K(s,t_1)C(s)C^T(s)K^T(s,t_1)ds \tag{3}$$

where K(s,t) is the matrix valued solution to the equations

$$\frac{\partial}{\partial s} K(s,t) = -K(s,t)A(s)-K(s+\tau,t)B(s+\tau),\ t_o \leq s \leq t-\tau \tag{4}$$

$$\frac{\partial}{\partial s} K(s,t) = -K(s,t)A(s),\ t-\tau \leq s \leq t \tag{5}$$

$$K(t,t) = I \tag{6}$$

$$K(s,t) = 0 \text{ elsewhere} \tag{7}$$

For $t \geq t_o$, $t_o \leq s \leq t$, define

$$V(t_o-\tau,t_o) = \int_{t_o-\tau}^{t_o} K(s+\tau,t)B(s+\tau)B^T(s+\tau)K^T(s+\tau,t)ds \tag{8}$$

Theorem I,3.4: (Ono and Yamasaki [18])

The system (2) is completely euclidean controllable at time t_1 if and only if

(I) rank $V(t_o-\tau,t_o) = n$

(II) Rank $W(t_o,t_1) = n$

Theorem I.3.5: (Ono and Yamasaki [18])

System (2) is controllable to any $\psi \in H$ (the subspace of continuously differentiable functions on $[t_o-\tau,t_o]$) if and only if (I) and (II) hold at $t_1-\tau$ and also given $\varphi \in C$ then with $u[t_o,t_1-\tau]$ such that $x(t_1-\tau;t_o,\varphi,u[t_o,t_1-\tau]) =0$ the equation

$$\text{(III) } C(t)u(t) = \dot{\psi}(t-t_1+t_o) - A(t)\psi(t-t_1+t_o)$$

$$-B(t)x(t-\tau;t_o,\varphi,u[t_o,t_1-\tau]);\ t\in[t_1-\tau,t_1]$$

has an admissible solution for all $\psi \in H$

Define the operator ad_A for two nxn matrices by

$$ad_A B = [A,B]$$

where $[A,B] = AB - BA$, the lie bracket, and inductively define

$$ad_A^K B = [A, ad_A^{K-1} B]$$

Theorem I.3.6: (Brockett [1])

Consider the dynamical system

$$X(t) = \sum_{1}^{M} u_i(t)B_i \; X(t); \; X(t) \text{ an nxn matrix}$$

given time $t_a>0$ and given two nonsingular matrices X_1 and X_2, there exist piecewise continuous controls which steer X_1 to X_2 if and only if $X_2X_1^{-1}$ belong to $\left\{ \exp \left\{ B_i \right\}_A \right\}_G$, the lie group generated by exponentiating the smallest lie algebra containing the set B_i.

Theorem I,3.7.: (Brockett [1])

Consider the dynamical system

$$\dot{X}(t) = (A + \sum_{1}^{v} u_i(t)B_i)X(t), \; X(t) \text{ an nxn matrix, and suppose that}$$

$\left[ad_A^k B_i, B_j \right] = 0$ for i, j = 1, 2, ..., v and k = 0,1,...,n^2-1. Then given time $t_a>0$ and two nxn matrices X_1 and X_2 there exist controls which transfer the system from X_1 at t = 0 to X_2 at t = t_a if and only if there exists $L \in H$ (H the linear subspace of R^{nxn} matrices spanned by $ad_A^k B_i$ for i = 1, 2, ..., v and k = 0,1,...n^2-1 such that

$$X_2 = e^{At_a} e^{L} X_1$$

CHAPTER II

EXISTENCE AND UNIQUENESS OF SOLUTIONS

II.1 Existence and Uniqueness of Solutions; Dependence on Initial Conditions.

$$\dot{x}(t) = Ax(t) + Cx(t-\tau) + \sum_{1}^{p} B_i u_i(t)\, x(t) \qquad (1)$$

$$x(\lambda) = \varphi(\lambda)\,,\ \lambda \in [-\tau, 0]$$

Theorem II.1.1: (Bellman)

If there exists a constant $C_1 > 0$ such that

a) $$\int_0^{\tau} \left\| \left[A + \sum_{1}^{p} B_i u_i(s) \right] \right\| ds < \infty$$

and

b) $$\left\| \int_0^{s} C\varphi(t-\tau)dt \right\| < C_1\,, \quad \forall s \in [o, \tau]$$

then there exists a unique bounded solution to (1) on $[0, \infty)$, furthermore $x(t)$ is continuous if

$$f(s) = \int_0^{s} C\varphi(t-\tau)\, dt \text{ is continuous}$$

Proof:

let

$$x_o(s) = \int_o^s C\varphi(t-\tau)dt$$

$$x_{n+1}(s) = x_o(s) + \int_o^s \left[A + \sum_1^p B_i u_i(s) \right] x_n(s)ds$$

let $[0,t_1]$ be the interval such that

$$\int_o^{t_1} \left\| \left[A + \sum_1^n B_i u_i(s) \right] \right\| ds \leq b < 1$$

If $t_1 \geq \tau$ the Liouville Neuman solution obtained by straightforward iteration is valid via the contraction mapping theorem in $[o,\tau]$. If $t_1 < \tau$ proceed as follows let

$$v_n = \sup \left\| x_n(t) \right\| \quad , \; t \in [o,t_1]$$

$$\left\| x_{n+1}(t) \right\| \leq C_1 + v_n \int_o^{t_1} \left\| A + \sum_1^p B_i u_i(s) \right\| ds$$

$$\leq C_1 + bv_n$$

Hence if $A_{n+1} = C_1 + bA_n$, $A_o = C_1$ we have $v_{n+1} \geq A_{n+1}$ then $\left\{ A_n \right\}$ is monotome increasing and uniformly bounded by

$$A = C_1/1-b \text{ since } 0 \leq b < 1$$

Then it follows that each integral in the iteration scheme exists and that $\{x_n\}$ is uniformly bounded in $[0,t_1]$. To establish convergence write

$$x_{n+1}(t) - x_n(t) = \int_o^t \left[A + \sum_1^p C_i u_i(s)\right]\left[x_n(s) - x_{n-1}(s)\right] ds$$

and obtain for $n \geq 1$

$$W_{n+1} = \sup_{t \in [0,t_1]} \left\| x_{n+1}(t) - x_n(t) \right\|$$

$$\leq \left(\sup_{t\ [0,t_1]} \left\| x_n(t) - x_{n-1}(t) \right\| \int_o^{t_1} \left\| A + \sum B_i u_i(s) \right\| ds \right)$$

$$\leq bW_n$$

and the series $\sum_o^\infty \left[x_{n+1} - x_n\right]$ is uniformly convergent by comparison with the geometric series $\sum_o^\infty b^n$. Hence $\{x_n\} \rightarrow x(t)$ bounded. Employing the Lesbesque dominated convergence theorem we may pass the limit in the iteration and establish the solution on $(0,t_1)$ and then by repeating the process we can extend the solution to $(0,2t_1)$ etc. We see that if U, the set of admissible controls, is composed of functions measurable and bounded on any finite interval and $\varphi \in C\left\{\left[\tau- \ ,0\right];R^n\right\}$ then if the integrals are taken in the sense of Lebesque it is possible to extend the solution over any finite interval, thus the theorem is complete.

The theorem also establishes the uniqueness of the solution.

II.2 The Fundamental Matrix

Consider the system

$$\dot{x}(t) = \left[A + \sum_{1}^{p} B_i u_i(t)\right] x(t) + Cx(t-\tau)$$

$$x(\theta) = \varphi(\theta), \theta \in [-\tau, 0]$$

Lemma II.2.1:

The solution to the above system is given by

$$x(t) = K(t_o,t)\,\varphi(t_o) + \int_{t_o-\tau}^{t_o} K(s+\tau,t)C\,\varphi(s)ds$$

where K(s,t) is defined for $t_o \leq s \leq t$, $t \geq 0$, the matrix valued solution to the equations

$$\frac{\partial}{\partial s} K(s,t) = -K(s,t)\left[A + \sum_{i=1}^{p} B_i u_i(s)\right] - K(s+\tau,t)C$$

$$t_o - \tau \leq s \leq t - \tau$$

$$\frac{\partial}{\partial s} K(s,t) = -K(s,t)\left[A + \sum_{i=1}^{p} B_i u_i(s)\right] \quad t-\tau \leq s \leq t$$

$$K(t,t) = I$$

$$K(s,t) = 0 \text{ elsewhere}$$

Proof: See Bellman and Cooke [19].

Definition II.2.2: The matrix K(s,t) defined ve is the fundamental matrix of the bilinear delay differential system II.1(1), corresponding to controls u_i, $i = 1, \cdots p$.

Notice that a more correct (perhaps) notation should be K(s,t;u) to emphasize the dependence of the fundamental matrix on the controls u.

II.3 The State Of a Bilinear System With Delay

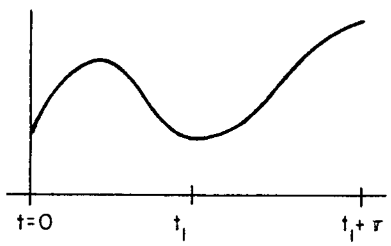
Consider the system

$$\dot{x}(t) = \left[A + \sum_{1}^{P} B_i u_i \right] x\ (t) + Cx(t-\tau) \qquad (1)$$

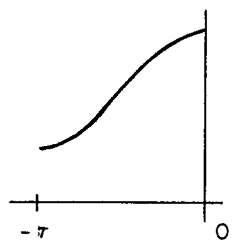
$$x(\theta) = \varphi\ (\theta)\ ,\ \theta \in [-\tau, 0]$$

Let $x(t; \varphi, u)$ be defined as the solution of (1) with initial condition $\varphi \in C \left\{ [-\tau, 0]\ ;\ R^n \right\}$ and control u.

The state of system (1) at time t_1 is defined as the trajectory of (1) on $[t_1-\tau, t_1]$ viewed as an element of $C \left\{ [-\tau, 0];\ R^n \right\}$



Trajectory of system 1



State of system 1.

That is using the definition of $x_t(\lambda)$ as in section II.1

$$x_t(\lambda) = x(t+\lambda)$$

it is possible to define the state at time t_1 of a system with trajectory

$$x(t, t_o, \psi, u)$$

as $\qquad x_{t_1}(\lambda) \quad \lambda \in [-\tau, 0]$

II.4 Controllability and Accessibility

Since the subject of this thesis is the investigation of properties of the attainable sets, both in R^n and in $C\left\{[-\tau,o];R^n\right\}$ the following definitions are presented here. For the system is again described here

$$\dot{x}(t) = \left[A + \sum_{1}^{p} B_i u_i(t)\right] x(t) + C\, x(t-\tau)$$

$$x(\theta) = \varphi(\theta), \theta \in [-\tau,o]$$

The reachable set in R^n from initial condition φ, at time $t > 0$ will be denoted by $R(t,\varphi)$, and it is the set of all $y \in R^n$ such that $x(t;\, o,\varphi,u) = y$ for some admissible control u. The reachable set in R^n from initial condition φ, in time $t > 0$ will be denoted by $\mathbb{R}(t,\varphi)$ and it is the set $\mathbb{R}(t,\varphi) = \bigcup_{s \leq t} R(s,\varphi)$. The reachable set in R^n from initial condition φ will be denoted by $\mathbb{R}(\varphi)$ and is the set $\mathbb{R}(\varphi) = \bigcup_{t \geq o} \mathbb{R}(t,\varphi)$. We have similar notions for function space reachability. For ease of notation we let C denote $C\left\{\left[-\tau,o\right] R^n\right\}$ and C^1 denote $C^1\left\{\left[-\tau,o\right];\, R^n\right\}$. Then the reachable set C^1 from initial condition φ, at time $t > o$, will be denoted by $R_C(t,\varphi)$, and it is the set of all $\lambda \in C^1$ such that $\lambda(\theta) = x_t(\theta),\, \theta \in [-\tau,o]$ for some admissible control u. Similarly the reachable set in C^1 from initial condition φ, in time $t > o$, is the set $\mathbb{R}_C(t,\varphi) - \bigcup_{o \leq s \leq t} R_C(s,\rho)$ and the reachable set in C^1 from initial condition φ, is the set $\mathbb{R}_C(\varphi) = \bigcup_{t \geq o} \mathbb{R}_C(t,\varphi)$.

We have the following set of definitions.

Definition II.4.1: Let $\lambda(\bullet) = x(\bullet;o,\varphi,u)$ be a trajectory of the system. The system has the local accessibility property along λ, in R^n, at time t_1 if there exists an R^n - neighborhood of $x(t_1;\, o,\varphi,u)$ which is included in $R(t_1,\varphi)$.

Definition II.4.2: Let λ be as above. The system has the local accessibility properly along λ , in function space, at time t_1 if there exists a C neighborhood of X_{t_1} which is included in $R_C\ (t_1,\varphi)$.

Definition II.4.3: The system is euclidean controllable (resp at time t_1, in time t_1) from initial condition φ if $\mathbb{R}(\varphi) = R^n$ (resp $R(t_1,\varphi) = R^n$, $\mathbb{R}(t_1,\varphi) = R^n$).

Definition II.4.4: The system is function space controllable to a subspace $H \subseteq C^1$ (resp at time t_1, in time t_1) from initial condition φ if $H \subseteq \mathbb{R}_C(\varphi)$ (resp $H \subseteq R_C(t_1,\varphi)$, $H \subseteq \mathbb{R}_C(t_1,\varphi)$)

Definition II.4.5: The system is completely euclidean controllable (at time t_1, in time t_1) if it is euclidean controllable (at time t_1, in time t_1) from every initial condition φ.

Definition II.4.6: The system is completely function space controllable to the subspace $H \subseteq C^1$ (at time t_1, in time t_1) if it is function space controllable to H (at time t_1, in time t_1) from every initial condition φ.

Definition II.4.7: The system has the euclidean accessibility property from φ (resp the accessibility property in function space if $\mathbb{R}(\varphi)$ (resp. $\mathbb{R}_C(\varphi)$) has non empty interior in R^n (resp in C).

Definition II.4.8: The system has the euclidean accessibility property (resp the accessibility property in function space) if it has the euclidean accessibility property (resp the accessibility property in function space) from every initial condition φ.

Definition II.4.9: If we replace $\mathbb{R}(\varphi)$ (resp $\mathbb{R}_C(\varphi)$) with $R(t,\varphi)$ (resp $R_C(T,\varphi)$) for some $t > 0$ in Definition II.4.7 and II.4.8 we have the strong euclidean accessibility property, (resp strong accessibility property in function space) from initial condition φ. Similarly for every φ.

CHAPTER III

ACCESSIBILITY PROPERTIES

III.1 Preliminaries

In this section certain conditions guaranteeing the accessibility property in bilinear systems with delay will be presented. These conditions will be dependent on the existence of controllability of linearized systems derived from the bilinear systems. The method used in the primary theorem of this section is based on a method used by Weiss [8] to derive certain controllability properties for general nonlinear systems.

We will only provide sufficient conditions. It is a consequence of the definitions given in the last section of the previous chapter that the system has the strong euclidean accessibility (resp. strong accessibility property in function space) from initial condition φ if and only if it has the euclidean accessibility property (resp. in function space) along all trajectories emanating from φ at some time $t>t_0$, the same for all trajectories. It may help to note that whether we are in R^n or in a function space if the system has the local accessibility along some trajectory at t_1, it certainly has the local accessibility property along the same trajectory for all $t_2 > t_1$. So conditions guaranteeing the local accessibility property also imply strong accessibility. Now local accessibility is very strongly related to controllability of linearized systems. The main thrust behind all of these ideas is that in some cases the accessibility property implies controllability, and that it is easier to check for accessibility. Of course note that controllability implies accessibility.

III.2 Local Accessibility via Controllability of Linearized Equations

In this section we make precise the relation between local accessibility and controllability of linearized equations. The method is a variation of a method previously used by Weiss [8]. Consider the general non-linear differential-delay system

$$\dot{x}(t) = f(t,x(t),x(t-\tau),u(t)) \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^p$, and u bounded and measurable on any finite interval, f is continuously differentiable in all its arguments and $f(t,0,0,0) = 0$. The linearized system about the trajectory $x_o(t) = x(t;0,\varphi,u_o)$ is defined as

$$\dot{y}(t) = A(t)y(t)+B(t)y(t-\tau)+C(t)u(t) \tag{2}$$

where

$$\begin{aligned} A(t) &= \frac{\partial}{\partial x} f(t,x_o(t),x_o(t-\tau),u_o(t)) \\ B(t) &= \frac{\partial}{\partial x_{-\tau}} f(t,x_o(t),x_o(t-\tau),u_o(t)) \\ C(t) &= \frac{\partial}{\partial u} f(t,x_o(t),x_o(t-\tau),u_o(t)) \end{aligned} \tag{3}$$

where φ is any admissible state, $u_o \in U$, the set of admissible controls and $x_{-\tau}(t) = x(t-\tau)$.

Theorem III .2.1:

Suppose system (2) is completely function space controllable at t_1 along trajectory $x(t,0,\varphi,u_o)$ then the system (1) has the euclidean local accessibility property along $x(t,0,\varphi,u_o)$ at $t\in\left[t_1-\tau,t_1\right]$

Proof:

Let $x_o(t;o,\varphi,u_o) = x_o(t)$ and substitute in (1)

$$x(t) = -z(t) + x_o(t) \tag{3}$$

where $x(t)$ is any other trajectory of system (1)

Then (1) becomes

$$\dot{z}(t) = -\dot{x}_o(t) + f(t,x(t),x(t-\tau),u(t)) \tag{4}$$

then

$$z(t) = -x_o(t) + \varphi(o) + \int_o^t f(\sigma,x(\sigma),x(\sigma-\tau),u(\sigma))d\sigma \tag{5}$$

Now introduce a parameter ξ into (5) by letting

$$u^{\xi}(t) = u(t,\xi) = \begin{cases} u_o(t) + C^T(t)K^T(t,t_1)\xi, & o\leq t\leq t_1-\tau \\ u_o(t) + \Gamma(t), \text{ where } \Gamma(t) \text{ is the solution to} \\ C(t)u(t) = -B(t)z(t-\tau,o,u^{\xi}[t_o,t_1-\tau]) \text{ for} \\ t_1-\tau\leq t\leq t_1 \end{cases} \tag{6}$$

Let the solution of (5) be $z(t,0,\xi)$ and define

$$J(t) = \frac{\partial z}{\partial \xi}(t,0,\xi)\Big|_{\xi = 0} \tag{7}$$

now noting that $z(t,0,0) = 0$ and $u^o(t) = u_o(t)$ for $t \in [o, t_1-\tau]$ we have

$$J(t) = \int_o^t \left[A(\sigma)J(\sigma) + B(\sigma)J(\sigma-\tau) + C(\sigma)\frac{\partial u^\xi}{\partial \xi}(\sigma) \right]_{\xi=0} d\sigma \tag{8}$$

Hence $\dot{J}(t) = A(t)J(t) + B(t)J(t-\tau)$

$$+ \begin{cases} C(t)C^T(t)K^T(t,t_1) & 0 \le t \le t_1-\tau \\ -B(t)J(t-J);\ t_1-\tau \le t < t_1 \end{cases} \tag{9}$$

where $K(s,t)$ is the kernel introduced in II.2

Then (9) becomes

$$J(t) = \int_o^t K(s,t)C(s)C^T(s)K^T(s,t)ds \tag{10}$$

$$t \in [0, t_1-\tau]$$

However if the system is assumed completely controllable at t_1 then this implies

$$\det J(t_1-\tau) \neq 0$$

also, on $[t_1-\tau, t_1]$ (9) becomes

$$\dot{J}(t) = A(t)\ J(t) \tag{11}$$

Hence it follows that $\det J(t) \neq o$ on $[t_1-\tau, t_1]$.

Consider the map.

$$g: R^n \times R^n \longrightarrow R^n$$

$$g(\xi, y) = x(t; o, \varphi, u^{\xi}) - y$$

Then clearly $g(o, x(t, o, \varphi, u^o)) = o$, and the Jacobian with respect to ξ is of full rank for $t \in [t_1-\tau, t_1]$. Then by the implicit function theorem there exists an open neighborhood N_o of $x(t; o, \varphi, u^o)$ in R^n, such that for every open neighborhood of $x(t; o, \varphi, u^o)$ $N \subset N_o$, there exists a unique continuous map $\pi : N \longrightarrow R^n$ such that $g(\pi(y), y) = o$ for all $y \in N$. But this is precisely the statement of local accessibility property along x_o at $t \in [t_1-\tau, t_1]$

Corollary III.2.2:

Suppose system (2) is completely euclidean space controllable at t_1 along trajectory $x_o(t)$. Then the system (1) has the euclidean local accessibility property along $x_o(t)$ at t_1.

Proof: Obvious.

III.3. An Algebraic Condition For Local Accessibility

Consider the bilinear differential delay system

$$\dot{x}(t) = \left[A + \sum_{1}^{p} B_i u_i(t) \right] x\ (t) + Cx(t-\tau) \tag{1}$$

$$x(\theta) = \varphi(\theta);\ \theta \in [-\tau, o]$$

Let $x_o(t)$ be the trajectory of system (1) with initial condition and controls u_{io}, where u_{io} has at least K continuous derivatives for each i. Define

$$\hat{B}_x(t) = \left[B_1x(t) : B_2x(t) : \ldots . : B_px(t) \right]$$

$$\hat{u}(t) = (u_1, u_2 \ldots . u_p)^T$$

then (1) becomes

$$x(t) = Ax(t) + \hat{B}_x(t)\hat{u}(t) + Cx(t-\tau)$$

then

$$\frac{\partial}{\partial x} \left[Ax(t) + \hat{B}_x(t)\hat{u}(t) + Cx(t-\tau) \right] \Bigg|_{(x_o, u_o)} = A + \sum_{i=1}^{p} u_{io}(t) B_i \overset{\Delta}{=} \hat{A}(t) \tag{2}$$

$$\frac{\partial}{\partial x_{-\tau}} \left[Ax(t) + \hat{B}_x(t)\hat{u}(t) + Cx(t-\tau) \right] \Bigg|_{x_o, u_o} = C \tag{3}$$

$$\frac{\partial}{\partial \hat{u}} \left[Ax(t) + \hat{B}_x(t)\hat{u}(t) + Cx(t-\ ) \right] \Bigg|_{x_o, u_o} = \hat{B}_{x_o}\ (t) \tag{4}$$

Now proceeding in a manner first used by Buckaloo [16] for linear differential delay systems define

$$P_o(t) = \widehat{B}_{xo}(t)$$

$$P_1(t) = -\widehat{A}(t)P_o(t) + \dot{P}_o(t)$$

$$P_k(t) = -\widehat{A}(t)P_{k+1}(t) + \dot{P}_{k-1}(t)$$

$$Q_c(t,xo,k) = [P_o(t):P_1(t):\ldots P_{k-1}(t)]$$

Theorem III.3.1

Let $x_o(t) = x(t;\varphi,u_o)$ be a trajectory of (1). Suppose $\exists\ t_1 > 0$ such that rank $Q_c(t_1,x_o,k) = n$. Then the system (1) has the local euclidean accessibility property along $x_o(t)$, at t_1.

Proof: Notice that if $K(s,t_1)$ is the fundamental matrix of the linearized system, then rank $Q_c(t,x_o,k) = n$ implies

$$\text{rank } W(o,t_1) = \text{rank}\int_o^{t_1} K(s,t_1)\widehat{B}_{x_o}(s)\widehat{B}^T_{x_o}(s)\ K^T(s,t_1)ds = n$$

To see this consider the Wronskian matrix

$$M(s,t_1) = \left[K(s,t_1)\widehat{B}_{x_o}(s): \frac{\partial}{\partial s}\ K(s,t_1)\widehat{B}_{x_o}(s):\ldots \frac{\partial^{k-1}}{\partial s^{k-1}}\ K(s,t_1)\widehat{B}_{x_o}(s)\right]$$

for $s \in [t_1-\tau,t_1]$.

But then by the defining properties of the fundamental matrix we have

$$K(s,t_1)\widehat{B}_{x_o}(s) = K(s,t_1)P_o(s)$$

$$\frac{\partial}{\partial s} K(s,t_1)\widehat{B}_{x_o}(s) = \frac{\partial}{\partial s}(K(s,t_1)P_o(s) = \left(\frac{\partial}{\partial s} K(s,t_1)\right) P_o(s) +$$

$$+ K(s,t_1)\dot{P}_o(s) = - K(s,t_1)\widehat{A}(s)P_o(s) +$$

$$+ K(s,t_1)\dot{P}_o(s) = K(s,t_1)P_1(s)$$

and similarly

$$\frac{\partial^{k-1}}{\partial s^{k-1}} K(s,t_1)\widehat{B}_{xo}(s) = K(s,t_1)P_{k-1}(s)$$

That is

$$M(s,t_1) = K(s,t_1)Q_c(s,x_o,k);\ t_1-\tau \leq s \leq t_1$$

Clearly though $K(s,t_1)$ is nonsingular for $s \in [t_1-\tau,t_1]$ since it satisfies the differential equation

$$\frac{\partial}{\partial s} K(s,t_1) = -K(s,t_1)\widehat{A}$$

$$K(t_1,t_1) = I$$

Therefore rank $M(s,t_1)$ = rank $Q_c(s,x_o,k)$ for $s \in [t_1-\tau,t_1]$. But since rank $Q_c(t_1,x_o,k) = n$ this implies that the rows of $K(s,t_1)\ B_{x_o}(s)$ are linearly independent time functions over $[t_1-\tau,t_1]$ and therefore over $[o,t_1]$.

Hence

$$\text{rank } W(o,t_1) = n.$$

Now this condition implies complete euclidean controllability of the linearized system for (1), and this inturn implies by Corollary III.2.2, that (1) has the local accessibility property along $x_o(t)$, at t_1. Notice that by introducing the operator

$$\Gamma(\bullet) = -\hat{A}(t)(\bullet) + \frac{d}{dt}(\bullet)$$

we can easily see that

$$Q_c(t_1,x_o,k) = \left[\hat{B}_{x_o}(t_1) : \Gamma\hat{B}_{x_o}(t_1) : \ldots : \Gamma^{k-1}\hat{B}_{x_o}(t_1)\right]$$

CHAPTER IV

CONTROLLABILITY RESULTS

IV.1 A Result For General Nonlinear Systems with Delay

Consider the system

$$\dot{x}(t) = A(x(t)) + B(x(t))u(t) + Cx(t-\tau) \tag{1}$$

$$x(\theta) = \varphi(\theta);\ \theta \in [-\tau, o]$$

Consider the linear system

$$\dot{x}(t) = A(x(t)) + Cx(t-\tau) + B(z(t))u(t)) \tag{2}$$

$$x(\theta) = \varphi(\theta);\ \theta \in [-\tau, o]$$

Where z is some fixed function $z \in C$ the Banach space of continuous R^n valued functions on $[0, t_f]$. For each fixed z system (2) is linear and the solution is given by

$$x(t) = K(0,t)\varphi(t_o) + \int_{-\tau}^{0} K(s+\tau, t)C\varphi(s) + \int_0^t K(s,t)\, B(z(s))u(s)ds \tag{3}$$

where the kernel K(s, t) is as defined before (see II. 2)

Define the controllability grammian G by

$$H(s,t,z(s)) = K(s,t)\, B(z(s)) \tag{4}$$

$$G(t,z) = \int_0^t H(s,t,z(s))\, H^T(s,t,z(s))ds \tag{5}$$

We present now the following theorem on controllability which is inspired by the method used in Kunze [20].

Theorem IV.1.1: The system

$$\dot{x}(t) = A(x(t)) + B(x(t))u(t) + Cx(t-\tau)$$

is completely euclidean controllable at $t_f > 0$ if the following conditions hold:

i) $M < \infty$ such that $|A_{ij}(x)| < M, |B_{ij}(x)| \leq M$ for all i, j and all x.

ii) C such that $\inf_{z \in C\{[o, t_f] : R^n\}} \det G(t_f, z) > C$

Proof: By hypothesis ii) given any z and final point $y \in R^n$ we may select a control u which derives the system (2) from the initial condition φ to $x(t_f) = y$. We may in fact write down the control, it is:

$$u(s, t_f, \varphi, z) = B^T(z(s))K^T(s, t_f)G^{-1}(t_f, z)\{y - K(o, t_f)\varphi(0) - \int_{-\tau}^{o} K(s+\tau, t_f)C\varphi(s)ds\} \quad (6)$$

This is easily seen by substitution in (3).

Thus define the operator

$$P: C\{[o, t_f]; R^n\} \to C\{[o, t_f] : R^n\}$$

by $P(z) = x_z(t)$, where x_z is the solution of (2) utilizing control (6). That is P sends any given function on $[o, t_f]$ into a trajectory of the linear system (2) moving from $\varphi(o)$ to y.

We now claim P(z) has a fixed point. First note that P is clearly continuous. Let $M(o, t, \varphi) = K(o, t)\varphi(o) + \int_{-\tau}^{o} K(s+\tau, t)C\varphi(s)ds$. Now note that by condition ii) of the hypothesis the matrix $G^{-1}(t_f, z)$ exists and that

$$C_2 = \sup_{z \in C\{[o, t_f]; \mathbb{R}^n\}} |G^{-1}(t_f, z)| < \infty$$

Where $|\cdot|$ is the induced matrix norm; now

$$\|P(z)\| = \sup_{t\in[o,t_f]} \|P(z)(t)\|_{R^n}$$

$$= \sup_{t\in[o,t_f]} \|M(o,t,\varphi) + \int_o^t K(s,t)B(z(s))B^T(z(s))K^T(s,t_f)$$

$$G^{-1}(t_f, z)\{y - M(o,t_f,\varphi\} ds\|_{R^n}$$

By i) M and K(s, t) are bounded on $[o, t_f]$ (compact) hence

$$\|P(z)\| \le C_1 + C_2 \int_o^{t_f} \|B(z(s))B^T(z(s))(y - M(o,t_f,\varphi)\|_{R^n} ds$$

but by i) this is also bounded hence

$$\|P(z)\| \le C_3 < \infty \ \forall z \in C\{[o,t_f]; \mathbb{R}^n\}$$

Consider the convex closed

$$A = \{z \in C\{[o,t_f];\mathbb{R}^n\} \|z\| \le C_3\}$$

P maps the set A into a subset of itself which is compact (easy to see this) hence by Schauder's theorem P has a fixed point.

We need only note that application of the control (6) drives the linear system (2) from $\varphi(0)$ to y for any given y. But for some $z^* \in C\{[o, t_f]; R^n\}$ (the fixed point of P) this trajectory will be duplicated by the trajectory of the system under the desired control hence a solution of the original system (1) and we are done.

IV.2 Bang-Bang Control

Consider the system

$$\dot{x}(t) = [A+ \sum_{1}^{n} B_i u_i(t)]\ x(t) + Cx(t-\tau) \qquad (1)$$

$$x(\theta) = \varphi(\theta), \theta \in [-\tau,0];\ \varphi \in C\ \{[-\tau,0]\ ;R^n\}$$

Let

V(T) = set of all measurable functions defined on [0,T], with values in the cube $\{(u_1,u_2,\ldots,u_m) \mid -1 \leq u_j \leq 1, j=i,2,\ldots,m\}$

$VB(T) = \{u \in V(T) \mid |u_i(t)| = 1\quad i = 1,2,\ldots,m\}$

$VBP(T) = \{u \in VB(T) \mid u(t) \text{ is piecewise constant}\}$

Lemma IV.2.1 : (Sussman [4]) VBP(T) is weakly dense in V(T) (in the L_2 sense).

It is sufficient to assume M = 1. Since every function can be approximated in the L_2 norm by piecewise constant functions, it follows that it will be sufficient to show that every consant function is a weak limit of elements of VBP(T).

Let $u(t) = r < 1$, for $0 \leq t \leq T$. We may assume $r > 0$. For each interval $I = [a,b]$ let the function f_I be defined as follows

$$f_I(t) = \begin{cases} -1 & , a \leq t \leq a+1/2(1-r)\ (b-a) \\ 1 & , a + 1/2(1-r)\ (b-a) < t \leq b \end{cases}$$

Then $\int_a^b f_I(t) = r(b-a)$. Now define u_K (for K=1,2,...) by partitioning

$[0,T]$ into K intervals $I_{K1},\ldots, I_{K_K}$ of length TK^{-1} and letting $u_K(t) = f_{I_{K_i}}(t)$, $t \in I_{K_i}$ $i = 1,2,\ldots,K$. Then the functions u_K belong to VBP(T) and their weak limit is u and we are done.

Let $u \in V(T)$. Let $x(\cdot;\varphi,u)$ be the solution of (1). The set of all elements of the form $x(T;\varphi,u)$ $u \in V(T)$ is the attainable set at time T and we call it $R(T,\varphi)$. Similarly we may define the sets $RB(T,\varphi)$, $RBP(T,\varphi)$. Similarly we have the set $\mathbb{R}(T,\varphi)$, $\mathbb{R}B(T,\varphi)$, $\mathbb{R}BP(T,\varphi)$.

Lemma IV 2.2:

Let the functions u_K converge weakly to u. Then $x(\cdot;\varphi,u_k)$ converges uniformly to $x(;\varphi u)$ for $o \leq t \leq T$.

Proof: For each $v \in V(T)$

$$x(t,\varphi,u) = \varphi(o) + \int_o^t [A + \sum_1^n B_i v_i(\sigma)]\ x(\sigma,\varphi,v)d\sigma + \int_o^t Cx(\sigma-\tau,\varphi v)d\sigma \qquad (2)$$

now since the functions A, B_i, C are bounded φ is bounded and $v_i(t) \leq 1$ then there exist constants C_1, C_2, such that

$$||x(t;\varphi,u)|| \leq ||\varphi(o)|| + C_1 \int_o^t ||x(\sigma;\varphi,v)|| d\sigma \qquad (3)$$

$$+ C_2 \int_o^t ||x(\sigma-\tau;\varphi,v)|| d\sigma$$

Now if $0 \leq t \leq \tau$ we have

$$\|x(t;\varphi,v)\| \leq \|\varphi(o)\| + C_1 \int_o^t \|x(\sigma;\varphi,v)\| \, d\sigma + C_2 \int_o^t \|\varphi(\sigma-\tau)\| \, d\sigma$$

$$\leq D_1 + C_1 \int_o^t \|x(\sigma;\varphi,v)\| \, d\sigma$$

where $D_1 = \|\varphi(o)\| + C_2 \tau \sup_{\sigma \in [-\tau,o]} \|\varphi(\sigma)\|$

Therefore be the usual argument

$$\|x(t;\varphi,v)\| \leq D_1 e^{C_1 t} \text{ for all v and } 0 \leq t \leq \tau$$

Similarly for $\tau \leq t \leq 2\tau$

$$\|x(t;\varphi,v)\| \leq \|\varphi(o)\| + C_1 \int_o^t \|x(\sigma;\varphi,v)\| \, d\sigma +$$

$$+ C_2 \int D_1 e^{C_1(\sigma-\tau)} d\sigma$$

$$\leq D_2 + C_1 \int_o^t \|x(\sigma;\varphi,v)\| \, d\sigma$$

with obvious identification of constants.

So again $\|x(t;\varphi,v)\| \leq D_2 e^{C_1 t}$ for all v and $\tau \leq t \leq \tau$. By a finite argument (since T is finite) we deduce

$$\|x(t;\varphi,v)\| \leq D e^{C_1 t} \text{ for all v and } 0 \leq t \leq T.$$

thus the functions $x(\bullet,$ $v_K \in V(T)$ are uniformly bounded. Equation (1) then implies the derivatives of these are uniformly bounded.

To show that $x(\bullet;\varphi,u_K)$ converges uniformly to $x(\varphi,u)$ it is sufficient to show that every subsequence has a subsequence that converges uniformly to $x(\varphi,u)$. By the previous paragraph and the Ascoli Arzela theorem every subsequence has a subsequence that converges uniformly to some function. Thus our lemma will be proved if we can show that if v_K converges weakly to v and if $x(\bullet,\varphi,u_K)$ converges uniformly to $x(\bullet)$ then $x(\bullet) = x(\bullet,\varphi,u)$

Equation (2) implies

$$x(t,\varphi,v_K) = \varphi(0) + \int_0^t (A + \sum_1^p B_i v_{K_i}(\sigma)) \, [x(\sigma,\varphi,v_K) - x(\sigma)] \, d\sigma$$

$$+ \int_0^t C(x(\sigma-\tau,\varphi,v_K) - x(\sigma-\tau)) \, d\sigma$$

$$+ \int_0^t \Big[[A + \sum_1^n B_i v_{K_i}(\sigma)] \, x(\sigma) + Cx(\sigma-\tau)\Big] \, d\sigma$$

using the weak convergence of v_K to v and the uniform convergence of $x(\bullet,\varphi,v_K)$ to $x(\bullet)$ if follows that

$$x(t) = \varphi(0) + \int_0^t (A + \sum_1^n B_i u_i(\sigma)) \, x(\sigma) + Cx(\sigma-\tau) \, d\sigma$$

Then

$x(t) = x(t;\varphi,u)$ and we are done.

Corollary IV.2.3:

The mapping $u \to x(\bullet,\varphi,u)$ is continuous from $V(T)$ with the weak topology into the space of continuous R^n valued functions in $[o,T]$ with the uniform topology.

Corollary IV.2.4: The sets $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ are compact.

Corollary IV. 2.5: The sets $RBP(T,\varphi)$, $\mathbb{R}BP(T,\varphi)$ are dense in $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ respectively.

Proof:

The 1st Corollary is a restatement of Lemma IV 2.2.Corrollary IV 2.4 follows from 1st Corollary and the fact that V(T) is weakly compact. Finally Corollary IV 2.5 follows from Lemma IV 2.1 and Corollary IV 2.3.

It is clear from the preceding that closedness of the attainable sets $RB(T,\varphi)$, (resp $RBP(T,\varphi)$, $\mathbb{R}B(T,\varphi)$ $\mathbb{R}BP(T,\varphi)$ is equivalent to $R(T,\varphi) = RB(T,\varphi)$, (resp. $R(T,\varphi) = RBP(T,\varphi)$, $\mathbb{R}(T,\varphi) = \mathbb{R}B(T,\varphi)$,$\mathbb{R}(T,\varphi) = \mathbb{R}BP(T,\varphi)$)
We consider now the reachable sets in function space

$$R_C(T,\varphi),\ R_CB(T,\varphi),\ R_CBP(T,\varphi) \text{ and } \mathbb{R}_C(T,\varphi),\ \mathbb{R}_CB(T,\varphi),\ \mathbb{R}_CBP(T,\varphi)$$

and we let $x_t(\varphi,u)$ be the state at time t starting at φ and using control u, i.e.,

$$x_t(\varphi,u)\ (\theta) = x(t+\theta,\varphi,u),\ \theta \in [-\tau,o]$$

Now suppose that $u_K \longrightarrow u$ weakly then $x_{(\cdot)}(\varphi,u_K)$ converges uniformly to $x_{(\cdot)}(\varphi,u)$. Indeed

$$\left\| x_{(\cdot)}(\varphi,u_K) - x_{(\cdot)}(\varphi,u) \right\|_{C([o,T];C)} =$$

$$= \sup_{t\in[o,T]} \left\| x_t(\varphi,u_K) - x_t(\varphi,u) \right\|_C =$$

$$= \sup_{t\in[o,T]} \left(\sup_{\theta\in[-\tau,o]} \left\| x(t+\theta;\varphi,u_K) - x(t+\theta;\varphi,u) \right\|_{R^n} \right)$$

and we are done by the result of Lemma IV 2.2. We have therefore the following corollaries.

Corollary IV.2.6: The mapping $u \longrightarrow x_{(\cdot)}(\varphi,u)$ is continuous from V(T) with the weak toplogy into the space of continuous C valued functions in (o,T) with the uniform topology.

Corollary IV.2.7: The sets $R_C(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$ are compact

Corollary IV. 2.8: The sets $R_C BP(T,\varphi)$; $\mathbb{R}_C BP(T,\varphi)$ are dense in $R_C(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$ respectively.

The following theorem provides an instance of a truly"Bang-Bang" result.

Theorem IV.2.9: If all the brackets $[B_i,B_j]$, $[A,B_i]$, vanish for all i, j, then $RB(T,\varphi)$ and $\mathbb{R}B(T,\varphi)$ are closed. More over the sets $RBP(T,\varphi)$,$[\mathbb{R}BP(T,\varphi)]$ are also closed.

Proof:

Over one delay interval the solution of

$$\dot{x}(t) = \left[A + \sum_{i=1}^{p} u_i(t) B_i \right] x(t) + C\,\varphi(t-\tau)$$

is

$$x(t) = \exp At \left[\prod_{1}^{p} \exp B_i \int_{o}^{t} u_i(\sigma) d\sigma \right] \varphi(0) +$$

$$+ \int_{o}^{t} \left[\exp A(t-\sigma) \prod_{2}^{p} \exp B_i \int_{\sigma}^{t} u_i(s) ds \right] C\varphi(\sigma-\tau) d$$

Now by Liapunov's theorem on the range of a vector valued measure [23] the set of matrices $B_i \int_{\sigma}^{t} u_i(s)ds$ where $u \in VB[T]$ is compact for each i and σ. Thus clearly the first component of the right hand side of the above equation generates a compact set and by Aumann's theorem [23,p. 24] the second component does also. Therefore $RB(t,\varphi)$ and $\mathbb{R}B(t,\varphi)$ are closed for $o < t \leq \tau$. Now for $\tau < t \leq 2\tau$ the solution is

$$x(t) = \exp A(t-\tau) \left[\prod_{1}^{p} \exp B_i \int_{\tau}^{t} u_i(\sigma) \right] d\sigma \; x(\tau) +$$

$$+ \int_{\tau}^{t} \exp A(t-\sigma) \prod_{1}^{p} \exp B_i \int_{\sigma}^{t} u_i(s) ds \Big] Cx(\sigma-\tau) d\sigma$$

Then by the previous result (i.e, $R(\tau,\varphi)$ compact the first component of the last equation generates a compact set and again by Aumann's theorem the second does also. For the general case a trivial induction argument similar to the above arguments establishes that $RB(T,\varphi)$ and $\mathbb{R}B(T,\varphi)$ are closed. Now to show that $RBP(T,\varphi)$ and $\mathbb{R}BP(T,\varphi)$ are closed we need only replace the use of Liapunov's theorem in the previous case by Halkin's theorem [21, 22] which establishes that the set of matrices $\int_{\sigma}^{t} B_i u_i(s)\, ds$ is compact for each σ, and i whenever $u \in VBP(T)$.

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