THESIS REPORT

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Robust H-infinity Output Feedback Control for Nonlinear Systems

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ABSTRACT

Title of Dissertation: Robust H-infinity Output Feedback Control for Nonlinear Systems

Carole Ann Teolis, Doctor of Philosophy, 1994

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The study of robust nonlinear control has attracted increasing interest over the last few years. Progress has been aided by the recent extension of the linear quadratic results which links the theories of L_2 gain control (nonlinear H_∞ control), differential games, and stochastic risk sensitive control. In fact, significant advances in both linear and nonlinear H_∞ theory have drawn upon results from the theories of differential games and stochastic risk sensitive control.

Despite these advances in H_∞ control theory, practical controllers for complex nonlinear systems which operate on basic H_∞ principles have not been realized to date. Issues of importance to the design of a practical controller include (i) computational complexity, (ii) operation solely with observable quantities, and (iii) implementability in finite time. In this dissertation we offer a design procedure which yields practical and implementable H_∞ controllers and meets the mandate of the above issues for general nonlinear systems. In particular, we develop a well defined and realistically implementable procedure for designing robust output feedback controllers for a large class of nonlinear systems. We analyze this problem in both continuous time and discrete time settings.

The robust output feedback control problem is formulated as a dynamic game problem. The solution to the game is obtained by transforming the problem into an equivalent full state feedback problem where the new state is called the *information state*. The information state method provides a separted control policy which involves the solution of a forward and a backward dynamic programming equation. Obtained from the forward equation is the information state, and from the backward equation is the value function of the game and the optimal information state control.

The computer implementation of the information state controller is addressed and several approximations are introduced. The approximations are designed to decrease the online computational complexity of controller.

Robust H-infinity Output Feedback Control for Nonlinear Systems

by

Carole Ann Teolis

Dissertation submitted to the Faculty of The Graduate School
of The University Of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1994

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Dedication

To my parents

Acknowledgments

As I end my graduate career I look forward to the future with excitement. I would like to take this opportunity to explicitly thank a few of the many people who have helped me along the way.

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Introduction

The study of robust nonlinear control has attracted increasing interest over the last few years. Progress has been aided by the recent extension [FM91, Jam92] of the linear quadratic results [Jac73, Whi81] which links the theories of L_2 gain control (nonlinear H_∞ control), differential games, and stochastic risk sensitive control. In fact, significant advances in both linear and nonlinear H_∞ theory have drawn upon results from the theories of differential games and stochastic risk sensitive control.

Despite these advances in H_∞ control theory, practical controllers for complex nonlinear systems which operate on basic H_∞ principles have not been realized to date. Issues of importance to the design of a practical controller include (i) computational complexity, (ii) operation solely with observable quantities, and (iii) implementability in finite time. In this dissertation we offer a design procedure which yields practical and implementable H_∞ controllers and meets the mandate of the above issues for general nonlinear systems. In particular, we develop a well defined and realistically implementable procedure for designing robust output feedback controllers for a large class of nonlinear systems. We analyze this problem in both continuous time and discrete time settings.

This dissertation is organized as follows. In Chapter 1, a precise statement of the robust H_∞ output feedback problem and an historical perspective of the developments in H_∞ control theory which have motivated this research are presented. The solution to the robust H_∞ output feedback problem on which the implementations in this dissertation have their roots is that of James, Baras and Elliott [JBE94, JBE93a, JB94b]. Chapter 2 discusses the formal extension of their discrete time results [JB94b] to the continuous time setting. Turning to implementation issues, Section 2.4 presents the conditions under which the information state is finite dimensional. It is only under these conditions that direct implementation of the information state controller is feasible. In cases where the information state is not finite dimensional an appropriate approximation must be made before implementation is possible. In Section 2.5 one such approximation is discussed. Chapter 3 presents the Certainty Equivalence Controller (CEC) which is shown to be equivalent to the information state controller under the assumptions of the Certainty Equivalence Principle. The CEC, though better suited for implementation than the information state controller, also suffers from the burden of requiring complex on line computations. Approximations of this controller which lend themselves to faster implementations are also discussed. In Chapter 4 numerical procedures for solving the Hamilton Jacobi inequalities and dynamic programming equations which arise in the solution to the robust H_∞ output feedback problem are discussed. In Chapter 5 examples are given which illustrate the effectiveness of the controllers discussed in Chapters 2 and 3. In Chapter 6 focus is turned to the discrete time robust H_∞ output feedback problem. Here the discrete version of the CEC is presented along with a numerical analysis of the discrete implementation. In addition several examples of systems with CEC control are presented. Finally in Chapter 7 a summary of this dissertation is given which highlights the most significant contributions of the work. In addition future directions for the research are discussed.

1.1 Mathematical Preliminaries

This section introduces the basic notation, concepts, and tools which we shall employ throughout this dissertation. In particular, we introduce the pertinent basic spaces and norms with which we shall deal. The notation used is fairly standard [Roy68], however, for completeness and clarity we explicitly describe our notation here.

1.1.1 Basic Symbols and Functional Spaces

$\mathbb{R}$ denotes the real numbers.

$\mathbf{R}^+ = \{t \in \mathbb{R} \ : \ t > 0\}$ denotes the positive real numbers.

$\mathbf{R}^* = \mathbb{R} \bigcup \{\pm\infty\}$ denotes the extended real numbers.

A function $f : \mathbf{R}^n \mapsto \mathbb{R}^m$ is called *locally Lipschitz continuous* if there exists a finite constant $K(x_0)$ with the property that

$$\|f(x) - f(y)\| \leq K(x_0)\|x - y\|$$

for all $x, y \in B$ where B is a ball in $\mathbf{R}^n$ of the form $\{x \in \mathbf{R}^n : \|x - x_0\| \leq r\}$. f is called *Lipschitz continuous* if $B = \mathbf{R}^n$.

$I_{[n,m]}$ is the $m \times n$ matrix which has ones on the diagonal and zeros elsewhere, i.e.,

$$I_{[n,m]} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}.$$

The *sup pairing* $(\cdot,\cdot)$ is defined by

$$(p,q) \triangleq \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}.$$

$\mathcal{E}$ is the function space of cost functionals

$$\mathcal{E} = \{p : \mathbb{R}^n \mapsto \mathbf{R}^*\}.$$

$\mathbb{C}$ denotes the complex numbers.

A function $f : \mathbb{C} \mapsto \mathbb{C}$ is *analytic* if its complex derivative f' exists.

$\mathbb{C}^n$ is the standard space of n element complex vectors. It is a Hilbert space with norm and inner product given as follows. If $z = \{z_i\}_{i=1}^n \in \mathbb{C}^n$ the norm $\|z\|$ of z is

$$\|z\| = \sum_{i=1}^n |z_i|^2.$$

The inner product of $\langle z, w \rangle$ of $z, w \in \mathbb{C}$ is then

$$\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w}_i$$

where $\overline{w}_i$ is the complex conjugate of w_i.

$L_2[a,b]$ is the Hilbert space of complex valued finite energy signals defined on the interval $[a,b]$. The *norm* of an element $f \in L_2[a,b]$ is

$$\|f\| = \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}} < \infty,$$

and the *inner product* of $f, g \in L_2[a,b]$ is

$$\langle f, g \rangle = \int_a^b f(t)\overline{g}(t) dt$$

where $\overline{g}$ is the complex conjugate of g.

$L_2^n[a,b]$ is the Hilbert space of complex n-vector valued finite energy signals defined on the interval $[a,b]$. A member $f \in L_2^n[a,b]$ is composed of an n-vector of functions where each function of the vector comes from $L_2[a,b]$, i.e., $f = (f_1 f_2 \cdots f_n)'$ where $f_i \in L_2[a,b]$, $i = 1, 2, \ldots, n$. The *norm* of an element $f \in L_2^n[a,b]$ is

$$\|f\| = \left(\int_a^b \|f(t)\|^2 dt \right)^{\frac{1}{2}} < \infty.$$

and the *inner product* of $f, g \in L_2^n[a,b]$ is

$$\langle f, g \rangle = \int_a^b \langle f(t), g(t) \rangle \, dt.$$

The reader should be aware that for notational convenience in contexts where the value of n is understood we drop the superscript n in $L_2^n[a,b]$ and write simply $L_2[a,b]$.

1.1.2 H_∞ Spaces

H_∞ control is fundamentally concerned with the Hardy space H_∞. It is a space which consists of all functions that are analytic and bounded in the open right half complex plane. The H_∞ norm of an analytic function is the least such bound. Definition 1.1.1 gives a formal definition of the space and its norm.

Definition 1.1.1 **(H_∞ norm)**

a. $H_\infty \triangleq \{F : \mathbb{C} \mapsto \mathbb{C} \mid F \text{ is analytic, } \sup_{\mathrm{Re}\{s\}>0} |F(s)| < \infty\}$,

b. $F \in H_\infty, \quad \|F\|_\infty \triangleq \sup_{\mathrm{Re}\{s\}>0} |F(s)|$.

For the case of a linear system, we are interested in the H_∞ norm associated with the transfer function of the system. Transfer functions for finite dimensional linear systems are real and rational, thus we may restrict ourselves to the subset RH_∞ of the Hardy space H_∞. $RH_\infty \subset H_\infty$ is the set of the real rational functions in H_∞. A transfer function $F(s)$ is a member of RH_∞ if and only if it is proper and stable, i.e., $\lim_{s\to\infty} |F(s)| < \infty$ and F has no poles in the closed right half plane. In terms of the Nyquist plot associated with a transfer function $F \in RH_\infty$ the H_∞ norm is the largest distance between a point on the Nyquist locus and the origin.

Fact 1.1.2 If F is a real rational function in H_∞ then its H_∞ norm is the supremum of the function evaluated along the imaginary axis, i.e.,

$$F \in RH_\infty \text{ then } \|F\|_\infty \triangleq \sup_{\omega \in \mathbb{R}} |F(jw)|$$

1.2 Problem Statement

In this dissertation we shall concern ourselves with the robust control of the broad class of nonlinear systems Σ described in state space by the following set of equations:

$$\Sigma \begin{cases} \dot{x}(t) &= f(x(t), u(t)) + w(t), \qquad x(t_0) = x_0, \\ y(t) &= h(x(t)) + v(t), \\ z(t) &= \ell(x(t), u(t)). \end{cases} \tag{1.2.1}$$

Here x is the state, y is the observable output, and z is a performance measure which is at our discretion. More specifically, the state $x(t) \in \mathbb{R}^n$ where the initial condition x_0 is unknown, the output or observation path $y(t) \in \mathbb{R}^p$, and the performance measure $z(t) \in \mathbb{R}^q$. The control signal $u(t) \in \mathbf{R}^m$ is subject to design under the constraint that u is a non-anticipating function of the observation path y. In symbols, $u \in \mathcal{O}$ where

$$\mathcal{O} \triangleq \left\{ u \in L_2[0,\infty] \ : \ u(t) = u(\{y(\tau)\}_{\tau=0}^{t}) \right\}.$$

Additionally, $w \in \mathbf{R}^n$ and $v \in \mathbf{R}^p$ are respectively the state and output disturbances which account for sensitivity to unmodeled dynamics associated with the real system and/or environment. These disturbances are assumed to be arbitrary but finite energy signals to which the system is inherently subject. The mapping $f : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n$ describes the evolution of the unperturbed system state ($w \equiv 0$), $h : \mathbf{R}^n \mapsto \mathbb{R}^p$ is the state to output mapping which describes the observable quantities on the system, and $\ell : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^q$ the performance measure mapping which is chosen to provide an indication of controller performance. These mappings all evaluate to zero at the origin, i.e., $f(0,0) = 0$, $h(0) = 0$, and $\ell(0,0) = 0$. Each of the state evolution mapping f, the system output mapping h, and the performance measure ℓ are constrained to be continuously differentiable, i.e., $f \in C^1(\mathbb{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$, $h \in C^1(\mathbb{R}^n, \mathbf{R}^p)$, $l \in C^1(\mathbf{R}^n \times \mathbf{R}^m, \mathbb{R}^q)$. We shall denote this situation as $\Sigma \in C^1$.

The constraint that $u \in \mathcal{O}$ is a practical one and reflects the notion that realistic control laws are restricted to operate based on knowledge of observable quantities. Thus, we are restricted to consider feedback control laws which are functions solely of the past outputs, i.e., output feedback control. In Figure 1 we depict the closed loop output feedback system Σ^u associated with the nonlinear system Σ. Note that the major difference between the open loop system Σ and the output feedback system Σ^u is that fact that in the latter system $u \in \mathcal{O}$ while in the former u is arbitrary.

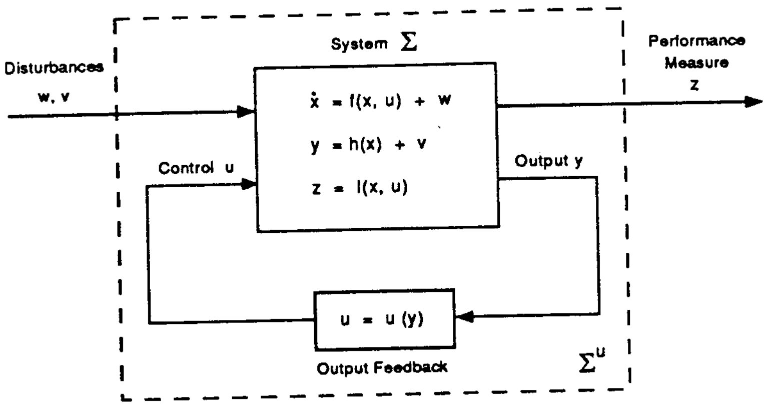

Figure 1: Closed loop nonlinear system with output feedback

The main problem which we address in this dissertation is the Robust H_∞ Output Feedback Control Problem. Simply stated the Robust H_∞ Output Feedback Control Problem is to determine an admissible control $u \in \mathcal{O}$ such that the closed loop system Σ^u

(i) is asymptotically stable when no disturbances are present, and

(ii) has finite L_2 gain from the disturbance inputs w and v to the performance output z.

The precise problem statement is given below in Problem 1.2.1.

Problem 1.2.1 Robust H_∞ Output Feedback Control Problem
Given $\gamma > 0$, find a control $u \in \mathcal{O}$ such that for all initial conditions $x_0 \in \mathbb{R}^n$,

(i) the closed loop system Σ^u is asymptotically stable when $w, v \equiv 0$, and

(ii) there exists a constant $\beta^u(x_0)$ where $0 \leq \beta^u(x_0) < \infty$ and $\beta^u(0) = 0$ such that

$$\int_0^t ||z(s)||^2 ds \leq \gamma^2 \int_0^t (||w(s)||^2 + ||v(s)||^2) ds + \beta^u(x_0)$$

for all $w, v \in L_2([0,t])$, for all $t > 0$.

Note that $\beta^u(x_0)$ also depends on γ.

1.3 Perspective

Although our main concern lies with the robust output feedback control of nonlinear systems, it is both interesting and important to trace the philosophy and developments in H_∞ control theory which have lead to the present study. In fact, the underlying roots of H_∞ theory can be traced to linear systems with very special performance measures. In its original formulation, the H_∞ control problem dealt with the design of controllers for linear systems which were to meet frequency-domain performance specifications [Zam81, Fra87].

1.3.1 Linear H_∞ Problem

For greater exposition of our ideas in later chapters and because it mirrors the development of the theory itself, we shall in this section introduce the fundamental concepts associated with H_∞ control in the context of linear systems. Consider the following linear system model:

$$\Sigma_L \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + w(t), \qquad x(t_0) = x_0, \\ y(t) & = & Cx(t) + v(t), \\ z(t) & = & \begin{pmatrix} \sqrt{Q}x(t) \\ \sqrt{u}(t) \end{pmatrix}. \end{array} \right.$$

Note that Σ_L is a special case of Σ as described in Section 1.2 where $f(x(t),u(t)) = Ax(t)+Bu(t)$, $h(x(t)) = Cx(t)$, and $\ell(x(t),u(t)) = (x(t)'\sqrt{Q}', u(t)')'$. In terms of the linear system Σ_L, the fundamental goal of H_∞ control is to minimize the H_∞ norm of the closed-loop transfer function from exogenous inputs w to controlled outputs z under a constraint of internal stability. Because internal stability means that bounded inputs to the closed loop system produce bounded internal signals, internal stability can not be determined solely by consideration of the input-output transfer functions.

Despite the fact that iterative algorithms for the solution to the H_∞-optimal problem for linear systems have been developed, the computational complexity of these algorithms precludes practical implementation [DGKF89]. Because of this, researchers have focused on the standard ***sub-optimal*** H_∞ control problem[1]: for a specified scalar $\gamma > 0$, find a control u such that the H_∞ norm of the closed-loop transfer function is bounded by γ and the system is internally stable. Clearly γ must be chosen greater than the H_∞-optimal level for a suboptimal solution to exist. The standard H_∞ design leads to controllers of the ***worst case***

[1]Henceforth we shall refer to the ***sub-optimal*** H_∞ control problem as the H_∞ control problem since this standard problem addressed in the literature.

type in the sense that emphasis is focused on minimizing the effect of the disturbances which produce the largest effect on the system output. Because of the worst case design strategy, H_∞ controllers tend to be conservative yet robust to disturbances.

The above formulation of the standard H_∞ problem does not generalize directly to the nonlinear setting since nonlinear systems are not described by transfer functions. However, there does exist an equivalent time domain formulation which does generalize to the nonlinear setting. In the time domain formulation one enforces a bound $\gamma > 0$ on the L_2 induced norm from exogenous inputs w to controlled outputs z (finite gain) again with an internal stability constraint [Fra87, vdS91]. Much of the later results in linear H_∞ theory are due to the use of such time domain methods.

Problem 1.3.1 Linear H_∞ Output Feedback Control Problem
Given $\gamma > 0$, and assuming $[A, \sqrt{Q}]$ observable and $[A, I_{[n,m]}]$ controllable, find a control $u \in \mathcal{O}$ such that for initial condition $x_0 = 0$, the closed loop system Σ_L^u is asymptotically stable when $w, v \equiv 0$, and

$$\int_0^\infty ||z(s)||^2 ds \leq \gamma^2 \int_0^\infty (||w(s)||^2 + ||v(s)||^2) ds$$

for all $w, v \in L_2([0, \infty])$.

For linear systems observability and controllability are easily verified and the addition of these assumptions[2] removes the need for concern about initial conditions.

By the late 80's the linear H_∞ problem had been completely solved [DGKF89, GD88], viz. Theorem 1.3.2, and several intersting connections had been made. The min-max nature of the H_∞ problem led easily to connections with deterministic game theory [DGKF89, LAKG92, BB91, RS89]. For linear systems with perfect state observation, Jacobson [Jac73] showed that the controllers for the stochastic risk-sensitive control problem and the dynamic game problem were identical. This result was the first to establish a link between deterministic and stochastic optimal control problems. It was later shown that for linear systems with imperfect state observation the controller for the infinite horizon stochastic risk-sensitive control problem enforces a bound on the H_∞ norm of the closed loop transfer function. Moreover, the controller minimizes the entropy integral over the set of all controllers meeting the H_∞ norm bound [GD88, BH89b, Whi81, RS89, BvS85].

Theorem 1.3.2 gives the solution to the Linear H_∞ Output Feedback Problem 1.3.1 which involves the solution to a pair of Riccati equations and a coupling condition.

[2]The assumption of observability and controllability can be relaxed to stabilizability and detectability.

Theorem 1.3.2 ([BB91]) Given $\gamma > 0$, consider the Linear H_∞ Output Feedback Problem 1.3.1. If the algebraic Riccati equations

$$ZA + A'Z - Z(BB' - \gamma^2 I)Z + Q = 0$$

$$AK + KA' - K(C'C - \gamma^2 Q)K + I = 0$$

both have minimal positive definite solutions Z^+ and K^+, and if they further satisfy the coupling condition

$$Z^+ - \gamma^2 K^{+^{-1}} < 0$$

then a controller which solves the linear H_∞ output feedback problem is given by

$$u^* = -B'Z^+(I - \gamma^{-2}K^+Z^+)^{-1}\bar{x}$$

where

$$\dot{\bar{x}}(t) = (A + \gamma^{-2}K^+Q)\bar{x} + Bu^* + K^+C'(y - C\bar{x}).$$

If any one of the above conditions fails then the given γ is less than the H_∞-optimal level and thus the problem has no solution.

1.3.2 Large Deviation Limits

It turns out that the connections between H_∞ control/dynamic game theory and risk sensitive control theory are a consequence of the linear-quadratic context in which they are formulated. Only when the optimization problem involves a linear system with quadratic (or exponential of quadratic) cost are the solutions coincident. The nature of the connection for general nonlinear systems was only recently made clear with the aid of asymptotic analysis. Figure 2 illustrates the asymptotic connections between the risk sensitive stochastic control problem, the dynamic game problem, the risk neutral stochastic control problem, and the familiar deterministic optimal control problem. The value function for the risk sensitive stochastic control problem is denoted $S^{\mu,\epsilon}(x,t)$ where x_t^ϵ is a controlled diffusion process, μ is the risk sensitivity parameter, and ϵ is the noise variance. Applying a logarithmic transform, we define $W^{\mu,\epsilon}(x,t) = \frac{\epsilon}{\mu}\log(S^{\mu,\epsilon}(x,t))$. Then taking the large deviation limit of $W^{\mu,\epsilon}(x,t)$ as $\epsilon \to 0$ yields the value function W^μ of the dynamic game[3]. Similarly, taking the large deviation limit of $W^{\mu,\epsilon}(x,t)$ as $\mu \to 0$ yields the value function W^ϵ of the risk neutral stochastic control problem. And finally, taking either the limit as $\epsilon \to 0$ of W^ϵ or as $\mu \to 0$ of W^μ yields the value function W of the deterministic optimal control problem. For more details on these relations for

[3] μ is related to γ by $\mu = \gamma^{-2}$.

systems with complete state information see [Jam92, CJ92, Whi91, FM91] and for the partially observed case see [JBE94, JBE93b].

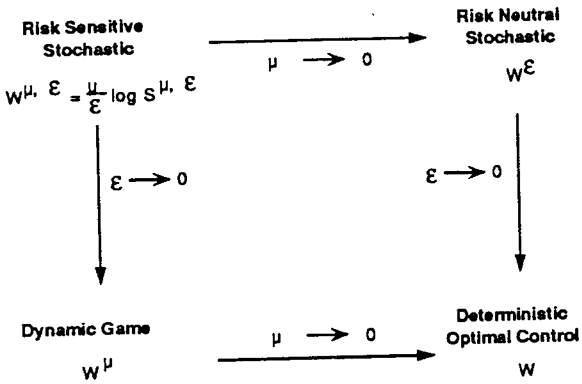

Figure 2: Large Deviations Limit Relations

1.3.3 Nonlinear Full State Feedback

In this next section we take a thorough look at the Robust H_∞ State Feedback Problem 1.3.3. In the state feedback problem it is assumed that the controller has direct access to the state variables. This assumption simplifies the problem considerably. Although our main interest lies with the Robust H_∞ Output Feedback Problem, the solution to the full state feedback problem offers much insight into its solution.

The theory for nonlinear H_∞/L_2 gain control problem is not nearly as developed as it is for linear systems. A general framework for dealing with L_2 gains in both the linear and nonlinear settings may be found in Willems' theory of dissipative systems [Wil72a, Wil72b]. The major motivation for the study of dissipative systems in the context of control is their connection with stability [HM76, HM77, HM80, MA73]. The application of this theory leads to a nonlinear version of the Bounded Real Lemma (Theorem 1.3.8) which equates the properties of *finite gain* and *finite gain dissipativity*. The property of finite gain dissipativity can be expressed in terms of a dynamic programming inequality or a partial differential inequality (PDI), known as the dissipation inequality [HM76, HM80, Jam93c].

In terms of a controlled version of the dissipation inequality a *theoretical* solution to the state feedback nonlinear H_∞ problem can be derived [JB94b]. To see how to arrive at that solution consider the system

$$\Sigma_S \begin{cases} \dot{x}(t) = f(x(t), u(t)) + w(t), & x(t_0) = x_0, \\ z(t) = \ell(x(t), u(t)) \end{cases}$$

where Σ_S is a special case of Σ given in equation (1.2.1) with $y = x$, i.e., full state information is available. Let S denote the set of state feedback controllers, $S \triangleq \{u \in L_2[0,\infty) : u(t) = u(x(t))\}$. We formally state the Robust H_∞ State Feedback below.

Problem 1.3.3 Robust H_∞ State Feedback Control Problem
Given $\gamma > 0$, find a control $u \in S$ such that for all initial conditions $x_0 \in \mathbf{R}^n$,

(i) the closed loop system Σ_S^u is asymptotically stable when $w \equiv 0$, and

(ii) there exists a constant $\beta^u(x_0)$ where $0 \leq \beta^u(x_0) < \infty$ and $\beta^u(0) = 0$ such that

$$\int_0^t ||z(s)||^2 ds \leq \gamma^2 \int_0^t ||w(s)||^2 ds + \beta^u(x_0)$$

for all $w \in L_2([0,t])$, for all $t > 0$.

Definition 1.3.4 (Supply Rate) A locally Lipschitz continuous function $r : \mathbf{R}^n \times \mathbf{R}^q \mapsto \mathbf{R}$ satisfying the growth condition $|r(w,z)| \leq C(1 + |w|^2 + |z|^2)$ is called a *supply rate.*

Here we are interested in the particular supply rate associated with finite gain dissipativity, $r(w,z) = \gamma^2\|w\|^2 - \|z\|^2$. Henceforth, we continue the development considering only this supply rate.

Definition 1.3.5 (Finite Gain Dissipative)

a. A function $V : \mathbf{R}^n \mapsto \mathbb{R}^+$ is said to be a *storage function* for the closed loop system Σ_S^u if it satisfies the *integral dissipation inequality*

$$V(x) \geq \sup_{t\geq 0} \sup_{w\in L_2[0,t]} \left\{ V(x(t)) - \int_0^t (\gamma^2\|w(s)\|^2 - \|z(s)\|^2)ds : x(0) = x \right\}. \tag{1.3.1}$$

b. The closed loop system Σ_S^u is called *finite gain dissipative* if a storage function exists which satisfies $V(0) = 0$.

A storage function for a particular closed loop system is not necessarily unique. The minimal and maximal storage functions are called the available storage V_a and the required supply V_r [Wil72a]. The required supply is finite if the system is reachable from the origin [Wil72a]. Although storage functions are not necessarily differentiable or even continuous, James has shown [Jam93c] that every dissipative system must possess a lower semicontinuous storage function.

Definition 1.3.6 (Available Storage) For all initial conditions $x(0) = x \in \mathbb{R}^n$ the available storage $V_a : \mathbb{R}^n \mapsto \mathbb{R}^+ \bigcup \{\infty\}$ is defined by

$$V_a(x) = \sup_{t\geq 0} \sup_{w \in L_2[0,t]} \int_0^t (\|z(s)\|^2 - \gamma^2 \|w(s)\|^2) ds.$$

When V_a is finite for all $x \in \mathbf{R}^n$ and $V_a(0) = 0$ we have automatically that the system is finite gain, i.e., Σ_S has finite L_2 gain from disturbance input w to the performance output z. In fact $V_a(x)$ is the minimal value of $\beta^u(x)$.

The following theorem [Jam93c] characterizes dissipativity in terms of a partial differential inequality (PDI) called the dissipation inequality. The dissipation inequality is meaningful even when the storage function is not differentiable so long as the inequality is regarded in the viscosity sense [CL83, CEL84, FS93, Jam93c].

Theorem 1.3.7

a. If the closed loop system Σ_S^u is finite gain dissipative with storage function V, then V satisfies

$$< \nabla_x V, f(x,u) > + \sup_{w \in \mathbf{R}^n} \left\{ < \nabla_x V, w > + \|z\|^2 - \gamma^2 \|w\|^2 \right\} \leq 0. \qquad (1.3.2)$$

The PDI of Equation (1.3.2) is called the *dissipation inequality.*

b. If a non-negative function V satisfies the Dissipation Inequality (1.3.2) and $V(0) = 0$ then Σ_S^u is finite gain dissipative and V is a storage function.

Although the Robust H_∞ State Feedback Problem has previously been solved [BH89a, vdS91] the following results are presented in a format which mirrors the development of the Robust H_∞ Output Feedback Problem. It is hoped that an understanding of these simpler proofs will provide insight into the proofs of the output feedback results described in Chapter 2. The discrete time equivalents of Theorems 1.3.8-1.3.12 can be found in [JB94b].

Theorem 1.3.8 (Bounded Real Lemma) The closed loop system Σ_S^u is finite gain if and only if its finite gain dissipative.

Proof: Assume Σ_S^u is finite gain dissipative. Then there exists a storage function V satisfying the integral dissipation inequality (1.3.1). That is, there exists a non-negative function V such that

$$V(x_0) + \int_0^t \gamma^2 \|w(s)\|^2 ds \geq V(x(t)) + \int_0^t \|z(s)\|^2 ds$$

for all $t > 0$, and for all $w \in L_2[0,t]$. Thus Σ_S^u is finite gain with $\beta^u(x_0) = V(x_0)$.

Conversely, assume Σ_S^u is finite gain. Then

$$\sup_{t>0} \sup_{w \in L_2[0,t]} \left\{ \int_0^t \|z(s)\|^2 ds - \int_0^t \gamma^2 \|w(s)\|^2 ds \right\} \leq \beta^u(x_0)$$

Thus the available storage V_a exists and is finite. Thus Σ_S^u is finite gain dissipative. ■

Thus if we can find a control u such that the Dissipation Inequality (1.3.2) is satisfied for some non-negative function V with $V(0) = 0$ then the closed loop system Σ_S^u is finite gain. Under the additional assumption of zero state detectability stability results can be obtained.

Definition 1.3.9 (Zero State Detectability) The closed loop system Σ_S^u is zero state detectable if $w \equiv 0$ and $z \in L_2[0,\infty]$ implies $\lim_{t\to\infty} x(t) = 0$.

Theorem 1.3.10 If closed loop system Σ_S^u is zero state detectable and finite gain dissipative, then Σ_S^u is asymptotically stable.

Proof: Setting $w \equiv 0$ in the integral dissipation inequality (1.3.1) and using the non-negativity of V we have

$$V(x_0) \geq \sup_{t\geq 0} \int_0^t \|z(s)\|^2 ds$$

for any initial condition x_0. Thus $z \in L_2[0,\infty]$, so by zero state detectability $\lim_{t\to\infty} x(t) = 0$. ■

Zero state detectability is a key property required for proving asymptotic stability but in general this property is difficult to check. Zero state detectability is also dependent on the particular controller $u \in S$.

Theorem 1.3.11 (Necessity) If a controller $u^* \in S$ solves the Robust H_∞ State Feedback Problem 1.3.3 then there exists a non-negative function $V(x)$ where $V(0) = 0$ and

$$\inf_{u \in U} \sup_{w \in \mathbf{R}^n} \left\{ < \nabla_x V, f(x,u) + w > -\gamma^2 \|w\|^2 + \|\ell(x,u)\|^2 \right\} \leq 0. \tag{1.3.3}$$

Proof: Since $\Sigma_S^{u^*}$ is finite gain, the Bounded Real Lemma 1.3.8 implies the existence of a storage function V satisfying the dissipation inequality 1.3.2. Therefore, V satisfies inequality (1.3.3). ∎

Equation (1.3.3) is a Hamilton Jacobi Inequality.

Theorem 1.3.12 (Sufficiency) Assume that $V \in C^1$ is a non-negative solution of inequality (1.3.3) satisfying $V(0) = 0$. Let $\bar{u}^*(x)$ be a control value which achieves the minimum in (1.3.3). Then the controller $u^* \in S$ defined by $\bar{u}^*(x)$ solves the Robust H_∞ State Feedback Problem if the closed loop system $\Sigma_S^{u^*}$ is detectable.

Proof: The closed loop system $\Sigma_S^{u^*}$ is finite gain dissipative since inequality (1.3.3) implies the dissipation inequality (1.3.2) is satisfied for for the controller u^*. Then the Bounded Real Lemma 1.3.8 implies $\Sigma_S^{u^*}$ is finite gain. Theorem 1.3.10 implies $\Sigma_S^{u^*}$ is asymptotically stable. Hence u^* solves the Robust H_∞ State Feedback Problem. ∎

1.3.4 Existence of Solutions to Hamilton Jacobi Inequalities

In the previous section, Section 1.3.3, we have given necessary and sufficient conditions for a solution to exist to the state feedback robust control problem which depend on a solution to the Hamilton Jacobi Inequality (1.3.3). Thus the problem remains to establish the existence of a solution to the Hamilton Jacobi Inequality. This problem is still an open problem though some insight as to the solvability of Hamilton Jacobi equations can be gained by using nonlinear geometric techniques [vdS91, vdS92]. Clearly if a solution exists to the equality then this solution is also a solution to the inequality.

Introduced by Lukes in [Luk69], the relation between Hamilton-Jacobi equations and invariant manifolds of Hamiltonian vector fields provides information about the global solvability of Hamilton Jacobi equations.

In fact, this relation has found extended use in the literature recently [Byr92, vdS91, vdS92]. Van der Schaft gives an overview of the results related to invariant manifolds of Hamiltonian vector fields in [vdS91] and the appendix of [vdS92] where an assumption is made that the linearized system is asymptotically stabilizable. This leads to the study of hyperbolic Hamiltonian vector fields.

A hyperbolic[4] Hamiltonian vector field of dimension $2N$ has an N dimensional stable and an N dimensional unstable manifold. In natural coordinates the Hamiltonian vector field is described by the familiar Hamiltonian equations for the state, x, and costate, p which arise in optimal control when applying Pontriagin's Maximum Principle (PMP).

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}, \qquad i = 1, \cdots n \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i} \end{aligned} \tag{1.3.4}$$

If the stable manifold is the graph of a function, i.e. if, on the stable manifold, the costate can be written as a function of the state, then the feedback controller is well defined. If, on the other hand, the stable manifold is not the graph of a function, the the control will be multivalued.

If the assumption of asymptotic stability of the linearization is relaxed to allow uncontrollable imaginary axis eigenvalues of the linearization, the results on invariant manifolds will not hold as given in [vdS91, vdS92]. The presence of imaginary axis eigenvalues forces the Hamiltonian vector field to have a center manifold, in addition to stable and unstable manifolds. Since center manifolds are only local objects, unlike the stable and unstable manifolds, the relaxation of the assumption eliminates the possibility of clean global results using these techniques.

1.3.5 Nonlinear Output Feedback

Most of the previous research conducted in the area of robust nonlinear control to date has focused on the case where full state information is available. Thus, previously little attention has been given to the problem of robust nonlinear control via output feedback. As a generalization of the results from linear theory, the solution to the output feedback problem has been postulated to involve a nonlinear observer combined with a controlled dissipation inequality for an augmented system. By postulating such a structure and solving an augmented game problem, several researchers [IA92, vdS93, DBB93] have established results yielding sufficient conditions for the existence of a solution to the output feedback robust control problem.

[4]Hyperbolic refers to the fact that the linearization of the vector field about the origin has no purely imaginary eigenvalues.

In [JBE94] a theoretical solution is obtained for the finite-horizon partially observed dynamic game problem for discrete-time nonlinear systems. The approach taken there is motivated by ideas from stochastic control; in fact, the controller is obtained as an asymptotic limit of the controller for the risk sensitive stochastic control problem, c.f. Section 1.3.2. In [JB94b] this work is extended to the infinite horizon case and stability issues are addressed. In the latter paper a purely deterministic viewpoint is maintained. Both papers present results which are necessary and sufficient. The continuous time problem can be solved, in principle, using a similar approach [JBE93b, JBE93a]; however, these results have not yet been proved rigorously for general systems. Work is currently in progress in this direction and preliminary results have been obtained [JB94a].

The novelty of the approach of [JBE93b, JBE94] is the reformulation of the partially observed dynamic game problem as a fully observed problem in terms of an *information state*. The information state is derived using only past outputs and is determined as the solution of a dynamic programming equation. Thus using this approach, the original problem is restated in terms of an equivalent problem in which there is full knowledge of a new state. This new state is called the *information state*. Although common in stochastic control, this type of separation approach had not been previously applied to deterministic game problems. For linear systems, the information state can be related to the "past stress" used by Whittle [Whi81] in solving the risk sensitive problem for linear systems (see also [Whi91]). Başar and Bernhard [BB91] have also employed the "past stress" in their solution to the game problem. Their use of the past stress is very different from James, Baras and Elliott's use of the information state. Başar and Bernhard use the past stress to determine a state estimate, while James, Baras and Elliott control the information state directly.

Building on the results of James, Baras and Elliott we shall in this dissertation address the practical implementational and theoretical issues associated with the Robust H_∞ Output Feedback Problem.

Chapter 2

Information State Feedback Control

For the case of discrete-time nonlinear systems, James et. al [JBE94] have solved the finite-horizon partially observed dynamic game problem. Their solution involves in a fundamental way the notion of the so called *information state*. Through the specification of the information state the partially observed dynamic game problem may be restated as an equivalent fully observed problem.

In this chapter we *formally* derive continuous time robustness results which are analogous to the discrete time results of [JB94b]. In order to provide rigorous proofs of the results discussed below it is necessary to extend the theory of viscosity solutions of partial differential equations and inequalities to allow solutions which may be functions of infinite dimensional arguments. Currently work is being done by James and Baras to obtain rigorous results [JB94a]. In cases where the information state can be identified with a finite dimensional quantity which evolves according to ordinary differential equations the formal results presented here hold rigorously, viz. Section 2.4. The consideration of formal arguments is important since the formal results give an heuristic confirmation of the form of the theoretical solution. Considering special cases, such as when the information state is finite dimensional, serves to further support the formal results.

2.1 Problem Statement

For the readers' convenience we briefly restate the Robust H_∞ Output Feedback Problem, viz. 1.2. Recall that we are interested in the robust control of the broad class of nonlinear systems Σ described in state space by the following set of equations:

$$\Sigma \begin{cases} \dot{x}(t) &= f(x(t),u(t)) + w(t), \qquad x(t_0) = x_0, \\ y(t) &= h(x(t)) + v(t), \\ z(t) &= \ell(x(t),u(t)). \end{cases} \tag{2.1.1}$$

Recall also that x is the state, y is the observable output, and z is a performance measure which is at our discretion. The initial condition x_0 is unknown. The control signal u is subject to design under the constraint that u is a non-anticipating function of the observation path y, i.e., $u \in \mathcal{O}$ where

$$\mathcal{O} \triangleq \left\{ u \in L_2[0,\infty] \; : \; u(t) = u(\{y(\tau)\}_{\tau=0}^{t}) \right\}.$$

Additionally, w and v are respectively the state and output disturbances which account for sensitivity to unmodeled dynamics associated with the real system and/or environment. These disturbances are assumed to be arbitrary but finite energy signals to which the system is inherently subject. Recall that each of the state evolution mapping f, the system output mapping h, and the performance measure ℓ are constrained to be continuously differentiable and evaluate to zero at the origin. The problem we wish to solve is the Robust H_∞ Output Feedback Problem stated below in Problem 2.1.1

Problem 2.1.1 Robust H_∞ Output Feedback Control Problem
Given $\gamma > 0$, find a control $u \in \mathcal{O}$ such that for all initial conditions $x_0 \in \mathbb{R}^n$,

(i) the closed loop system Σ^u is asymptotically stable when $w, v \equiv 0$, and

(ii) there exists a constant $\beta^u(x_0)$ where $0 \leq \beta^u(x_0) < \infty$ and $\beta^u(0) = 0$ such that

$$\int_0^t ||z(s)||^2 ds \leq \gamma^2 \int_0^t (||w(s)||^2 + ||v(s)||^2) ds + \beta^u(x_0)$$

for all $w, v \in L_2([0,t])$, for all $t > 0$.

As discussed in Section 1.3.2, the solution to the Robust H_∞ Output Feedback Problem is equivalent to the solution of a related zero sum game problem [FM91, Jam92]. We shall in this chapter employ this equivalence to solve the Robust H_∞ Output Feedback Problem 2.1.1. We begin in Section 2.2 with a simplified version of the Robust H_∞ Output Feedback Problem for which the asymptotic stability requirement is not an issue, i.e. the finite time horizon problem. The finite time horizon results are then extended to the infinite horizon case in Section 2.3. Turning to implementation issues Section 2.4 presents the conditions under which the information state is finite dimensional. Under these conditions the results of Section 2.2

and Section 2.3 hold rigorously and the information state controller may be implemented directly. In cases where the information state is not finite dimensional an appropriate approximation must be made before implementation is possible. In Section 2.5 one such approximation is discussed.

2.2 Finite Time Horizon

In the finite time horizon problem asymptotic stability is not an issue. Thus the Finite Time Robust H_∞ Output Feedback Problem entails finding an $\mathbf{R}^m$-valued control $u(t)$ which may be any non-anticipating function of the observation path y that renders the closed loop system finite gain. Let $\mathcal{O}_{t,t_f}$ denote the set of non-anticipating output feedback controllers defined on the time interval $[t, t_f]$. The Finite Time Robust H_∞ Output Feedback Problem is stated formally as Problem 2.2.1 below.

Problem 2.2.1 Finite Time Robust H_∞ Output Feedback Control Problem
Given $\gamma > 0$, find a control $u \in \mathcal{O}_{0,t_f}$ such that for all initial conditions $x_0 \in \mathbf{R}^n$, there exists a constant $\beta^u_{t_f}(x_0)$ where $0 \le \beta^u_{t_f}(x_0) < \infty$ and $\beta^u_{t_f}(0) = 0$ such that

$$\int_0^t ||z(s)||^2 ds \le \gamma^2 \int_0^t (||w(s)||^2 + ||v(s)||^2) ds + \beta^u_{t_f}(x_0)$$

for all $w, v \in L_2([0,t])$, for all $t \in [0, t_f]$.

2.2.1 Equivalent Game Problem

In this section we establish the equivalence between the Finite Time Robust H_∞ Output Feedback Problem and a finite time horizon game problem. First we state clearly the game problem. The equivalence of the two problems is then made clear in Lemma 2.2.2 below.

Recall the function space $\mathcal{E}$ of cost functionals

$$\mathcal{E} = \{p : \mathbb{R}^n \mapsto \mathbf{R}^*\},$$

where $\mathbf{R}^*$ denotes the extended reals. The finite time horizon game associated with the Finite Time Robust H_∞ Output Feedback Problem has cost

$$\mathcal{J}_{p_0,t_f}(x_0, u, v, w) = p_0(x_0) + \frac{1}{2} \int_0^{t_f} (||\ell(x(s), u(s))||^2 - \gamma^2(||w(s)||^2 + ||v(s)||^2)) ds \qquad (2.2.1)$$

where the initial cost $p_0 \in \mathcal{E}$ is chosen in such a way as to reflect any a priori knowledge of the initial state x_0 of the closed loop system Σ^u.

In the context of a game there are two players: (i) the control system designer, and (ii) nature. The control system designer's objective is to minimize the given cost (2.2.1), while nature, acting in direct opposition to that objective, strives to disturb the system so as to maximize that cost. The game is played as follows:

(i) Player 1 (designer) selects an $\mathbf{R}^m$-valued control $u(t)$ which may be any non-anticipating function of the observation path y, i.e. $u \in \mathcal{O}_{0,t_f}$. This selection is designed to minimize the cost.

(ii) Player 2 (nature) selects the initial condition x_0 and a disturbance (w, v), which is a square integrable open loop sequence. Nature's selection is assumed to be made to maximize the cost.

More precisely, if $J_{p_0,t_f}(u)$ denotes the effect of nature's selection so that

$$J_{p_0,t_f}(u) = \sup_{w,v \in L_2[0,t_f]} \sup_{x_0 \in \mathbf{R}^n} \mathcal{J}_{p_0,t_f}(x_0, u, v, w), \tag{2.2.2}$$

then the partially observed dynamic game problem is to find an admissible $u^* \in \mathcal{O}_{0,t_f}$ such that

$$J_{p_0,t_f}(u^*) = \inf_{u \in \mathcal{O}_{0,t_f}} J_{p_0,t_f}(u).$$

Now we define a function $\delta_x \in \mathcal{E}$ for which choosing the initial cost $p_0(x_0) = \delta_x(x_0)$ corresponds to having full knowledge of the initial condition, i.e., $x_0 = x$. Define for each $x \in \mathbf{R}^n$ a function $\delta_x \in \mathcal{E}$ by

$$\delta_x(\xi) \triangleq \begin{cases} 0 & \text{if } \xi = x, \\ -\infty & \text{if } \xi \neq x. \end{cases}$$

The finite gain property can be expressed in terms of the cost $J_{p_0,t_f}(u)$, Equation (2.2.2), as described in Lemma 2.2.2.

Lemma 2.2.2 For any output feedback controller $u \in \mathcal{O}_{0,t_f}$, the closed loop system Σ^u is finite gain if and only if

$$J_{\delta_{x_0},t_f}(u) \leq \beta^u_{t_f}(x_0),$$

for some finite $\beta^u_{t_f}(x_0)$ with $\beta^u_{t_f}(0) = 0$.

From Lemma 2.2.2 it is clear that if we find a control policy $u \in \mathcal{O}_{0,t_f}$ which minimizes each functional $J_{\delta_{x_0},t_f}$ then this control will solve the Finite Time Robust H_∞ Output Feedback Problem. Also note that when Σ^u is finite gain for each initial condition x_0, the cost $J_{p,t_f}(u)$ satisfies

$$(p, 0) \leq J_{p,t_f}(u) \leq (p, \beta^u_{t_f}),$$

where recall the "sup pairing" $(\cdot,\cdot)$ is defined by

$$(p,q) \triangleq \sup_{x\in\mathbf{R}^n} \{p(x)+q(x)\}.$$

Thus we define

$$\text{dom } J_{p,t_f}(u) \triangleq \{p\in\mathcal{E} : (p,0), (p,\beta^u_{t_f}) < \infty\}.$$

2.2.2 Information State Formulation

Following [JBE94, JBE93a], we shall solve the game problem by introducing a new state variable, the *information state*, and then replacing the original game problem with an equivalent problem expressed in terms of the information state. The information state can be thought of as a deterministic ***sufficient statistic*** in that it contains all the information needed to control the system with respect to the given performance measure. It is important to emphasize that the information state is *not* a state estimator. The information state p is given by

$$p_t(x) = \sup_{w\in L_2[0,t]} \sup_{x_0\in\mathbf{R}^n} \{p_0(x_0) + \frac{1}{2}\int_0^t (\|\ell(x(s),u(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2))ds : x(t) = x\} \tag{2.2.3}$$

where past observations and controls $\{u(s), y(s) : s\in[0,t]\}$ are known. From the definition it is seen that the information state is the cost accumulated up to the time t, consistent with the available information at time t under the assumption that state at time t is x. Thus the information state is the value of the accrued cost for each possible state x at time t.

Using dynamic programming it can be shown that the information state is the solution to the Hamilton-Jacobi Equation (c.f., [JBE93a])

$$\begin{cases} \frac{\partial p_t}{\partial t} &= F(p_t, u(t), y(t)) \\ p_0 &= p_0, \end{cases} \tag{2.2.4}$$

where

$$\begin{aligned} F(p,u,y) &\triangleq \sup\nolimits_{w\in\mathbf{R}^n}\{-<\nabla_x p, f(x,u)+w> - \tfrac{\gamma^2}{2}(\|w\|^2 + \|h(x)-y\|^2) + \tfrac{1}{2}\|\ell(x,u)\|^2\} \\ &= -<\nabla_x p, f(x,u)> + \tfrac{1}{2\gamma^2}\|\nabla_x p\|^2 + \tfrac{1}{2}\|\ell(x,u)\|^2 - \tfrac{\gamma^2}{2}\|h(x)-y\|^2. \end{aligned}$$

The maximizing disturbance in this case is $\widehat{w} = -\frac{1}{\gamma^2}\nabla_x p$.

The cost (Equation (2.2.2)) can be rewritten in terms of the information state as

$$J_{p,t_f}(u) = \sup_{y\in L_2[0,t_f]} \{(p_{t_f}, 0) : p_0 = p\}. \tag{2.2.5}$$

Thus the original output feedback game is equivalent to a new game where the information state dynamics (2.2.4) are new *infinite dimensional* state dynamics Ξ with control u and disturbance y. The state p_t and disturbance y_t are available to the controller, so the original output feedback game is equivalent to a new one with *full information*. Let $\mathcal{I}$ denote the class of information state feedback controllers u such that $u(t) = \bar{u}(p_t)$ for some function $\bar{u} : \mathcal{E} \mapsto U$. Note that since the information state p depends only on observable information, the class of information state feedback controllers $\mathcal{I}$ is contained in the class of output feedback controllers $\mathcal{O}$.

2.2.3 Solution to the Finite Time Robust H_∞ Output Feedback Problem

In this section we present necessary and sufficient conditions for obtaining a solution to the Finite Time Robust H_∞ Output Feedback Problem. The reformulation, described in Sections 2.2.1 and 2.2.2, of the Finite Time Robust H_∞ Output Feedback Problem as an output feedback game with full, albeit infinite dimensional, information, allows us to formally apply the same arguments as those used in solving the nonlinear state feedback problem, viz. Section 1.3.3. The information state solution we present below constitutes a separation principle since the problem has been split into two simpler problems: (i) computing the information state, and (ii) computing the optimal control as a function of the information state.

The value of the full information game is given by

$$W_{t_f}(p,t) = \inf_{u \in \mathcal{O}_{t,t_f}} \sup_{y \in L_2} \{(p_{t_f}, 0) : p_t = p\}. \tag{2.2.6}$$

We shall henceforth drop the subscript t_f for notational convenience. Applying dynamic programming, we can determine an optimal control $u(t)$ which is a function of the past observations $\{y(s) : s \in [t_0, t]\}$ through the information state p. Formally the value function W satisfies the dynamic programming equation (c.f., [JBE93a])

$$\begin{cases} \frac{\partial W}{\partial t} + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{< \nabla_p W, F(p,u,y) >\} = 0 \\ W(p, t_f) = (p, 0) \end{cases} \tag{2.2.7}$$

The optimal control is given by the minimizing value of u in this equation. Note that this control is an *information state feedback control*.

We define the domain dom W_{t_f} of a value function $W_{t_f} : \mathcal{E} \times [0, t_f] \mapsto \mathbf{R}^*$

$$\text{dom } W_{t_f} \triangleq \{p \in \mathcal{E} : W(p,t) < \infty \text{ for all } t \in [0, t_f]\}. \tag{2.2.8}$$

Theorem 2.2.3 (Necessity) Assume that there exists a controller $u^o \in \mathcal{O}$ which solves the Finite Time Robust H_∞ Output Feedback Problem. Then there exists a function W which is finite on dom $J_{p,t_f}(u^o)$, satisfies $W(p,t) \geq (p,0)$ and $W(\delta_0,t)=0$ for all $t \in [0,t_f]$, and the Dynamic Programming Equation (2.2.7).

Proof: Define

$$W(p,t) = \inf_{u \in \mathcal{O}_{t,t_f}} \sup_{w,v \in L_2[0,t_f]} \sup_{x_t \in \mathbf{R}^n} \{p(x_t) + \frac{1}{2}\int_t^{t_f} (\|\ell(x(s),u(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|v(s)\|^2))ds\}$$

From this definition it is clear that $W(p,0) \geq W(p,t) \geq (p,0)$. Also note that $W(p,0) = \inf_{u \in \mathcal{O}_{0,t_f}} J_{p,t_f}(u)$. Since Σ^{u^o} is finite gain we have

$$W(p,t) \leq W(p,0) \leq J_{p,t_f}(u^o) \leq (p, \beta_{t_f}^{u^o}),$$

thus W is finite on dom $J_p(u^o)$, and $W(p,t) \geq (p,0)$ and $W(\delta_0,t)=0$ for all $t \in [0,t_f]$. It remains to show that W satisfies the dynamic programming equation (2.2.7).

From the representation result Equation (2.2.5) we know that W is also defined by

$$W(p,t) = \inf_{u \in \mathcal{O}_{t,t_f}} \sup_{y \in L_2[0,t_f]} \{(p_{t_f},0) : p_0 = p\}.$$

It is then a common result of dynamic programming that W solves the dynamic programming equation (2.2.7). We include a formal argument here for completeness. For more details refer to [Tit87, Ell93].

Consider the problem

$$\inf_{u \in \mathcal{O}_{t,t_f}} \sup_{y \in L_2[0,t_f]} \{\int_0^{t_f} f_0(t,p_t,y(t),u(t))dt + \Phi(p(t_f)) : p_0 = p\}.$$

Let $V(p,t)$ be the optimal value of the objective function starting from state p at time t.

$$\begin{cases} V(p,t) &= \inf_{u \in \mathcal{O}_{t,t_f}} \sup_{y \in L_2[0,t_f]} \{\int_t^{t+\delta} f_0(\tau,p_\tau,y(\tau),u(\tau))d\tau + V(t+\delta,p_{t+\delta})\} \\ V(p,t_f) &= \Phi(p(t_f)). \end{cases}$$

Suppose that $V(p,t)$ is differentiable in both its arguments. Taking a first order expansion in δ gives

$$V(p,t) = \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{f_0(t,p_t(x),y(t),u(t))\delta + V(p,t) + \frac{\partial V(p,t)}{\partial t}\delta + < \nabla_p V(p,t), F(p,u,y) > \delta + o(\delta)\}$$

where $\frac{o(\delta)}{|\delta|} \to 0$ as $\delta \to 0$. Then dividing by δ and letting $\delta \to 0$, we get

$$\begin{cases} \frac{\partial V(p,t)}{\partial t} + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{f_0(t,p_t(x),y(t),u(t)) + < \nabla_p V(p,t), F(p,u,y) >\} = 0 \\ V(p,t_f) = \Phi(p(t_f)). \end{cases}$$

In our case we take $f_0 \equiv 0$ and the result follows. ∎

Theorem 2.2.4 (Sufficiency) Assume that $W \in C^1$ is a solution of the Dynamic Programming Equation (2.2.7) satisfying $\delta_x \in \text{dom } W_{t_f}$ for all $x \in \mathbf{R}^n$, and $W(p,t) \geq (p,0)$, $W(\delta_0,t) = 0$ for all $t \in [0,t_f]$. Let $\bar{u}^*(p)$ be a control value which achieves the minimum in (2.2.7). Then the controller $u^* \in \mathcal{I}$ defined by $u^* = \bar{u}^*(p)$ solves the finite time output feedback robust control problem.

Proof: By assumption $\delta_x \in \text{dom } W_{t_f}$ for all $x \in \mathbb{R}^n$ so $W(\delta_x,t)$ is finite for all $t \in [0,t_f]$. Define $\beta^u_{t_f}(x) = W(\delta_x, 0)$. Then by assumption $\beta^u_{t_f}(0) = 0$.

$$\begin{aligned} \beta^u_{t_f}(x) &= \inf\nolimits_{u \in \mathcal{O}_{t,t_f}} \sup\nolimits_{y \in L_2} \{(p_{t_f}, 0) : p_t = \delta_x\} \\ &= \sup\nolimits_{y \in L_2} \{(p_{t_f}, 0) : p_t = \delta_x, u = u^*\} \\ &= J_{\delta_x, t_f}(u^*) \end{aligned}$$

Thus $\beta^u_{t_f}(x) \geq J_{\delta_x,t_f}(u^*)$, so by Lemma 2.2.2 Σ^{u^*} is finite gain for all $x \in \mathbf{R}^n$ and thus u^* solves the finite time output feedback robust control problem. ∎

Corollary 2.2.5 If the finite time Output Feedback Robust Control Problem is solvable by an output feedback controller $u^o \in \mathcal{O}$, then it is also solvable by an information state feedback controller $u^* \in \mathcal{I}$.

Thus we have presented necessary and sufficient conditions for obtaining an information state feedback solution to the Finite Time Robust H_∞ Output Feedback Problem. The information state solution is obtained by means of a separation principle.

2.3 Infinite Time Horizon

In order to solve the Robust H_∞ Output Feedback Problem on the infinite time horizon we would like to find a stationary version of the Dynamic Programming Equation (2.2.7). To do this we will minimize over $u \in \mathcal{O}$ the functional

$$J_p(u) \triangleq \sup_{t \geq 0} J_{p,t}(u). \tag{2.3.1}$$

This makes sense in view of Lemma 2.3.1 below which expresses the finite gain property in terms of this new cost (and equivalently in terms of the information state).

Lemma 2.3.1 For any output feedback controller $u \in \mathcal{O}$, the closed loop system Σ^u is finite gain if and only if

$$J_{\delta_{x_0}}(u) = \sup_{t \geq 0} \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = \delta_{x_0}\} \leq \beta^u(x_0),$$

for some finite $\beta^u(x_0)$ with $\beta^u(0) = 0$.

From this Lemma 2.3.1 and the definition of $J_p(u)$, Equation (2.3.1), it is clear that when Σ^u is finite gain for each initial condition x_0, the cost $J_p(u)$ satisfies

$$(p, 0) \leq J_p(u) \leq (p, \beta^u).$$

Thus we define the domain dom $J_p(u)$ of the cost functional $J_p(u)$ to be

$$\text{dom } J_p(u) \triangleq \{p \in \mathcal{E} : (p, 0), (p, \beta^u) < \infty\}.$$

Also define for a value function $W : \mathcal{E} \mapsto \mathbb{R}^*$ its domain dom W as

$$\text{dom } W \triangleq \{p \in \mathcal{E} : W(p) < \infty\},$$

c.f. Equation 2.2.8.

Definition 2.3.2 (Finite Gain Dissipative) Let $u \in \mathcal{I}$. The information state system Ξ^u is finite gain dissipative if there exists a storage function $W(p)$ such that dom W contains δ_x for all $x \in \mathbf{R}^n$, $W(p) \geq (p, 0)$, $W(\delta_0) = 0$, and which satisfies the dissipation inequality

$$\sup_{y \in \mathbf{R}^p} \{< \nabla_p W, F(p, u, y) >\} \leq 0 \tag{2.3.2}$$

where $u = u(p)$.

Theorem 2.3.3 (Bounded Real Lemma) Let $u \in \mathcal{I}$. Then the information state system Ξ^u is finite gain dissipative if and only if it is finite gain.

Proof: Assume Ξ^u is finite gain dissipative then the dissipation inequality (2.3.2) implies that formally W is nonincreasing along trajectories of Ξ^u, i.e.,

$$W(p_t) \leq W(p_0)$$

for all $y \in L_2[0,t]$, for all $t > 0$. Setting $p_0 = \delta_{x_0}$ and using the inequality $W(p) \geq (p,0)$ we get

$$(p_t, 0) \leq W(p_t) \leq W(\delta_{x_0})$$

for all $y \in L_2[0,t]$, for all $t > 0$. Thus by Lemma 2.3.1 Ξ^u is finite gain with $\beta^u(x_0) \triangleq W(\delta_{x_0})$.

Conversely, assume that Ξ^u is finite gain. Then for all $p \in \text{dom } J_p(u)$, and for all $t \geq 0$

$$(p, 0) \leq J_{p,t}(u) = \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = p\} \leq (p, \beta^u).$$

By definition (Equation (2.2.2)) we know that $J_{p,t}(u)$ is nondecreasing, i.e.,

$$J_{p,t_1}(u) \geq J_{p,t_0}(u)$$

when $t_1 \geq t_0$. Define

$$W_a(p) \triangleq \lim_{t \to \infty} J_{p,t}(u)$$

which exists and is finite on dom W_a. Note that dom W_a contains dom $J_p(u)$ since $J_{p,t}(u)$ is monotone nondecreasing and bounded above on dom $J_p(u)$. From the finite gain property we have dom W_a contains δ_x for all $x \in \mathbf{R}^n$ and $W_a(\delta_0) = 0$ since $\beta^u(0) = 0$. Also we have $W_a(p) \geq (p, 0)$.

Now we must show that W_a satisfies the dissipation inequality (2.3.2). Pick $u = \bar{u} \in \mathcal{I}$ and $y = \bar{y}$ then define

$$p_1(x) = \sup_{w \in L_2[0,t_1]} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \frac{1}{2}\int_0^{t_1} (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|h(x(s)) - \bar{y}(s)\|^2))ds : x(t_1) = x\}.$$

Thus p_1 is the information state at time t_1 starting from information state p_0 at time 0 and with inputs $\bar{u}$ and $\bar{y}$.

$$\begin{aligned}
J_{p_0,t}(\bar{u}) &= \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = p_0\} \\
&= \sup_{y,w \in L_2[0,t]} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \tfrac{1}{2}\textstyle\int_0^t (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2))ds\} \\
&\geq \sup_{y,w \in L_2[t_1,t]} \sup_{x_0 \in \mathbf{R}^n} \{p_1(x_0) + \tfrac{1}{2}\textstyle\int_{t_1}^t (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2))ds\} \\
&= J_{p_1, t-t_1}(\bar{u})
\end{aligned}$$

Taking the limit as $t \to \infty$

$$W_a(p_0) \geq W_a(p_1).$$

Since this is true for any arbitrary $\bar{y}$ we have that W_a is nonincreasing along trajectories of Ξ^u. Thus formally W_a satisfies the dissipation inequality. ■

For the remainder of this section we assume that the output function h defined in Equation (2.1.1) satisfies the linear growth condition:

$$\|h(x)\| \leq C\|x\|. \tag{2.3.3}$$

Definition 2.3.4 (Zero State Detectability)

The closed loop system Σ^u is *zero state detectable* if $w, v \equiv 0$ and $z \in L_2[0,\infty]$ implies $\lim_{t\to\infty} x(t) = 0$.

The closed loop system Σ^u is *L_2-zero state detectable* if $w, v \equiv 0$ and $z \in L_2[0,\infty]$ implies $x \in L_2[0,\infty]$.

The definition of uniformly reachable, Definition 2.3.5 below, is a technical condition which is needed for stability of the information state system. The idea represented by the definition is that the closed loop system, regardless of the output, can be driven by finite energy disturbances to any state given enough time. The assumption of uniform reachability is used in Theorem 2.3.7 below to obtain stability of the information state.

Definition 2.3.5 (Uniformly Reachable) For $u \in \mathcal{O}$ and $y \in L_2[0,\infty]$, the closed loop system Σ^u is uniformly reachable if for all $x \in \mathbb{R}^n$ and for all $t > 0$ sufficiently large there exists $x_0 \in \mathbb{R}^n$, $w, v \in L_2[0,\infty]$, and $0 \le \alpha(x) < +\infty$ such that $x(0) = x_0$, $x(t) = x$, $y = h(x) + v$ and

$$\|x_0\|^2 + \int_0^t (\|w(s)\|^2 + \|v(s)\|^2)ds \le \alpha(x).$$

Next we give a definition of the stability of an information state system. The idea is that for stability we require eventual boundedness of the information state.

Definition 2.3.6 (Stability) Given $u \in \mathcal{I}$ and $y \in L_2[0,\infty]$, the information state system Ξ^u is stable if for each $x \in \mathbb{R}$ there exists $T_x, C_x \ge 0$ such that

$$\|p_t(x)\| \le C_x \quad \text{for all} \quad t \ge T_x,$$

provided the initial value p_0 satisfies the growth condition

$$-a_1'\|x\|^2 - a_2' \le p_0(x) \le -a_1\|x\|^2 + a_2$$

where $a_1, a_1', a_2, a_2' \ge 0$.

Theorem 2.3.7 Let $u \in \mathcal{I}$. If Ξ^u is finite gain dissipative and Σ^u is zero state detectable, then Σ^u is asymptotically stable. If Ξ^u is finite gain dissipative and Σ^u is L_2-zero state detectable and uniformly reachable, then Ξ^u is stable.

Proof: The dissipation inequality implies that formally $W(p_t) \leq W(p_0)$, thus

$$\sup_{y,w \in L_2[0,t]} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \frac{1}{2}\int_0^t (\|\ell(x(s),u(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2))ds\} \leq W(p_0)$$

for all $t \geq 0$. So if we select $p_0 = \delta_{x_0}$, $w \equiv 0$, $y = h(x)$, and recalling $z = \ell(x,u)$ we have

$$\frac{1}{2}\int_0^t \|z(s)\|^2 ds \leq W(p_0) < \infty$$

for all $t \geq 0$. Thus $z \in L_2[0,\infty]$ so by zero state detectability $\lim_{t\to\infty} x(t) = 0$, thus Σ^u is asymptotically stable.

By L_2-zero state detectability $x \in L_2[0,\infty]$. Then by assumption of linear growth of the output function (2.3.3), we have also that $y \in L_2[0,\infty]$. The assumption of finite gain dissipativity implies that

$$p_t(x) \leq (p_t, 0) \leq W(p_0) < \infty$$

for all $p_0 \in$ dom W, for all $t \geq 0$. Uniformly reachable implies for all $x \in \mathbf{R}^n$ and for all $t > 0$ sufficiently large there exists $x_0 \in \mathbf{R}^n$, $w, v \in L_2[0,\infty]$, and $0 \leq \alpha(x) < +\infty$ such that $x(0) = x_0$, $x(t) = x$ and

$$\|x_0\|^2 + \int_0^t (\|w(s)\|^2 + \|v(s)\|^2)ds \leq \alpha(x).$$

Thus

$$\begin{aligned} p_t(x) &\geq p_0(x_0) - \gamma^2 \textstyle\int_0^t (\|w(s)\|^2 + \|v(s)\|^2)ds \\ &\geq -\gamma^2\alpha(x) + (\gamma^2 - a_1')\|x_0\|^2 - a_2' \end{aligned}$$

for all t sufficiently large. Thus $p_t(x)$ is eventually bounded and therefore Ξ^u is stable. ■

Note that the stability conditions are difficult to check in practice. Also that it is feasible that Σ^u is stable, with Ξ^u unstable. This corresponds to an unstable stabilizing controller.

The Robust H_∞ Information State Feedback Problem is stated next. This problem will be shown to be equivalent to the Robust H_∞ Output Feedback Problem 2.1.1.

Problem 2.3.8 Robust H_∞ Information State Feedback Control Problem
Given $\gamma > 0$, find a control $u \in \mathcal{I}$ such that for all initial conditions $x_0 \in \mathbf{R}^n$,

(i) the closed loop system Ξ^u is stable when $w, v \equiv 0$, and

(ii) there exists a constant $\beta^u(x_0)$ where $0 \leq \beta^u(x_0) < \infty$ and $\beta^u(0) = 0$ such that

$$\sup_{t \geq 0} \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = \delta_{x_0}\} \leq \beta^u(x_0),$$

Theorem 2.3.9 (Necessity) Assume that there exists a controller $u^o \in \mathcal{O}$ which solves the Robust H_∞ Output Feedback Problem. Then there exists a function $W(p)$ which is finite on dom $J_p(u^o)$, satisfies $W(p) \geq (p, 0)$, $W(\delta_0) = 0$, and the dissipation inequality

$$\inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{< \nabla_p W, F(p, u, y) >\} \leq 0. \tag{2.3.4}$$

Proof: Define

$$W(p) = \inf_{u \in \mathcal{O}} \lim_{t \to \infty} J_{p,t}(u)$$

where $J_{p,t}(u) = \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = p\}$. Since Σ^{u^o} is finite gain we have

$$(p, 0) \leq W(p) \leq \lim_{t \to \infty} J_{p,t}(u^o) \leq (p, \beta^{u^o}),$$

thus W is finite on dom $J_p(u^o)$, $W(p) \geq (p, 0)$, and $W(\delta_0) = 0$. It remains to show that W satisfies inequality (2.3.4). This proof is almost identical to the second half of the proof of the Bounded Real Lemma 2.3.3. The difference is that here we allow any output feedback controller $\bar{u} \in \mathcal{O}$, we are not restricted to information state feedback controllers $\bar{u} \in \mathcal{I}$.

Pick $u = \bar{u} \in \mathcal{O}$ and $y = \bar{y}$ then define

$$p_1(x) = \sup_{w \in L_2[0,t_1]} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \frac{1}{2} \int_0^{t_1} (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2 (\|w(s)\|^2 + \|h(x(s)) - \bar{y}(s)\|^2)) ds : x(t_1) = x\}.$$

Thus p_1 is the information state at time t_1 starting from information state p_0 at time 0 and with inputs $\bar{u}$ and $\bar{y}$.

$$\begin{aligned} J_{p_0,t}(\bar{u}) &= \sup_{y \in L_2[0,t]} \{(p_t, 0) : p_0 = p_0\} \\ &= \sup_{y,w \in L_2[0,t]} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \tfrac{1}{2} \textstyle\int_0^t (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2 (\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2)) ds\} \\ &\geq \sup_{y,w \in L_2[t_1,t]} \sup_{x_0 \in \mathbf{R}^n} \{p_1(x_0) + \tfrac{1}{2} \textstyle\int_{t_1}^t (\|\ell(x(s), \bar{u}(s))\|^2 - \gamma^2 (\|w(s)\|^2 + \|h(x(s)) - y(s)\|^2)) ds\} \\ &= J_{p_1, t-t_1}(\bar{u}) \end{aligned}$$

Taking the limit as $t \to \infty$ and the $\inf_{u \in \mathcal{O}}$

$$W(p_0) \geq W(p_1).$$

Since this is true for any arbitrary $\bar{y}$ we have that W is nonincreasing along trajectories of the closed loop system. Thus formally W satisfies the inequality (2.3.4). ∎

The Bounded Real Lemma 2.3.3 gives us immediately that if the Output Feedback Robust Control Problem is solvable by an information state feedback controller $u^i \in \mathcal{I}$, then there exists a solution to the inequality (2.3.4). However this result is not adequate for a necessity theorem since it is expressed a priori in terms of an information state feedback controller.

Theorem 2.3.10 (Sufficiency) Assume that $W \in C^1$ is a solution of the dissipation inequality (2.3.4) satisfying $\delta_x \in \text{dom } W$ for all $x \in \mathbb{R}^n$, $W(p) \geq (p, 0)$, $W(\delta_0) = 0$. Let $\bar{u}^*(p)$ be a control value which achieves the minimum in (2.3.4). Then the controller $u^* \in \mathcal{I}$ defined by $u^* = \bar{u}^*(p)$ solves the information state feedback robust control problem if the closed loop system Σ^{u^*} is L_2-zero state detectable and uniformly reachable.

Proof: The information state system Ξ^{u^*} is finite gain dissipative since (2.3.4) implies (2.3.2) for the controller u^*. Hence by the Bounded Real Lemma 2.3.3, Ξ^{u^*} is finite gain. Theorem 2.3.7 then shows that Σ^{u^*} is stable. Hence u^* solves the information state feedback robust control problem. ■

Corollary 2.3.11 If the Output Feedback Robust Control Problem is solvable by an output feedback controller $u^o \in \mathcal{O}$, then it is also solvable by an information state feedback controller $u^* \in \mathcal{I}$.

2.4 Finite Dimensional Information State

In this section we discuss conditions under which the infinite dimensional information state p evolving according to the Dynamic Programming Equation (2.2.4) can be identified with a finite dimensional quantity which evolves according to ordinary differential equations. It is important to characterize the finite dimensional nature of the information state because under these circumstances

I. the results of Sections 2.2 and 2.3 hold rigorously,

II. the controller is directly implementable.

2.4.1 General Systems

Consider the class of systems described by the following set of state space equations

$$\Sigma_F \quad \begin{cases} \dot{x}(t) &= f(x(t)) + A(u(t))x(t) + B(u(t)) + w(t), \qquad x(t_0) = x_0, \\ y(t) &= Cx(t) + v(t), \\ z(t) &= \begin{pmatrix} \sqrt{Q(u(t))}x(t) \\ \sqrt{R(u(t))} \end{pmatrix}. \end{cases} \tag{2.4.1}$$

where Σ_F is a special case of Σ given in Equation (2.1.1) with $f(x(t), u(t)) = f(x(t)) + A(u(t))x(t) + B(u)$, $h(x(t)) = Cx(t)$, and $\ell(x(t), u(t)) = (x(t)'\sqrt{Q(u(t))}', \sqrt{R(u(t))}')'$. We make the following additional assumptions:

(A1) f is Lipschitz continuous and satisfies

$$f(x) = \nabla_x F(x)$$

for some $F : \mathbf{R}^n \mapsto \mathbf{R}$ and

$$\frac{1}{2}\|f(x)\|^2 + < f(x), A(u)x + B(u) > = \frac{1}{2}x'\Sigma(u)x + \Lambda(u)x + \frac{1}{2}\Gamma(u).$$

(A2) The matrix functions $A(u)$, $B(u)$, $\Sigma(u), \Lambda(u)$, and $Q(u)$ are Lipschitz continuous, and $\Gamma(u)$, and $R(u)$ are locally Lipschitz continuous with at most quadratic growth. In addition $Q(u) \geq 0$ and $R(u) > 0$.

(A3) The initial cost is

$$\bar{p}(x) = -\frac{\gamma^2}{2}(x - \bar{x})'\bar{P}^{-1}(x - \bar{x}) + \bar{\phi} + \gamma^2 F(x).$$

For this class of systems the information state p is given by

$$\begin{aligned} p_t(x) = \sup_{w\in L_2} \sup_{x_0 \in \mathbf{R}^n} \{ \quad & \bar{p}(x_0) + \tfrac{1}{2}\int_0^t (< x(s), Q(u(s))x(s) > + R(u(s)) \\ & -\gamma^2(\|w(s)\|^2 + \|Cx(s) - y(s)\|^2))ds : x(t) = x\} \end{aligned}$$

where past observations and controls $\{u(s), y(s) : s \in [0, t]\}$ are known. Using dynamic programming it can be shown that the information state is the solution to the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial p_t}{\partial t} &= -< \nabla_x p, f(x) + A(u)x + B(u) > + \frac{1}{2\gamma^2}\|\nabla_x p\|^2 - \frac{\gamma^2}{2}\|Cx - y\|^2 + \frac{1}{2}(< x, Q(u)x > + R(u)) \\ p_0 &= \bar{p}. \end{cases} \tag{2.4.2}$$

In general the information state is infinite dimensional, i.e., $p_t = p_t(x)$ evolves in a general class of functions which cannot be parameterized by finite dimensional quantities. In [Jam93a] James shows that for the special class of nonlinear systems Σ_F under the assumptions (A1)-(A3), the special form of the dynamics and cost function allow the information state to be described in terms of finite dimensional quantities. The exact formula for the information state are given in Theorem 2.4.1 [Jam93a]. The fact that the information state satisfies Equation (2.4.2) can be shown by direct differentiation and substitution.

Theorem 2.4.1 For Σ_F, Equation (2.4.1), under the assumptions (A1)-(A3) the information state is given by

$$p_t(x) = \phi(t) - \frac{\gamma^2}{2} < x(t) - \widehat{x}(t), P^{-1}(t)(x(t) - \widehat{x}(t)) > +\gamma^2 F(x)$$

where $P = P' > 0$ and $\widehat{x}(t)$, $P(t)$ and $\phi(t)$ satisfy the ODE's

$$\begin{aligned}
\dot{\widehat{x}}(t) &= (A(u(t)) + \tfrac{1}{\gamma^2}P(t)Q(u(t)) - P(t)\Sigma(u(t)))\widehat{x}(t) + B(u(t)) - P(t)\Lambda(u(t)) \\
&\quad +P(t)C'(y(t) - C\widehat{x}(t)), \\
\widehat{x}(0) &= \bar{x},
\end{aligned}$$

$$\begin{aligned}
\dot{P}(t) &= P(t)A(u(t))' + A(u(t))P(t) - P(t)(C'C - \tfrac{1}{\gamma^2}Q(u(t)))P(t) + \Sigma(u(t))P(t) + I, \\
P(0) &= \bar{P},
\end{aligned}$$

$$\begin{aligned}
\dot{\phi}(t) &= \tfrac{1}{2}(< \widehat{x}(t), Q(u(t))\widehat{x}(t) > +R(u(t)) - \gamma^2||y(t) - C\widehat{x}(t)||^2) \\
&\quad -\tfrac{\gamma^2}{2}(< \widehat{x}(t), \Sigma(u(t))\widehat{x}(t) > +2\Lambda(u(t))\widehat{x}(t) + \Gamma(u(t)), \\
\phi(0) &= \bar{\phi}.
\end{aligned}$$

Theorem 2.4.1 implies that for any system of the form Σ_F under the assumptions listed above the information state can be identified with the finite dimensional quantity $\rho \triangleq (\widehat{x}, P, \phi)$. We denote the finite dimensional information state by p_ρ, i.e., $p_\rho = \phi - \frac{\gamma^2}{2} < x - \widehat{x}, P^{-1}(x - \widehat{x}) > +\gamma^2 F(x)$. Using this expression of the information state, the representation (2.2.5) becomes

$$\begin{aligned}
J_{p_\rho, t_f}(u) = \; & \sup_{y \in L_2[0,t_f]}\{\tfrac{1}{2}\int_0^{t_f}(< \widehat{x}(s), Q(u(s))\widehat{x}(s) > +R(u(s)) - \gamma^2(||y(s) - C\widehat{x}(s)||^2) \\
& -\gamma^2(< \widehat{x}(s), \Sigma(u(s))\widehat{x}(s) > +2\Lambda(u(s))\widehat{x}(s) + \Gamma(u(s)))ds + \phi(0) : \rho(0) = \rho\}.
\end{aligned} \quad (2.4.3)$$

Thus the output feedback robust control problem is equivalent to a new finite dimensional state feedback game with state ρ. We now regard the value function W as a function of ρ:

$$W(\rho, t) = W(p_\rho, t).$$

Remark 2.4.2 From the representation (2.4.3) we can see that W is only dependent on ϕ at the initial time, thus we can immediately write $W(\widehat{x}, P, \phi, t) = \widetilde{W}(\widehat{x}, P, t) + \phi(t)$. This also implies that $\nabla_\phi W = 1$.

Before we give the dynamic programming equation for $W(\rho, t)$, however, we must first define some pertinent operators. Let $A,\ B \in \mathbb{R}^{n\times m}$ where $A = [a_{ij}]$ and $B = [b_{ij}]$.

Definition 2.4.3 The matrix dot product $<< \cdot, \cdot >>: \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \longrightarrow \mathbb{R}$ is defined by

$$<< A, B >> \triangleq \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}.$$

Definition 2.4.4 The derivative of a function $W(\cdot) : \mathbf{R}^{n\times m} \longrightarrow \mathbf{R}$ with respect to its matrix argument is defined by

$$\nabla_A W \triangleq \left[\frac{\partial W}{\partial a_{ij}}\right].$$

Thus $\nabla_A W$ is the matrix of first partial derivatives with respect to the elements of A. The dynamic programming equation is now

$$\begin{cases} \frac{\partial W}{\partial t}(\rho, t) + \sup_{y\in\mathbf{R}^p}\inf_{u\in U}\{< \nabla_{\hat{x}} W, (A(u(t)) + \frac{1}{\gamma^2}P(t)Q(u(t)) - P(t)\Sigma(u(t)))\hat{x}(t) \\ \quad + B(u(t)) - P(t)\Lambda(u(t)) + P(t)C'(y(t) - C\hat{x}(t)) > \\ \quad + << \nabla_P W, P(t)A(u(t))' + A(u(t))P(t) \\ \quad - P(t)(C'C - \frac{1}{\gamma^2}Q(u(t)))P(t) + \Sigma(u(t))P(t) + I >> \\ \quad + \frac{1}{2}(< \hat{x}(t), Q(u(t))\hat{x}(t) > + R(u(t)) - \gamma^2\|y(t) - C\hat{x}(t)\|^2) \\ \quad - \frac{\gamma^2}{2}(< \hat{x}(t), \Sigma(u(t))\hat{x}(t) + 2\Lambda(u(t))\hat{x}(t) + \Gamma(u(t))) >\} = 0 \\ W(\rho, t_f) = (p_{\rho_{t_f}}, 0) \end{cases} \tag{2.4.4}$$

where y plays the role of a competing disturbance. Note that since Isaacs condition is satisfied the order in which the inf and sup are applied is inconsequential.

An interesting and novel feature of this problem is that the value function need not be finite for all values of ρ, t. In the linear case, this is closely related to the coupling condition described in Theorem 1.3.2 between the minimal solutions Z^+ and K^+ of the control and estimation Riccati equations, i.e., $Z^+ - \gamma^2 K^{+^{-1}} < 0$ [YJ93]. Let us write

$$\mathcal{D} = \{(\hat{x}, P, \phi, t) \in \mathbb{R}^n \times \mathrm{S}^n \times \mathbf{R} \times [0, T] \ : \ W(\hat{x}, P, \phi, t) < \infty\}. \tag{2.4.5}$$

In general, $\mathcal{D}$ is a nontrivial subset of $\mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R} \times [0,T]$.

The partially observed game problem can now be solved using the Dynamic Programming Equation (2.4.4) [Fri71, BB91, FS93], as stated in the following theorem.

Theorem 2.4.5 (Verification) Assume there exists a smooth solution $W \in C^1(\mathcal{D})$ of the Hamilton-Jacobi Equation (2.4.4). Then the control $u^*(\rho, t)$ which attains the infimum in (2.4.4) defines an optimal controller $u^* \in \mathcal{O}_{t_0,t_f}$ which minimizes the cost functional (6.1.2) for the partially observed game. The optimal control at time t is $u_t^* = u^*(\rho_t, t)$.

In general, the value function need not be C^1, and Equation (2.4.4) must be interpreted in a generalized sense. This is typically the case in optimal control and game theory. However, it can happen that W is smooth in certain regions, as in the next theorem.

Corollary 2.4.6 For each final condition $(\rho_f, t_f) \in \mathcal{D}$ there exists an open neighborhood V of this point such that u^* is optimal on V.

Proof: Using the method of characteristics [Eva92, Joh82], a solution may be obtained to the Hamilton Jacobi Equation (HJE) (2.4.4) by solving a related system of ordinary differential equations (ODE's) called the characteristic equations. The initial conditions for the characteristic ODE's are determined by the initial conditions for the HJE in such a way that they remain compatible with the original problem. The solution to the original HJE is then the union of the solutions to the characteristic equations starting from the various initial conditions.

Before giving the characteristic equations we first define

$$\begin{aligned} z(t) &\triangleq W(\rho(t), t), \quad \text{and} \\ q(t) &\triangleq \nabla_\rho W(\rho(t), t), \end{aligned}$$

so that Equation (2.4.4) can be rewritten as

$$\frac{\partial W}{\partial t}(\rho, t) + H(q, \rho) = 0.$$

The characteristic equations [Eva92] are

$$\begin{aligned} \dot{\rho}(t) &= \nabla_q H(q(t), \rho(t)) \\ \dot{q}(t) &= -\nabla_\rho H(q(t), \rho(t)) \\ \dot{z}(t) &= < \nabla_q H(q(t), \rho(t)), q(t) > - H(q(t), \rho(t)) \end{aligned}$$

Note that in Equation (2.4.4) the maximizing y is $y^* = \frac{1}{\gamma^2} DP\nabla_{\hat{x}} W + C\hat{x}$ and the minimizing u, u^* is defined in Theorem 2.4.5. Thus $H(q,\rho)$ is a polynomial function of q, ρ and we are thus guaranteed smooth solutions of the characteristic ODE's [Kha92] at least for small time. In order to guarantee a smooth solution to the HJE (2.4.4), we must, however, also guarantee that the solutions to the characteristic equations starting from different initial conditions do not intersect. Because of this we only have a locally smooth solution to the HJE (2.4.4), i.e., starting from the final condition $(\rho_f, t_f) \in \mathcal{D}$ there exits an open neighborhood V of (ρ_f, t_f) such that a smooth solution exists for all $(\rho, t) \in V$ [Eva92]. Invoking the Verification Theorem, we conclude u^* is optimal on V.

■

2.4.2 Bilinear Systems

An important class of systems which satisfy the assumptions for having a finite dimensional information state is that of bilinear systems. Consider the class of bilinear systems described by the following set of state space equations

$$\Sigma_B^u \begin{cases} \dot{x}(t) = [A_0 + \sum_{i=1}^m A_i u_i(t)]x(t) + Bu(t) + w(t), \qquad x(t_0) = x_0, \\ y(t) = Cx(t) + v(t), \\ z(t) = \begin{pmatrix} \sqrt{Q}x(t) \\ \sqrt{R}u(t) \end{pmatrix}. \end{cases}$$

Note that Σ_L is a special case of Σ as described in Section 1.2 where $f(x(t), u(t)) = [A_0 + \sum_{i=1}^m A_i u_i(t)]x(t) + Bu(t)$, $h(x(t)) = Cx(t)$, and $\ell(x(t), u(t)) = (x(t)'\sqrt{Q}', u(t)'\sqrt{R}')'$.

This class of systems is noteworthy because, although the class is only mildly nonlinear, there are well know strategies for controlling nonlinear systems which may not apply to this class of systems. For example, when $B = 0$, any method which relies on the ability to stabilize the linearization will not apply if the drift term A_0 is not already an asymptotically stable matrix.

Below are stated the finite dimensional information state results specialized to bilinear systems. In Section 5.1 examples of information state control for bilinear systems are given which demonstrate that information state controllers can be robust and stabilizing even in cases when linearization techniques fail. For further discussion and examples see [TYJB94].

Theorem 2.4.7 For Σ_B, the information state is given by

$$p_t(x) = \phi(t) - \frac{\gamma^2}{2} < x(t) - \hat{x}(t), P^{-1}(t)(x(t) - \hat{x}(t)) >$$

where $P = P' > 0$ and $\widehat{x}(t)$, $P(t)$ and $\phi(t)$ satisfy the ODE's

$$\begin{aligned}
\dot{\widehat{x}}(t) &= (A_u(t) + \gamma^{-2}P(t)Q)\widehat{x}(t) + Bu(t) + P(t)C'(y(t) - C\widehat{x}(t)), \\
\widehat{x}(0) &= \widehat{x}, \\
\\
\dot{P}(t) &= P(t)A_u(t)' + A_u(t)P(t) - P(t)(C'C - \gamma^{-2}Q)P(t) + I, \\
P(0) &= P, \\
\\
\dot{\phi}(t) &= \tfrac{1}{2}(< \widehat{x}(t), Q\widehat{x}(t) > + < u(t), Ru(t) > -\gamma^2 \|y(t) - C\widehat{x}(t)\|^2), \\
\phi(0) &= \phi.
\end{aligned}$$

Again the output feedback robust control problem is equivalent to a new finite dimensional state feedback game with the state $\rho = (\widehat{x}, P, \phi)$, and the value W is a function of this state. Precisely

$$W(\rho, t) = W(p_\rho, t).$$

The dynamic programming equation for $W(\rho, t)$ is

$$\begin{cases}
\frac{\partial W}{\partial t}(\rho, t) + \sup_{y \in \mathbf{R}^p} \inf_{u \in U}\{< \nabla_{\widehat{x}} W, (A_u(t) + \gamma^{-2}P(t)Q)\widehat{x}(t) + Bu(t) + P(t)C'(y(t) - C\widehat{x}(t)) > \\
\quad + << \nabla_P W, P(t)A_u(t)' + A_u(t)P(t) - P(t)(C'C - \gamma^{-2}Q)P(t) + I >> \\
\quad + \tfrac{1}{2}(< \widehat{x}(t), Q\widehat{x}(t) > + < u(t), Ru(t) > -\gamma^2 \|y(t) - C\widehat{x}(t)\|^2)\} = 0 \\
W(\rho, t_f) = (p_{\rho_{t_f}}, 0)
\end{cases}$$

2.5 Approximation of the Information State

In this section we describe an approximation to the solution of the Robust H_∞ Output Feedback Problem which is accomplished by approximating the information state by a concave quadratic function with its maximum at a *nonconstant* point $\widetilde{x}_t$,

$$p(x, t) \approx P_t - \frac{\gamma^2}{2}(x - \widetilde{x}_t)^T Q_t^{-1}(x - \widetilde{x}_t), \tag{2.5.1}$$

where $Q = Q^T > 0$. The ordinary differential equations (ODEs) by which the unknown parameters $\widetilde{x}_t$, Q_t, and P_t evolve are derived below.

Approximations such as this are important since the *theoretical* solution of the Robust H_∞ Output Feedback Problem, given in Section 2.3, is not implementable for general nonlinear systems. The barrier to the implementation is the solution of the Dissipation Inequality (2.3.4). Theoretically a solution to this inequality requires the determination of a function $W(p)$ and a control $u^*(p)$ for each feasible information state $p_t(x)$, where $p_t(x)$ evolves in a general class of functions. Such a computation is not practically possible except under the special circumstances in which the information state can be identified with a finite dimensional quantity which evolves according to ODEs as in Section 2.4. As described in Section 2.4 these circumstances are quite restrictive. Thus it is of interest to approximate the information state by a function whose finite dimensional parameters, ρ, evolve according to ODEs.

Such approximations are well suited to implementation. The determination of the control which is now a function of the finite dimensional *approximated* information state, $u^*(\rho)$, is still a computationally intensive process but it can be performed *off line*. Only the solutions to the ODEs must be performed in *real time* and various numerical methods exist for this. In Section 5.1.2 examples are given which demonstrate that control systems which control the finite dimensional approximated information state can be stabilizing and robust to noise.

Now we derive ODEs which approximate the evolution of the unknown parameters $\widetilde{x}_t$, Q_t, and P_t. We'll start with $\widetilde{x}_t$. We have assumed that $\widetilde{x}_t = \arg\max_x p(x,t)$, thus we know $\nabla_x p(\widetilde{x}_t, t) = 0$. Differentiating with respect to t yields

$$\dot{\widetilde{x}}_t = \frac{1}{\gamma^2} Q_t \nabla_x (\frac{\partial p}{\partial t}),$$

where, from our approximation, $\nabla_x^2 p \approx -\gamma^2 Q_t^{-1}$. Next substituting for $\frac{\partial p}{\partial t}$ from Equation (2.2.4) and evaluating at $\widetilde{x}_t$ yields the evolution equation for $\widetilde{x}_t$,

$$\dot{\widetilde{x}}_t = f(\widetilde{x}_t, u) + \frac{1}{2\gamma^2} Q_t \nabla_x \|\ell(\widetilde{x}_t, u)\|^2 - Q_t \nabla_x h(h(\widetilde{x}_t) - y(t)). \tag{2.5.2}$$

In order to find the equation for the evolution of Q_t we make use of the identity $Q_t \dot{Q_t^{-1}} = -\dot{Q}_t Q_t^{-1}$ along with the approximation $\dot{Q_t^{-1}} \approx -\frac{1}{\gamma^2} \nabla_x^2 (\frac{\partial p}{\partial t})$ yielding

$$\dot{Q}_t \approx \frac{1}{\gamma^2} Q \nabla_x^2 (\frac{\partial p}{\partial t}) Q.$$

Thus we differentiate Equation (2.2.4) twice with respect to x and evaluate at $\widetilde{x}_t$. We also drop all terms which involve third order and higher derivatives of the information state p, and all terms which involve second order and higher derivatives of the measurement function h. By this procedure we arrive at an approximate evolution equation for Q_t,

$$\dot{Q}_t \approx 2\nabla_x f(\widetilde{x}_t, u) Q_t + I + Q_t (\frac{1}{2\gamma^2} \nabla_x^2 \|\ell(\widetilde{x}_t, u)\|^2 - \|\nabla_x h(\widetilde{x}_t)\|^2) Q_t. \tag{2.5.3}$$

The equation for P_t can be found by substituting our approximation of the information state, Equation (2.5.1), into Equation (2.2.4) and equating terms without $\tilde{x}_t$ in them. Then evaluation at $\tilde{x}_t$ yields

$$\dot{P}_t \approx \|\ell(\tilde{x}_t, u)\|^2 - \frac{\gamma^2}{2}\|h(\tilde{x}_t) - y(t)\|^2.$$

Theorem 2.5.1 below summarizes the approximation just derived.

Theorem 2.5.1 For a general nonlinear system Σ, the information state has the quadratic approximation

$$p(x,t) \approx P_t - \frac{\gamma^2}{2} < x - \tilde{x}_t, Q_t^{-1}(x - \tilde{x}_t) >,$$

where $Q = Q' > 0$ and $\tilde{x}_t$, Q_t and P_t satisfy the ODE's

$$\begin{aligned}
\dot{\tilde{x}}(t) &= f(\tilde{x}_t, u) + \tfrac{1}{2\gamma^2} Q_t \nabla_x \|\ell(\tilde{x}_t, u)\|^2 - Q_t \nabla_x h(h(\tilde{x}_t) - y(t)),\\
\tilde{x}(0) &= \tilde{x},\\
\\
\dot{Q}_t &= 2\nabla_x f(\tilde{x}_t, u) Q_t + I + Q_t(\tfrac{1}{2\gamma^2}\nabla_x^2 \|\ell(\tilde{x}_t, u)\|^2 - \|\nabla_x h(\tilde{x}_t)\|^2) Q_t,\\
Q_0 &= Q,\\
\\
\dot{P}_t &= \|\ell(\tilde{x}_t, u)\|^2 - \tfrac{\gamma^2}{2}\|h(\tilde{x}_t) - y(t)\|^2,\\
P_0 &= P.
\end{aligned}$$

Theorem 2.5.1 implies that for any system Σ the information state can be approximated by the finite dimensional quantity $\rho \triangleq (\tilde{x}, Q, P)$. We denote the finite dimensional information state by p_ρ, i.e., $p_\rho = P - \frac{\gamma^2}{2} < x - \tilde{x}, Q^{-1}(x - \tilde{x}) >$. Using this expression of the information state, the representation (2.2.5) becomes

$$J_{p_\rho, t_f}(u) = \quad \sup_{y \in L_2[0,t_f]} \{ \tfrac{1}{2} \int_0^{t_f} (\|\ell(\tilde{x}_t, u)\|^2 - \tfrac{\gamma^2}{2}\|h(\tilde{x}_t) - y(t)\|^2) ds + P(0) : \rho(0) = \rho \}.$$

Thus the dynamic programming equation is

$$\begin{cases}
\frac{\partial W}{\partial t}(\rho, t) &+ \quad \sup_{y \in \mathbf{R}^p} \inf_{u \in U} \{ < \nabla_{\tilde{x}} W, f(\tilde{x}_t, u) + \tfrac{1}{2\gamma^2} Q_t \nabla_x \|\ell(\tilde{x}_t, u)\|^2 - Q_t \nabla_x h(h(\tilde{x}_t) - y(t)) > \\
&+ \quad << \nabla_P W, 2\nabla_x f(\tilde{x}_t, u) Q_t + I + Q_t(\tfrac{1}{2\gamma^2}\nabla_x^2 \|\ell(\tilde{x}_t, u)\|^2 - \|\nabla_x h(\tilde{x}_t)\|^2) Q_t >> \\
&+ \quad \|\ell(\tilde{x}_t, u)\|^2 - \tfrac{\gamma^2}{2}\|h(\tilde{x}_t) - y(t)\|^2 \} = 0 \\
W(\rho, t_f) &= \quad (p_{\rho_{t_f}}, 0)
\end{cases}$$

where y plays the role of a competing disturbance.

For examples of information state control systems which employ this approximation see Section 5.1.2.

Chapter 3

Certainty Equivalence

In this Chapter we examine a suboptimal controller, the Certainty Equivalence Controller (CEC) [BB91, JBE94], which has received much attention in the literature [BHW91, IA92, vdS93, DBB93]. This particular controller is appealing because it is a generalization of the linear case in the sense that it involves the solution of two uncoupled dynamic programming equations, one for control and one for estimation. This controller, however, is not optimal in general [Jam93b]. For the case of linear systems it has been shown that the Certainty Equivalence Controller (CEC) is optimal [BB91]. For nonlinear systems, some general conditions have been found under which a Certainty Equivalence Principle (CEP) [Whi81, BB91, JBE93b] holds, i.e., the CEC is optimal. Here we give an alternative proof of the nonlinear result by application of a verification theorem.

If the assumptions of the Certainty Equivalence Principle (CEP) are valid, the CEC is identical to the optimal information state control policy. This is beneficial from the perspective of implementations since the CEC is computed using a computationally simpler algorithm. It is hoped that the CEQ controller, although in many cases suboptimal, will be stabilizing and robust to noise for many nonlinear systems. It is interesting to note that all other research in the area of nonlinear robust H_∞ output feedback control appears to be limited exclusively to certainty equivalence control [BHW91, IA92, vdS93, DBB93].

3.1 Certainty Equivalence Controller

The CEC is defined in terms of a minimum stress estimate $\bar{x}_t$ and the optimal state feedback controller $\tilde{u}(x,t)$:

$$u_{CE}(t) = \tilde{u}(\bar{x}_t, t). \tag{3.1.1}$$

The value function of the full state feedback game is the solution to the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial V}{\partial t} &= -\inf_u \sup_w \{< \nabla_x V, f(x,u) + w > + \frac{1}{2}\|\ell(x,u)\|^2 - \frac{\gamma^2}{2}\|w\|^2\} \\ V_{t_f} &= 0, \end{cases} \tag{3.1.2}$$

and $\tilde{u}(x,t)$ is the value of u achieving the minimum in (3.1.2), assuming V is sufficiently smooth. The *minimum stress estimate* [Whi81] of the state is given by

$$\bar{x} \triangleq \arg\max_x \{p(x) + V(x)\} \tag{3.1.3}$$

where p is the information state defined in Equation (2.2.4) Note that in general, $\bar{x}$ is set valued. If the CEP holds then the optimal value function is given by

$$W_{CE}(\rho, t) = (p_\rho, V_t). \tag{3.1.4}$$

3.2 Optimality of the CEC

For systems with finite dimensional information state, it is shown in this section via the verification principle that the CEC is optimal provided the minimum stress estimate is unique and the information state and value function are sufficiently smooth. This has since been show by James [JB94a] for more general systems and constitutes an alternative proof to the ones given in [BB91, DBB93].

Lemma 3.2.1 Suppose $A, B \in \mathbb{R}^{n\times n}$ and A is symmetric, then

(i) $\nabla_A < x, A^{-1}x >= -A^{-1}xx'A^{-1}$, and

(ii) $<< \nabla_A < x, A^{-1}x >, B >>= - < A^{-1}x, BA^{-1}x >$.

Proof: Part (i) is a direct consequence of a well know result from linear systems theory [GL93]: given a linear system with transfer function $H_{A,b,c}(s) = c'(sI - A)^{-1}b$, where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, and s is a complex valued scalar, then

$$\nabla_A H = (sI - A)^{-1}cb'(sI - A)^{-1}.$$

Part (ii)[1] can be shown as follows:

$$\begin{aligned} << \nabla_A < x, A^{-1}x >, B >> &= << -A^{-1}xx'A^{-1}, B >> \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}[-A^{-1}xx'A^{-1}]_{ij}b_{ij} \\ &= \text{trace}(-A^{-1}xx'A^{-1}B) \\ &= -A^{-1}xBx'A^{-1}. \end{aligned}$$

■

Theorem 3.2.2 If the system has a finite dimensional information state, i.e., the system is given by Equation (2.4.1) and assumptions (A1)-(A3) are satisfied, and

(i) the minimum stress estimate $\bar{x}(\rho, t)$ is unique for all $(\rho, t) \in \mathcal{D}$, and

(ii) the full state information value function V satisfying (3.1.2) is continuously differentiable,

then the function

$$W_{CE}(\rho, t) = (p_\rho, V_t)$$

is a solution to the Dynamic Programming Equation (2.4.4) and the Certainty Equivalence Controller (3.1.1) is optimal.

Proof:

Recall that the finite dimensional information state is given by $p_\rho(x) = \phi - \frac{\gamma^2}{2} < x - \widehat{x}, P^{-1}(x - \widehat{x}) >$ $+F(x)$ where $\rho = (\widehat{x}, P, \phi)$ satisfy the ODE's given in Theorem 2.4.1. Thus $\bar{x} \triangleq \bar{x}(\rho, t) = \arg\max_x\{p_\rho(x) + V_t(x)\}$ and $\bar{x}_t = \bar{x}(\rho_t, t)$.

Consider $(\rho, t) \in \mathcal{D}$, then by assumption (i) the minimum stress estimate $\bar{x}$ is unique. This together with assumption (ii), that the full state information value function V is continuously differentiable, allow us to equate $W_{CE} = p_\rho(\bar{x}) + V_t(\bar{x})$ and to differentiate naturally with $\bar{x}$ as a parameter.

By first order optimality condition and assumption (A1)

$$\nabla_x V(\bar{x}) = -\nabla_x p(\bar{x}) = \gamma^2 P^{-1}(\bar{x} - \widehat{x}) - \gamma^2 f(x).$$

The CEC is obtained by determining the minimizing control $\widetilde{u}$ in Equation (3.1.2) with $f(x, u) = f(x) + A(u)x + B(u)$ and $\ell(x, u) = \frac{1}{2}(< x, Q(u)x > + R(u)$ and evaluating at the minimum stress estimate $\bar{x}$. Thus

[1]Thanks to S. Yuliar for suggesting this more elegant proof!

the CEC satisfies:

$$\begin{aligned}
0 &= <\nabla_x V, \frac{\partial A(\widetilde{u})}{\partial u}\bar{x} + \frac{\partial B(\widetilde{u})}{\partial u} > + \tfrac{1}{2}(<\bar{x}, \frac{\partial Q(\widetilde{u})}{\partial u}\bar{x}> + \frac{\partial R(\widetilde{u})}{\partial u}) \\
&= <\gamma^2 P^{-1}(\bar{x}-\widehat{x}), \frac{\partial A(\widetilde{u})}{\partial u}\bar{x} + \frac{\partial B(\widetilde{u})}{\partial u} > \\
&\quad + \tfrac{1}{2}(<\bar{x}, \frac{\partial Q(\widetilde{u})}{\partial u}\bar{x}> + \frac{\partial R(\widetilde{u})}{\partial u}) - \tfrac{\gamma^2}{2}(<\bar{x}, \frac{\partial \Sigma(\widetilde{u})}{\partial u}\bar{x}> + 2\frac{\partial \Lambda(\widetilde{u})}{\partial u}\bar{x} + \frac{\partial \Gamma(\widetilde{u})}{\partial u}).
\end{aligned}$$

The optimal control u^* in the case that $W = W_{CE}$ is obtained by determining the minimizing control in Equation (2.4.4) when $W = W_{CE}$. Similarly the optimal output y^* is the maximizing output.

$$y^* = C\widehat{x} + \tfrac{1}{\gamma^2}CP\nabla_{\widehat{x}}W_{CE} = C\bar{x}$$

$$\begin{aligned}
0 &= <\nabla_{\widehat{x}}W, (\frac{\partial A(u^*)}{\partial u} + \tfrac{1}{\gamma^2}P\frac{\partial Q(u^*)}{\partial u} - P\frac{\partial \Sigma(u^*)}{\partial u})\widehat{x}(t) + \frac{\partial B(u^*)}{\partial u} - P\frac{\partial \Lambda(u^*)}{\partial u} > \\
&\quad + << \nabla_P W, P\frac{\partial A(u^*)'}{\partial u} + \frac{\partial A(u^*)}{\partial u}P + \tfrac{1}{\gamma^2}P\frac{\partial Q(u^*)}{\partial u}P + \frac{\partial \Sigma(u^*)}{\partial u}P >> \\
&\quad + \tfrac{1}{2}(<\widehat{x}, \frac{\partial Q(u^*)}{\partial u}\widehat{x}> + \frac{\partial R(u^*)}{\partial u}) - \tfrac{\gamma^2}{2}(<\widehat{x}, \frac{\partial \Sigma(u^*)}{\partial u}\widehat{x}> + 2\frac{\partial \Lambda(u^*)}{\partial u}\widehat{x} + \frac{\partial \Gamma(u^*)}{\partial u}) \\
&= <\gamma^2 P^{-1}(\bar{x}-\widehat{x}), (\frac{\partial A(u^*)}{\partial u} + \tfrac{1}{\gamma^2}P\frac{\partial Q(u^*)}{\partial u} - P\frac{\partial \Sigma(u^*)}{\partial u})\widehat{x}(t) + \frac{\partial B(u^*)}{\partial u} - P\frac{\partial \Lambda(u^*)}{\partial u} > \\
&\quad + \tfrac{\gamma^2}{2} < P^{-1}(\bar{x}-\widehat{x}), (P\frac{\partial A(u^*)'}{\partial u} + \frac{\partial A(u^*)}{\partial u}P + \tfrac{1}{\gamma^2}P\frac{\partial Q(u^*)}{\partial u}P + \frac{\partial \Sigma(u^*)}{\partial u}P)P^{-1}(\bar{x}-\widehat{x}) > \\
&\quad + \tfrac{1}{2}(<\widehat{x}, \frac{\partial Q(u^*)}{\partial u}\widehat{x}> + \frac{\partial R(u^*)}{\partial u}) - \tfrac{\gamma^2}{2}(<\widehat{x}, \frac{\partial \Sigma(u^*)}{\partial u}\widehat{x}> + 2\frac{\partial \Lambda(u^*)}{\partial u}\widehat{x} + \frac{\partial \Gamma(u^*)}{\partial u}) \\
&= <\gamma^2 P^{-1}(\bar{x}-\widehat{x}), \frac{\partial A(u^*)}{\partial u}\bar{x} + \frac{\partial B(u^*)}{\partial u} > \\
&\quad + \tfrac{1}{2}(<\bar{x}, \frac{\partial Q(u^*)}{\partial u}\bar{x}> + \frac{\partial R(u^*)}{\partial u}) - \tfrac{\gamma^2}{2}(<\bar{x}, \frac{\partial \Sigma(u^*)}{\partial u}\bar{x}> + 2\frac{\partial \Lambda(u^*)}{\partial u}\bar{x} + \frac{\partial \Gamma(u^*)}{\partial u})
\end{aligned}$$

Thus the certainty equivalence control $u_{CE} = \widetilde{u}(\bar{x})$ and the optimal control $u^*(\rho, t)$ satisfy the same equation when $W = W_{CE}$. Thus when $W = W_{CE}$, the optimal controller and the CEC are equivalent, i.e.,

$$u^*(\rho, t) = u_{CE}.$$

By direct differentiation and substitution it can be shown that $W_{CE} = V_t(\bar{x}) + p_\rho(\bar{x})$ satisfies Equation (2.4.4). Now

$$\begin{aligned}
\text{LHS} &\triangleq \frac{\partial W_{CE}}{\partial t} \\
&= \frac{\partial V}{\partial t}(\bar{x}) \\
&= -\gamma^2 < P^{-1}(\bar{x}-\widehat{x}), (A + B\widetilde{u}(\bar{x}))\bar{x} + P^{-1}(\bar{x}-\widehat{x}) > -\tfrac{1}{2}(<\bar{x}, Q\bar{x}> + <\widetilde{u}(\bar{x}), R\widetilde{u}(\bar{x})>) \\
&\quad + \tfrac{\gamma^2}{2} < (\bar{x}-\widehat{x}), P^{-2}(\bar{x}-\widehat{x}) >
\end{aligned}$$

and also

$$\begin{aligned}
\text{RHS} \;\triangleq\;& -< \nabla_{\widehat{x}} W_{CE}, (A(u) + \tfrac{1}{\gamma^2}PQ(u) - P\Sigma(u))\widehat{x} + B(u) - P\Lambda(u) + PC'(y - C\widehat{x}) > \\
& - << \nabla_P W_{CE}, PA(u)' + A(u)P - P(C'C - \tfrac{1}{\gamma^2}Q(u))P + \Sigma(u)P + I >> \\
& + \tfrac{1}{2}(< \widehat{x}, Q(u)\widehat{x} > + R(u) - \gamma^2 \|y - C\widehat{x}\|^2) - \tfrac{\gamma^2}{2}(< \widehat{x}, \Sigma(u)\widehat{x} > + 2\Lambda(u)\widehat{x} + \Gamma(u)) \\
\\
=\;& -\gamma^2 < P^{-1}(\bar{x} - \widehat{x}), (A(u) + \tfrac{1}{\gamma^2}PQ(u) - P\Sigma(u))\widehat{x} + B(u) - P\Lambda(u) + PC'(y - C\widehat{x}) > \\
& - \tfrac{\gamma^2}{2} < P^{-1}(\bar{x} - \widehat{x}), (PA(u)' + A(u)P - P(C'C - \tfrac{1}{\gamma^2}Q(u))P + \Sigma(u)P + I)P^{-1}(\bar{x} - \widehat{x}) > \\
& + \tfrac{1}{2}(< \widehat{x}, Q(u)\widehat{x} > + R(u) - \gamma^2 \|y - C\widehat{x}\|^2) - \tfrac{\gamma^2}{2}(< \widehat{x}, \Sigma(u)\widehat{x} > + 2\Lambda(u)\widehat{x} + \Gamma(u)) \\
\\
=\;& -\gamma^2 < P^{-1}(\bar{x} - \widehat{x}), (A + Bu^*)\bar{x} + P^{-1}(\bar{x} - \widehat{x}) > - \tfrac{1}{2}(< \bar{x}, Q\bar{x} > + < u^*, Ru^* >) \\
& + \tfrac{\gamma^2}{2} < (\bar{x} - \widehat{x}), P^{-2}(\bar{x} - \widehat{x}) >
\end{aligned}$$

Thus since $u^* = \widetilde{u}(\bar{x})$ when $W = W_{CE}$ we are done. ■

As previously mentioned the assumption that the information state is finite dimensional can be relaxed. The remaining assumptions given in Theorem 3.2.2 are essentially those of [BB91], and are difficult to verify in general.

3.3 Filter Equation for Minimum Stress Estimate

The minimum stress estimate as defined in Equation (3.1.3) is not given in a form which is familiar in controls. It would be more typical to have an estimate defined in terms of a nonlinear filtering equation, e.g., the Extended Kalman Filter [Kha92]. It is interesting that under a few assumptions the minimum stress estimate can be expressed in the form of a filtering equation. In fact this filter turns out to be the nonlinear central controller of van der Schaft [vdS93]. In this section we derive a filtering equation which describes the evolution of the minimum stress estimate.

The minimum stress estimate is not a typical state estimate and this is reflected in the filtering Equation (3.3.6) derived below. The minimum stress estimate is an estimate of the state under the assumption that the disturbance is the worst possible from the point of view of control. The filtering Equation (3.3.6) for the minimum stress estimate thus has an extra term corresponding to the worst case disturbance. By allowing $\gamma \to \infty$, however, the extra term drops out and the estimate takes the form of a more typical state estimate. This is related to the large deviations limit results discussed in Section 1.3.3. Recall that by allowing

$\gamma \to \infty$ the solution of dynamic game problem approaches that of the deterministic optimal control problem [Jam92, CJ92, JBE94, JBE93b].

Assuming that there exists a unique minimum stress estimate $\bar{x}_t$ for each time t and for all observation histories $y_{[0,t_f]}$ and that the information state (2.2.4) and the value function (3.1.2) are twice continuously differentiable, we can derive a filter type equation for the evolution of the minimum stress estimate. The uniqueness assumption implies that the CEP holds. Consider the input affine nonlinear system Σ_{IA} below:

$$\Sigma_{IA} \quad \begin{cases} \dot{x}(t) &= f(x(t)) + g(x(t))u + w(t), \qquad x(t_0) = x_0, \\ y(t) &= h(x(t)) + v(t), \\ z(t) &= \begin{pmatrix} \sqrt{\phi(x(t))} \\ u(t) \end{pmatrix}. \end{cases} \tag{3.3.1}$$

Note that Σ_{IA} is a special case of Σ as described in Section 1.2 with $f(x(t), u(t)) = f(x(t)) + g(x(t))u$ and $\ell(x(t), u(t)) = (\sqrt{\phi(x(t))}', u(t)')'$.

For the system Σ_{IA} the information state satisfies the dynamic programming equation

$$\begin{cases} \frac{\partial p_t}{\partial t} &= F(p_t, u(t), y(t)) \\ p_0 &= \bar{p}, \end{cases} \tag{3.3.2}$$

where F is given by

$$\begin{aligned} F(p, u, y) &\triangleq \sup_{w \in \mathbf{R}^n}\{- < \nabla_x p, f(x) + g(x)u + w > -\tfrac{\gamma^2}{2}(||w||^2 + ||h(x) - y||^2) + \tfrac{1}{2}(\phi(x) + \|u\|^2)\} \\ &= - < \nabla_x p, f(x) + g(x)u > + \tfrac{1}{2\gamma^2}||\nabla_x p||^2 + \tfrac{1}{2}(\phi(x) + \|u\|^2) - \tfrac{\gamma^2}{2}||h(x) - y||^2, \end{aligned}$$

and the past control inputs u and observations y are known. The maximizing disturbance in this case is $\widehat{w} = -\frac{1}{\gamma^2}\nabla_x p$.

The value function, V, of the full state feedback game is the solution to the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial V}{\partial t} &= G(V_t, u) \\ V_{t_f} &= 0, \end{cases} \tag{3.3.3}$$

where G is given by

$$\begin{aligned} G(v, u) &\triangleq -\inf_u \sup_w\{< \nabla_x V, f(x) + g(x)u + w > + \tfrac{1}{2}(\phi(x) + \|u\|^2) - \tfrac{\gamma^2}{2}\|w\|^2)\} \\ &= - < \nabla_x V, f(x) > + \tfrac{1}{2}\|< \nabla_x V, g(x) >\|^2 - \tfrac{1}{2\gamma^2}\|\nabla_x V\|^2 - \tfrac{1}{2}\phi(x). \end{aligned}$$

The minimizing control u^* and the maximizing disturbance w^* are respectively $u^*(x,t) = - < g(x), \nabla_x V(x,t) >$ and $w^*(x,t) = \frac{1}{\gamma^2}\nabla_x V(x,t)$

Thus assuming the CEP holds[2], the optimal output feedback controller (CEQ controller) is given by

$$u_{CEQ} = - < g(\bar{x}_t), \nabla_x V(\bar{x}_t, t) >,$$

where $\bar{x}_t$ is the minimum stress estimate (3.1.3).

Now we will derive a filter type equation for the minimum stress estimate, $\bar{x}$. First, we define

$$S(x,t) \triangleq p(x,t) + V(x,t).$$

Since the minimum stress estimate maximizes S, viz. Equation (3.1.3), we know by the first order optimality condition that

$$\nabla_x S(\bar{x},t) = \nabla_x p(\bar{x},t) + \nabla_x V(\bar{x},t) = 0.$$

Notice that this implies that the maximizing disturbance for both the information state and the value function are the same, i.e., $\widehat{w}(\bar{x}) = w^*(\bar{x})$. This result will be needed shortly. Differentiating with respect to t yields

$$\nabla_x^2 S(\bar{x},t)\dot{\bar{x}} + \nabla_x \frac{\partial S(\bar{x},t)}{\partial t} = 0. \tag{3.3.4}$$

Thus we see that in order to obtain a useful differential equation for $\bar{x}$ we must find an equation for $\nabla_x \frac{\partial S}{\partial t}$. To do this we add together Equations (3.3.2) and (3.3.3) where $\widehat{w}$, w^*, and u^* denote the optimal values of these parameters as defined above.

$$(\frac{\partial p}{\partial t} + \frac{\partial V}{\partial t}) + \nabla_x^T p(f(x) + g(x)u + \widehat{w}) + \nabla_x^T V(f(x) + g(x)u^* + w^*) + \frac{\gamma^2}{2}|h(x) - y|^2 \frac{\gamma^2}{2}(|\widehat{w}|^2 - |w^*|^2) = 0$$

By differentiating this equation with respect to x and evaluating at $\bar{x}$ we obtain the needed equation.

$$\nabla_x \frac{\partial S(\bar{x},t)}{\partial t} + \nabla_x^2 S(f(\bar{x}) + g(\bar{x})u^* + w^*) + \gamma^2 < \nabla_x h(\bar{x}), h(\bar{x}) - y >= 0 \tag{3.3.5}$$

Assuming $\nabla_x^2 S$ is invertable, Equations (3.3.4) and (3.3.5) combine to give

$$\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})u^* + \frac{1}{\gamma^2}\nabla V_x(\bar{x},t) + K(\bar{x},t)(h(\bar{x}) - y) \tag{3.3.6}$$

where $u^*(x,t) = - < g(\bar{x}), \nabla_x V(\bar{x},t) >$ and $K(\bar{x},t) = \gamma^2[\nabla_x^2 S(\bar{x},t)]^{-1}\nabla_x^T h(\bar{x})$.

As mentioned above, the minimum stress estimate is an estimate of the state under the assumption that the disturbance is the worst possible from the point of view of control. This is reflected in the filtering Equation (3.3.6) for the minimum stress estimate by the term $\frac{1}{\gamma^2}\nabla V_x(\bar{x},t)$, corresponding to the worst case disturbance input. By allowing $\gamma \to \infty$ the disturbance term drops out and the estimate takes the form of a more typical state estimate.

[2]Although the conditions under which a Certainty Equivalence Principle holds may be quite restrictive we contend that in many practical cases the CEC is a good approximation to the optimal control.

3.4 Approximations

The computation of the CEC, although employing a computationally simpler algorithm than that of the information state controller, is still very computationally expensive. The Dynamic Programming Equation (3.1.2) which gives the full state feedback control can be computed off line. However, the computation of the CEC still involves the online solution of the Dynamic Programming Equation (2.2.4) for the information state. Thus the hope of direct implementation of the CEC for use in a *real-time* control system is not well founded. However, knowledge of the optimal solution can be exploited to guide the choice of a suboptimal controller which would be better suited for implementation.

Again, as in Section 2.5, we can approximate the information state by a quadratic which evolves according to ODE's. Examples are given in Section 5.2.1 which demonstrate that the approximate CEC, i.e., the CEC with an approximated information state, can be stabilizing and robust to noise.

A further approximation for input affine systems Σ_{IA}, motivated by the CEC filtering Equation (3.3.6), is given below. This controller, though well suited for real-time implementation, is essentially a modified extended Kalman Filter. As with the other approximations to the optimal control some performance is most likely lost by neglecting the higher order nonlinearities. This case in particular is a very rough approximation and is likely to give local results only for mildly nonlinear systems. Examples which demonstrate the performance of this controller are given in Section 5.2.1.

Again we approximate the information state by a concave quadratic function with its maximum at a *nonconstant* point $\tilde{x}_t$, as in Section 2.5,

$$p(x,t) \approx P_t - \frac{\gamma^2}{2}(x - \tilde{x}_t)^T Q_t^{-1}(x - \tilde{x}_t).$$

The value function is approximated by a convex quadratic with its minimum at the origin,

$$V \approx \frac{1}{2}x^T \Pi x, \tag{3.4.1}$$

where $\Pi = \Pi^T > 0$. Substituting these estimates into Equation (3.1.3) for the minimum stress estimate gives

$$\bar{x}_t \approx \arg\max_x \{x^T \Pi x - \gamma^2 (x - \tilde{x}_t)^T Q_t^{-1}(x - \tilde{x}_t)\}.$$

Solving for $\bar{x}_t$ gives the approximation

$$\bar{x}_t \approx (I - \frac{1}{\gamma^2} Q_t \Pi)^{-1} \tilde{x}_t.$$

Note that $u_{CEQ} \approx -g^T(\bar{x}_t)\Pi\bar{x}_t$.

The next step is to approximate the evolution of the unknown parameters $\tilde{x}_t$, Q_t, and the value of Π. Equations for $\tilde{x}_t$ and Q_t have already been derived in Section 2.5 and are given by Equations (2.5.2) and (2.5.3) respectively. Here we give the equations for $\tilde{x}_t$ and Q_t for the specific case of input affine systems Σ_{IA},

$$\begin{aligned}\dot{\tilde{x}}_t &= f(\tilde{x}_t) + g(\tilde{x}_t)u + \tfrac{1}{2\gamma^2}Q_t\nabla_x\phi(\tilde{x}_t) - Q_t\nabla_x h(h(\tilde{x}_t) - y(t)),\\ \dot{Q}_t &= 2(\nabla_x f(\tilde{x}_t) + \nabla_x g(\tilde{x}_t)u)Q_t + I + Q_t(\tfrac{1}{2\gamma^2}\nabla_x^2\phi(\tilde{x}_t) - \|\nabla_x h(\tilde{x}_t)\|^2)Q_t.\end{aligned}$$

The determination of the value of Π is accomplished by iterating an approximation of Equation (3.3.3) describing the evolution equation for Π_t until a steady state value is reached. For convenience we rewrite Equation (3.3.3) to evolve forward in time

$$\frac{\partial V}{\partial t} =< \nabla_x V, f(x) > -\frac{1}{2}\|< \nabla_x V, g(x) >\|^2 + \frac{1}{2\gamma^2}\|\nabla_x V\|^2 + \frac{1}{2}\phi(x). \tag{3.4.2}$$

From here the procedure for finding the evolution equation for Π_t follows a similar procedure to that of Q_t. From our approximation of the full state value function, Equation (3.4.1), we know $V(0,t) = 0$, $\nabla_x V(0,t) = 0$, $\nabla_x^2 V(0,t) = \Pi_t$, and so finally $\dot{\Pi}_t = \nabla_x^2 \frac{\partial V(0,t)}{\partial t}$. Thus differentiating Equation (3.4.2) with respect to x twice and evaluating at zero yields the evolution equation for Π_t

$$\dot{\Pi}_t = 2\Pi_t \cdot \nabla_x f(0) - (\Pi_t \cdot g(0))^2 + \frac{1}{\gamma^2}|\Pi_t|^2 + \nabla_x^2\phi.$$

All terms which involve second order and higher derivatives have been dropped. This equation is now integrated forward in time until a steady state value Π is obtained.

Theorem 3.4.1 below summarizes these results.

Theorem 3.4.1 A quadratic approximation of the Certainty Equivalence Controller for input affine systems Σ_{IA} is given by

$$u_{CEQ} \approx -g^T(\bar{x}_t)\Pi\bar{x}_t.$$

The minimum stress estimate $\bar{x}$ is approximated by

$$\bar{x}_t \approx (I - \frac{1}{\gamma^2}Q_t\Pi)^{-1}\tilde{x}_t$$

and the variables $\tilde{x}$ and Q evolve according to the ordinary differential equations as follows

$$\begin{aligned}\dot{\tilde{x}}_t &= f(\tilde{x}_t) + g(\tilde{x}_t)u + \tfrac{1}{2\gamma^2}Q_t\nabla_x\phi(\tilde{x}_t) - Q_t\nabla_x h(h(\tilde{x}_t) - y(t)),\\ \dot{Q}_t &= 2(\nabla_x f(\tilde{x}_t) + \nabla_x g(\tilde{x}_t)u)Q_t + I + Q_t(\tfrac{1}{2\gamma^2}\nabla_x^2\phi(\tilde{x}_t) - \|\nabla_x h(\tilde{x}_t)\|^2)Q_t.\end{aligned}$$

The value of Π is the steady state value of the following ODE

$$\Pi_t = 2\Pi_t \cdot \nabla_x f(0) - (\Pi_t \cdot g(0))^2 + \frac{1}{\gamma^2}|\Pi_t|^2 + \nabla_x^2\phi.$$

Chapter 4

Implementation

In this chapter we describe the numerical methods used to compute a solution to dissipation inequalities of the type which arise in the information state feedback solution, viz. Chapter 2, and the Certainty Equivalence solution, viz. Chapter 3, of the Robust H_∞ Output Feedback Problem. We employ a finite difference approximation technique developed by Kushner and Dupuis [KD92]. The finite difference approximation is termed a Markov chain approximation because of its interpretation as that of a controlled Markov chain in which movement of the state through the discretized state space is described by Markov transition probabilities. The approximating chain is parameterized by the finite difference interval Δ such that as $\Delta \to 0$ the *local* properties of the chain approach those of the original process. It is shown that the optimal cost functions for the sequence of approximating chains converges to that for the underlying original process as $\Delta \to 0$. The Markov chain approximation technique was introduced to the author by James and Yuliar [JY93].

The Markov chain approximation technique is applicable for computing the solution of any dissipation inequality with finite dimensional state/information state domain. In the case that the information state is not finite dimensional a preliminary approximation must be made before the Markov chain approximation technique can be applied. In this dissertation we consider two possible types of preliminary approximations of the information state controller: (i) the approximation of the information state by a finite dimensional quantity, viz Section 2.5, or (ii) the use of the Certainty Equivalence Control[1], viz. Chapter 3.

[1]In the case that the CEP holds this is not an approximation, it is equivalent to implementing the information state feedback controller

4.1 The Dissipation Inequality

In Chapter 2 we give necessary and sufficient conditions for solving both the finite time horizon and the infinite time horizon Robust H_∞ Output Feedback Problem. In both cases the value function and optimal control may be computed off line. Off line computation has obvious advantages in terms of control system run time. However, in the finite time case, since the value function and the optimal control are functions of time, it is required that their values be stored for each instant of run time. As a result, the memory requirements to implement the finite time solution may be excessive especially for large systems. For this reason we have chosen to implement only the infinite horizon solution.

The goal of the infinite time horizon Robust H_∞ Output Feedback Problem is to minimize over the admissible controls $u \in \mathcal{O}$ the cost functional

$$\sup_{t\geq 0} \sup_{x_0\in\mathbf{R},\ w,v\in L_2[0,t]} \mathcal{J}_{\bar{p},t}(x_0,u,v,w)$$

where

$$\mathcal{J}_{\bar{p},t}(x_0,u,v,w) = \bar{p}(x_0) + \frac{1}{2}\int_0^t (\|\ell(x(s),\bar{u}(s))\|^2 - \gamma^2(\|w(s)\|^2 + \|v(s)\|^2))ds. \tag{4.1.1}$$

In Section 2.3 it is shown under detectability and reachability assumptions, that if a solution can be found to the dissipation inequality

$$\inf_{u\in U}\sup_{y\in\mathbf{R}^p}\{<\nabla_p W, F(p,u,y)>\} \leq 0, \tag{4.1.2}$$

where $\dot{p} = F(p,u,y)$ describes the dynamics of the information state, such that $\delta_x \in \text{dom } W$ for all $x \in \mathbb{R}^n$, $W(p) \geq (p,0)$, $W(\delta_0) = 0$ then the control $\bar{u}^*(p)$ which achieves the minimum in (4.1.2) solves the information state feedback robust control problem. So in order to find the control $u^*(p)$ we are faced with the problem of numerically solving the Dissipation Inequality (4.1.2).

4.2 Value Space Iterations

Using the method of value space iterations the solution to Dissipation Inequality (4.1.2) is approximated by considering an associated finite horizon dynamic programming equation for t large enough. We consider the finite horizon solution $W(p,t)$, which is a solution in the viscosity sense of the partial differential equation (PDE)

$$\begin{cases} -\frac{\partial W}{\partial t} + \inf_{u\in U}\sup_{y\in\mathbf{R}^p}\{<\nabla_p W, F(p,u,y)>\} = 0, \\ W(p,0) = 0. \end{cases} \tag{4.2.1}$$

Note that this is exactly the PDE which arises in solving the Finite Time Robust H_∞ Output Feedback Problem with the exception that *for convenience* we allow it to evolve forward in time.

Define $W_a(p) \triangleq \sup_{t\geq 0} W(p,t)$. Then W_a solves the equality

$$\inf_{u\in U} \sup_{y\in \mathbf{R}^p} \{< \nabla_p W_a, F(p,u,y) >\} = 0. \tag{4.2.2}$$

Clearly a solution to this equality is also a solution to the Dissipation Inequality (4.1.2).

Theorem 4.2.1 Suppose there exists a finite solution $W_a(p)$ to the Dissipation Equality (4.2.2) then

$$\lim_{t\to\infty} W(p,t) = W_a(p)$$

where $W(p,t)$ is a solution of the Dynamic Programming Equation (4.2.1).

Proof: The key to the proof is the demonstration that $W(p,t)$ is non-decreasing in t. By definition $W_a(p) = \sup_{t\geq 0} W(p,t)$ we know $W(p,t) \leq W_a(p)$ for all $t \geq 0$. Thus since we have assumed the existence of a finite solution $W_a(p)$ to the Dissipation Equality (4.2.2), $\lim_{t\to\infty} W(p,t)$ exists and is finite. Then by definition of $W_a(p)$ and the non-decreasing property of $W(p,t)$ we will get that $\lim_{t\to\infty} W(p,t) = W_a(p)$. So all we have left to show is that $W(p,t)$ is non-decreasing in t.

$$W(p,t) = \inf_{u\in O} \sup_{x_0\in \mathbf{R},\ w,v\in L_2[0,t]} \mathcal{J}_{\bar{p},t}(x_0,u,v,w)$$

where $\mathcal{J}$ is defined in Equation (4.1.1). Define W_t, V_t to be the set of disturbances $w, v \in L_2[0,\infty]$ such that $w(\tau), v(\tau) = 0$ for all $\tau > t$. Let $t_2 \geq t_1$ then $W_{t_1}, V_{t_1} \subset L_2[0,t_2]$.

$$\begin{aligned}
\sup_{x_0\in\mathbf{R},\ w,v\in L_2[0,t_2]} \mathcal{J}_{\bar{p},t_2}(x_0,u,v,w) &\geq \sup_{x_0\in\mathbf{R},\ w\in W_{t_1},\ v\in V_{t_1}} \mathcal{J}_{\bar{p},t_2}(x_0,u,v,w)\\
&= \sup_{x_0\in\mathbf{R},\ w,v\in L_2[0,t_1]} \mathcal{J}_{\bar{p},t_1}(x_0,u,v,w) + \tfrac{1}{2}\int_{t_1}^{t_2} (\|\ell(x(s),\bar{u}(s))\|^2 ds\\
&\geq \sup_{x_0\in\mathbf{R},\ w,v\in L_2[0,t_1]} \mathcal{J}_{\bar{p},t_1}(x_0,u,v,w)
\end{aligned}$$

This inequality holds for all controls u as long as both sides use the same control over the interval $[0,t_1]$. By applying the principle of optimality we know that when we take the infimum on both sides over the admissible controls the optimal choice u^* is the same on both sides over the interval $[0,t_1]$. Thus $W(p,t_2) \geq W(p,t_1)$.
■

Theorem 4.2.1 implies that for large t the finite horizon solution eventually reaches a steady state solution which is a solution to the infinite horizon problem. Thus when computed for t large enough the finite horizon solution to the Dynamic Programming Equation (4.2.1) serves as a good approximation to the solution of the Dissipation Inequality (4.1.2).

4.3 Markov Chain Approximation Method

The computer implementation of a dynamic programming equation such as Equation (4.2.1) requires the discretization of both the time and the information state space. We use a finite difference scheme which is similar to that presented in [KD92, JY93].

To emphasize its finite dimensional nature, we denote of the information state by ρ where $\rho \triangleq (\widehat{x}, P, \phi)$ is the finite dimensional vector with which the information state $p_\rho = \phi - \frac{\gamma^2}{2} < x - \widehat{x}, P^{-1}(x - \widehat{x}) > +\gamma^2 F(x)$ can be identified under the assumptions given in Section 2.4. Referring to Remark 2.4.2 it is clear that by setting the initial value $\phi = 0$, we can rewrite the Dynamic Programming Equation (2.4.4) for W such that W is no longer a function of ϕ:

$$\begin{cases} -\frac{\partial W}{\partial t}(\widetilde{\rho}, t) & + \quad \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{ < \nabla_{\widetilde{\rho}} W, F(\widetilde{\rho}, u, y) > + C(\widetilde{\rho}, u, y) = 0, \\ W(\widetilde{\rho}, 0) & = \quad (p_{\rho_{t_f}}, 0). \end{cases} \tag{4.3.1}$$

Here we define $\widetilde{\rho} \triangleq (\widehat{x}, P)$, $\dot{\widetilde{\rho}} \triangleq F(\widetilde{\rho}, u, y)$, and $C(\widetilde{\rho}, u, y)$ is the cost integrand from Equation (2.4.3). Note we have allowed W to evolve forward in time.

Before we describe the Markov transition probability we must make a few definitions. We consider a uniformly discretized grid of the information state space of size Δ. Denote this grid by $(R^{\widetilde{n}})^\Delta$ where $\widetilde{n} = \frac{n(n+3)}{2}$ is the dimension of $\widetilde{\rho}$, and n is the dimension of the state space of the original system Σ_F. Define the neighborhood $N_\Delta(\widetilde{\rho})$ of a point $\rho \in (R^{\widetilde{n}})^\Delta$ by the $2\widetilde{n} + 1$ Δ-adjacent points on the discretized grid, i.e.,

$$N_\Delta(\widetilde{\rho}) = \left\{ q \in (R^{\widetilde{n}})^\Delta : q = \widetilde{\rho} \text{ or } q = \widetilde{\rho} \pm \Delta e_i \text{ for some } i \in \{1, \cdots, \widetilde{n}\} \right\}$$

where $e_i \in \mathbb{R}^{\widetilde{n}}$ denotes the ith unit vector $i = 1, \cdots, \widetilde{n}$. Define a normalization constant λ by

$$\lambda = \sup_{\widetilde{\rho} \in (R^{\widetilde{n}})^\Delta, y \in (R^p)^\Delta, u \in (R^m)^\Delta} \|F(\widetilde{\rho}, u, y)\|_1$$

where the 1-norm of a vector $v \in \mathbf{R}^n$ is the sum of the absolute value of the components, i.e., $\|v\|_1 = \sum_{i=1}^n |v_i|$. The transition probability from state $\widetilde{\rho} \in (R^{\widetilde{n}})^\Delta$ to $q \in (R^{\widetilde{n}})^\Delta$ is given by

$$\mathcal{P}^\Delta(q, \widetilde{\rho}; y, u) = \begin{cases} 1 - |F(\widetilde{\rho}, u, y)|_1 / \lambda, & \text{if } q = \widetilde{\rho}, \\ F_i^\pm(\widetilde{\rho}, u, y)/\lambda, & \text{if } q = \widetilde{\rho} \pm \Delta e_i \quad i = 1, \cdots, n, \\ 0, & \text{if } q \notin N_\Delta(\widetilde{\rho}), \end{cases} \tag{4.3.2}$$

where

$$F_i^+(\widetilde{\rho}, u, y) = \begin{cases} F_i(\widetilde{\rho}, u, y), & \text{if } F_i(\widetilde{\rho}, u, y) \geq 0, \\ 0, & \text{elsewhere,} \end{cases}$$

$$F_i^-(\widetilde{\rho}, u, y) = \begin{cases} -F_i(\widetilde{\rho}, u, y), & \text{if } F_i(\widetilde{\rho}, u, y) \leq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

This discretization is depicted for a 2-dimensional information state in Figure 1.

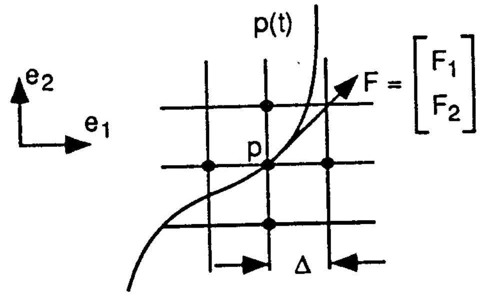

Figure 1: Markov Chain Transition Probabilities

The time discretization is defined by $t_k = k\Delta/\lambda, \quad k = 0, 1, \cdots$. The discretization of Dynamic Programming Equation (4.3.1) is then given by

$$\begin{cases} W^{\Delta}(\tilde{\rho}, k) &= \inf_{u\in(R^m)^{\Delta}} \sup_{y\in(R^p)^{\Delta}} \left\{ \sum_{q\in N_{\Delta}(\tilde{\rho})} W^{\Delta}(q, k-1)\mathcal{P}(q, \tilde{\rho}; u, y) + C(\tilde{\rho}, u, y)\frac{\Delta}{\lambda} \right\} \\ W^{\Delta}(\tilde{\rho}, 0) &= 0. \end{cases} \tag{4.3.3}$$

By iterating Equation (4.3.3) we get the representation

$$W^{\Delta}(\tilde{\rho}, k) = \inf_{u\in\mathcal{O}_{0,k-1}} \sup_{y\in\ell_2[0,k-1]} E_{\tilde{\rho}} \left[\sum_{i=0}^{k-1} C(\tilde{\rho}_i, u_i, y_i)\frac{\Delta}{\lambda} \right]$$

where $E_{\tilde{\rho}}$ denotes the expectation with respect to the Markov chain of information states with initial condition $\tilde{\rho}$, and the admissible controls $u \in \mathcal{O}_{0,k-1}$ are any $\mathbb{R}^m$-valued sequence which is a non-anticipating function of the observation path.

Now we will show that if we iterate Equation (4.3.3) we approach a steady state solution,

$$W^{\Delta}(\tilde{\rho}) = \sup_{t\geq 0} \inf_{u\in\mathcal{O}_{0,k-1}} \sup_{y\in\ell_2[0,k-1]} E_{\tilde{\rho}} \left[\sum_{i=0}^{k-1} C(\tilde{\rho}_i, u_i, y_i)\frac{\Delta}{\lambda} \right].$$

It turns out that the steady state solution of the discretized Dynamic Programing Equation (4.3.3) is exactly the finite difference analog of the steady state solution of the Dynamic Programming Equation (4.3.1) which is given by

$$W^{\Delta}(\tilde{\rho}) = \inf_{u\in(R^m)^{\Delta}} \sup_{y\in(R^p)^{\Delta}} \left\{ \sum_{q\in N_{\Delta}(\tilde{\rho})} W^{\Delta}(q)\mathcal{P}(q, \tilde{\rho}; u, y) + C(\tilde{\rho}, u, y)\frac{\Delta}{\lambda} \right\}. \tag{4.3.4}$$

Theorem 4.3.1 Assume that W^{Δ} exists and is finite. Then

$$\lim_{k\to\infty} W^{\Delta}(\tilde{\rho}, k) = W^{\Delta}(\tilde{\rho}).$$

Proof: The crux of proof of this theorem is showing that $W^\Delta(\tilde{\rho},k)$ is nondecreasing in k. For the purpose of this proof we will assume that Σ_F is such that $f(x) \equiv 0$. Define

$$J^\Delta(\tilde{\rho},k;u,y) = E_{\tilde{\rho}}\left[\sum_{i=0}^{k-1} C(\tilde{\rho}_i,u_i,y_i)\frac{\Delta}{\lambda}\right]$$

In this case $C(\tilde{\rho},u,y) = \frac{1}{2}(<\hat{x}(s),Q(u(s))\hat{x}(s)> + R(u(s)) - \frac{\gamma^2}{2}(||y(s)-C\hat{x}(s)||^2)$.

Define Y_k to be the set of outputs $y \in \ell_2[0,\infty]$ such that $y_i = C\hat{x}_i$ for all $i > k$. Let $k_2 \geq k_1$ then $Y_{k_1} \subset \ell_2[0,k_2]$.

$$\begin{aligned}
\sup_{y\in\ell_2[0,k_2]} J^\Delta(\tilde{\rho},k_2;u,y) &\geq \sup_{y\in Y_{k_1}} J^\Delta(\tilde{\rho},k_2;u,y)\\
&= \sup_{y\in\ell_2[0,k_1]}\{J^\Delta(\tilde{\rho},k_1;u,y) + \tfrac{1}{2}(<\hat{x}_i(s),Q(u_i(s))\hat{x}_i(s)> + R(u_i(s)))\}\\
&\geq \sup_{y\in\ell_2[0,k_1]} J^\Delta(\tilde{\rho},k_1;u,y)
\end{aligned}$$

The last inequality follows from the assumption that $Q(u) \geq 0$, and $R(u) > 0$, viz. Section 2.4. This inequality holds for all control sequences as long as both sides use the same control $\{u\}_{0,k_1-1}$. By applying the principle of optimality we know that when we take the infimum on both sides over the admissible controls the optimal choice $\{u^*\}_{0,k_1-1}$ is the same on both sides. Thus $W^\Delta(\tilde{\rho},k_2) \geq W^\Delta(\tilde{\rho},k_1)$.

By definition we know $W^\Delta(\tilde{\rho},k) \leq W^\Delta(\tilde{\rho})$ for all $k \geq 0$. Thus since we have assumed that $W^\Delta(\tilde{\rho})$ exists and is finite, $\lim_{k\to\infty} W^\Delta(\tilde{\rho},k)$ exists and is finite. Then by definition of $W^\Delta(\tilde{\rho})$ and the non-decreasing property of $W^\Delta(\tilde{\rho},k)$ we get that $\lim_{k\to\infty} W^\Delta(\tilde{\rho},k) = W^\Delta(\tilde{\rho})$. ■

Thus we have shown that by iterating Equation 4.3.3 until a steady state solution is achieved, we arrive at a good approximation to the discretization of the original Dissipation Inequality (4.3.4). Theorem 4.3.2 below further implies that when Δ is chosen small enough the discretized control u^*_Δ, found by iterating Equation 4.3.3 until a steady state solution is achieved, renders the system finite gain.

Theorem 4.3.2 Let $\gamma > 0$. If there exists $\Delta_0 > 0$ such that for all $0 < \Delta \leq \Delta_0$ the Dissipation Inequality

$$W^\Delta(\tilde{\rho}) \geq \inf_{u\in(R^m)^\Delta}\sup_{y\in(R^p)^\Delta}\left\{\sum_{q\in N_\Delta(\tilde{\rho})} W^\Delta(q)\mathcal{P}(q,\tilde{\rho};u,y) + C(\tilde{\rho},u,y)\frac{\Delta}{\lambda}\right\}$$

has a solution W^Δ satisfying $W^\Delta(\tilde{\rho}) \geq (p_\rho,0)$, $\lim_{\Delta\to 0} W^\Delta(0) = 0$, and

$$\sup_{0<\Delta\leq\Delta_0}\ \sup_{\rho\in(\mathbf{R}^n)^\Delta,\ |\rho_i|<R_i\quad i=1,\cdots,n} |W^\Delta(\tilde{\rho})| \leq \infty \qquad R \in \text{dom } W$$

then for $0 < \Delta \leq \Delta_0$, $\Sigma^{u^*_\Delta}$ is finite gain.

Proof: Define

$$W(\widetilde{\rho}) = \liminf_{\Delta\searrow 0, \widetilde{\rho}_\Delta \to \widetilde{\rho}, \widetilde{\rho}_\Delta \in (\mathbb{R}^n)^\Delta} W^\Delta(\widetilde{\rho}_\Delta).$$

Then W is lower semi-continuous on dom W and by assumption W satisfies $W^\Delta(\widetilde{\rho}) \geq (p_\rho, 0)$ and $W(0) = 0$. Next we show that W satisfies Dissipation Equality (4.2.2) in the viscosity sense.

Let $\psi \in C^1$ and assume that $W - \psi$ attains a strict local minimum at $\widetilde{\rho}_0$. There is a subsequence $\widetilde{\rho}_\Delta$, again indexed by Δ, such that

$$\lim_{\Delta\to 0, \widetilde{\rho}_\Delta \to \widetilde{\rho}_0} W^\Delta(\widetilde{\rho}_\Delta) = W(\widetilde{\rho}_0),$$

and $W^\Delta - \psi$ has a local minimum at $\widetilde{\rho}_\Delta \in (\mathbb{R}^n)^\Delta$. Then Equation (4.3.4) and

$$W^\Delta(\widetilde{\rho}) - W^\Delta(\widetilde{\rho}_\Delta) \geq \psi(\widetilde{\rho}) - \psi(\widetilde{\rho}_\Delta), \quad \widetilde{\rho} \in N_\Delta(\widetilde{\rho}_\Delta)$$

for Δ small imply

$$\inf_{u\in(R^m)^\Delta} \sup_{y\in(R^p)^\Delta} \left\{ \sum_{i=1}^{n} \frac{\psi(\widetilde{\rho}_\Delta \pm \Delta e_i) - \psi(\widetilde{\rho}_\Delta)}{\Delta} F_i^\pm(\widetilde{\rho}_\Delta, u, y) \right\} \leq 0.$$

Letting $\Delta \searrow 0$ we obtain

$$\inf_{u\in(R^m)^\Delta} \sup_{y\in(R^p)^\Delta} \left\{ \nabla_{\widetilde{\rho}} \psi F(\widetilde{\rho}_0, u, y) \right\} \leq 0.$$

Thus W satisfies Equation (4.2.2) in the viscosity sense, and hence by Theorem 2.2.4 $\Sigma^{u^*_\Delta}$ is formally finite gain. ■

4.4 Summary of Approximation Method for Information State and Certainty Equivalence Control

Given in this section is a description of the Markov Chain approximation method for both the information state controller and the CEC. Let Σ be a general nonlinear system, viz. Section 1.2,

$$\Sigma \begin{cases} \dot{x}(t) &= f(x(t), u(t)) + w(t), \qquad x(t_0) = x_0, \\ y(t) &= h(x(t)) + v(t), \\ z(t) &= \ell(x(t), u(t)). \end{cases}$$

Finite Difference Approximation of the Information State Controller:

For a general nonlinear system Σ, the information state has the quadratic approximation

$$p(x,t) \approx P_t - \frac{\gamma^2}{2} < x - \widetilde{x}_t, Q_t^{-1}(x - \widetilde{x}_t) >,$$

where $Q = Q' > 0$ and $\widetilde{x}_t$, Q_t and P_t satisfy the ODE's

$$\begin{aligned}
\dot{\widetilde{x}}(t) &= f(\widetilde{x}_t, u) + \tfrac{1}{2\gamma^2} Q_t \nabla_x \|\ell(\widetilde{x}_t, u)\|^2 - Q_t \nabla_x h(h(\widetilde{x}_t) - y(t)), \\
\widetilde{x}(0) &= \widetilde{x},
\end{aligned}$$

$$\begin{aligned}
\dot{Q}_t &= 2\nabla_x f(\widetilde{x}_t, u) Q_t + I + Q_t(\tfrac{1}{2\gamma^2}\nabla_x^2 \|\ell(\widetilde{x}_t, u)\|^2 - \|\nabla_x h(\widetilde{x}_t)\|^2) Q_t, \\
Q_0 &= Q,
\end{aligned}$$

$$\begin{aligned}
\dot{P}_t &= \|\ell(\widetilde{x}_t, u)\|^2 - \tfrac{\gamma^2}{2}\|h(\widetilde{x}_t) - y(t)\|^2, \\
P_0 &= P.
\end{aligned}$$

We identify the information state with the finite dimensional vector $\widetilde{p} = (\widetilde{x}, Q)$ and define $\dot{\widetilde{p}} \triangleq F(\widetilde{p}, u, y)$ and $C(\widetilde{p}, u, y) \triangleq \|\ell(\widetilde{x}_t, u)\|^2 - \frac{\gamma^2}{2}\|h(\widetilde{x}_t) - y(t)\|^2$. The Markov transition probability $\mathcal{P}$ which is the probability that information state $\widetilde{p}$ will move to a value of q given the current information state $\widetilde{p}$, control u, and output y is defined by

$$\mathcal{P}^\Delta(q, \widetilde{p}; y, u) = \begin{cases} 1 - |F(\widetilde{p}, u, y)|_1/\lambda, & \text{if } q = \widetilde{p}, \\ F_i^{\pm}(\widetilde{p}, u, y)/\lambda, & \text{if } q = \widetilde{p} \pm \Delta e_i \quad i = 1, \cdots, n, \\ 0, & \text{if } q \notin N_\Delta(\widetilde{p}), \end{cases}$$

where the normalization parameter λ is defined by $\lambda = \sup_{\widetilde{p} \in (\widetilde{R^n})^\Delta, y \in (R^p)^\Delta, u \in (R^m)^\Delta} \|F(\widetilde{p}, u, y)\|_1$. The time discretization is defined by $t_k = k\Delta/\lambda, \quad k = 0, 1, \cdots$. The discretization of the Dynamic Programming Equation is then obtained by iterating Equation (4.4.1) until a steady state value is obtained. The information state control $u^\Delta(\widetilde{p})$ is the steady state value of u achieving the minimum in

$$\begin{cases} W^\Delta(\widetilde{p}, k) = \inf_{u \in (R^m)^\Delta} \sup_{y \in (R^p)^\Delta} \left\{ \sum_{q \in N_\Delta(\widetilde{p})} W^\Delta(q, k-1) \mathcal{P}(q, \widetilde{p}; u, y) + C(\widetilde{p}, u, y) \frac{\Delta}{\lambda} \right\} \\ W^\Delta(\widetilde{p}, 0) = 0. \end{cases} \tag{4.4.1}$$

Finite Difference Approximation of the Certainty Equivalence Controller:

The CEC is

$$u_{CE}(k) = \widetilde{u}^\Delta(\bar{x}_k)$$

where

$$\bar{x}_k \triangleq \arg\max_x \{p^\Delta(x, k) + V^\Delta(x)\}$$

and the discretized information state p^Δ and the discretized infinite horizon full state information value function V^Δ along with the discretized full state feedback control $\tilde{u}^\Delta$ are defined below. First define the Markov transition probability $\mathcal{P}$ which is the probability that the state will move to state z given the current state x, control u, and output y.

$$\mathcal{P}^\Delta(z,x;y,u) = \begin{cases} 1 - |f(x,u,y)|_1/\lambda, & \text{if } z = x, \\ f_i^\pm(x,u,y)/\lambda, & \text{if } z = x \pm \Delta e_i \;\; i = 1, \cdots, n, \\ 0, & \text{if } z \notin N_\Delta(x), \end{cases}$$

where the normalization parameter λ is defined by $\lambda = \sup_{x\in(\tilde{R^n})^\Delta, y\in(R^p)^\Delta, u\in(R^m)^\Delta} \|f(x,u,y)\|_1$. The time discretization is defined by $t_k = k\Delta/\lambda, \quad k = 0, 1, \cdots$. The discretization of the information state is then given by

$$\begin{cases} p^\Delta(x,k) &= \sup_{w\in(\mathbf{R}^n)^\Delta} \Big\{ \sum_{z\in N_\Delta(x)} p^\Delta(x,k-1)\mathcal{P}(z,x;u,y) \\ & \qquad -(\frac{\gamma^2}{2}(\|w\|^2 + \|h(x)-y\|^2) + \frac{1}{2}\|\ell(x,u)\|^2)\frac{\Delta}{\lambda} \Big\} \\ p^\Delta(x,0) &= 0. \end{cases}$$

The discretization of the infinite horizon full state information value function V^Δ is obtained by iterating Equation (4.4.2) below until a steady state value is obtained. The quantity $\tilde{u}^\Delta(x)$ is the value of u achieving the minimum in Equation (4.4.2) at steady state where

$$\begin{cases} V^\Delta(x,k) &= \inf_{u\in(R^m)^\Delta} \sup_{y\in(R^p)^\Delta} \Big\{ \sum_{z\in N_\Delta(x)} V^\Delta(z,k-1)\mathcal{P}(z,x;u,y) + (\frac{1}{2}\|\ell(x,u)\|^2 - \frac{\gamma^2}{2}\|w\|^2)\frac{\Delta}{\lambda} \Big\} \\ V^\Delta(x,0) &= 0. \end{cases} \tag{4.4.2}$$

Chapter 5

Examples

In this chapter we present numerical examples to illustrate the digital controllers described in Chapters 2 and 3. We employ a finite difference scheme as described in Chapter 4.

In addition to the finite difference (Markov chain) approximation required to numerically solve the Hamiltion Jacobi equations and dissipation inequalities involved in the solution to the Robust H_∞ Output Feedback Control Problem, digital implementation requires preliminary approximations which are designed to accomplish the following main goals:

(i) Allow the information state control of general nonlinear systems not necessarily admitting a finite dimensional information state; and

(ii) Decrease the computational complexity of the process by which the control is computed.

As we shall show both of these goals may be met by considering appropriate approximations to the information state.

Although the continuous extension of the discrete time results discussed in Chapter 2 is valid only in a formal sense, it does hold rigorously for systems which have an associated finite dimensional information state, viz. Section 2.4. Thus, in the case of such systems the information state control is directly digitally implementable modulo the finite difference approximation discussed above. For systems in which the information state can not be identified with a finite dimensional quantity or that the CEP[1] does not hold, the digital implementation of an information state controller requires that some sort of preliminary numerical

[1]Recall that in the case that the CEP holds the information state controller and the CEC are equivalent.

approximation of the information state first be applied.

Here we consider two types of preliminary approximations of the information state controller which meet our goals: (i) approximation of the information state by a finite dimensional quantity, viz. Section 2.5, or (ii) use of the Certainty Equivalence Control (CEC), viz. Chapter 3. The first approximation type can also be applied to the CEC to further reduce its computational complexity. Although both of these preliminary approximations decrease the computational complexity of the process by which the control is computed, the computation of the CEC still suffers from the *curse of dimensionality*. This refers to the fact that the order of the computational complexity increases exponentially with the dimension of the system. In terms of real time implementation, the computational complexity of the information state is the critical issue since its computation must be performed on line, whereas the value function may be computed off line. Thus the preliminary approximation of the information state by a finite dimensional quantity may be the best choice when real time computational speed is an issue. In Figure 1 a block diagram of the information state controller is given which emphasizes the fact that the most computationally intensive calculations can be performed off line.

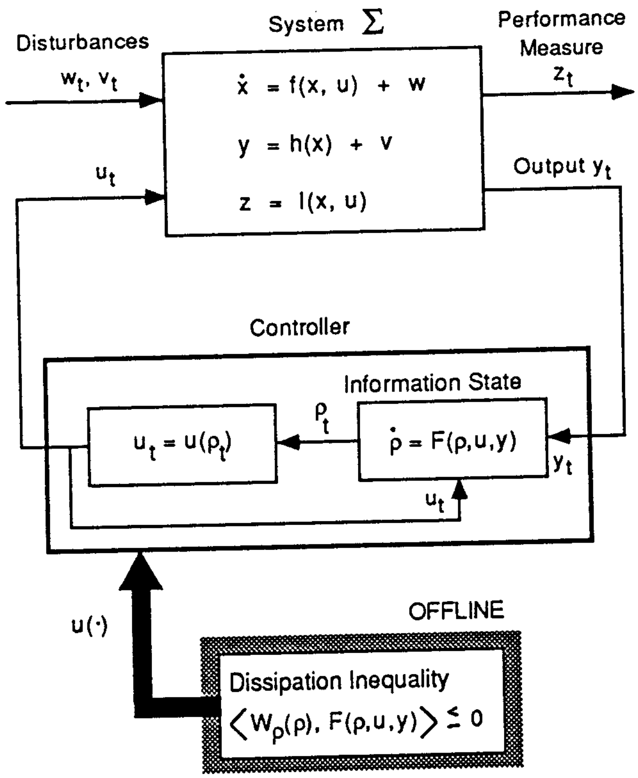

Figure 1: Information State Controller

5.1 Information State Feedback Control

Below in Section 5.1.1 we give examples which demonstrate that the information state controller is stabilizing and robust to noise for systems with finite dimensional information state. In addition we illustrate the nontrivial domain $\mathcal{D}$, Equation (2.4.5), of the information state value function and its relation to the Riccati Equation coupling condition in the case of linear systems, viz. Theorem 1.3.2.

In Section 2.5 we described an approximation of the information state by a convex quadratic. By using this approximation we can implement the information state controller for general nonlinear systems which do not satisfy the strict criteria, viz. Theorem 2.4.1, required for a finite dimensional information state to exist. In Section 5.1.2 examples are given which demonstrate that the approximate information state controller is stabilizing and robust to noise for nonlinear systems.

As indicated in Remark 2.4.2 $W(\widehat{x}, P, \phi, t) = \widetilde{W}(\widehat{x}, P, t) + \phi(t)$ which implies that $\nabla_\phi W = 1$, and that W is only dependent on ϕ at the initial time only. Thus in the examples below we set the initial condition $\phi(0) = 0$ and consider $\widetilde{W}(\widehat{x}, P, t) = W(\widehat{x}, P, \phi, t)$.

5.1.1 Finite Dimensional Information State

Example 1 [TYJB94]

Consider a linear system with the state space model

$$\Sigma_L \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + w(t), \\ y(t) &= Cx(t) + v(t), \\ z(t) &= \begin{bmatrix} \sqrt{Q}x(t) \\ u(t) \end{bmatrix}. \end{cases} \tag{5.1.1}$$

Using the standard H_∞ control theory for linear systems we can compute the optimal value γ^* of the L_2 gain parameter. The information state for Σ_L can be identified with the finite dimensional vector $\rho = (\widehat{x}, P, \phi, t)$

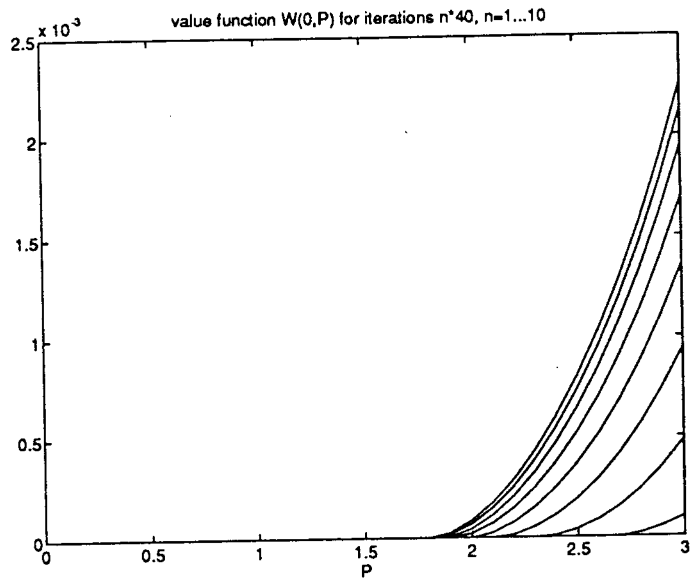

Figure 2: Domain of Value Function for Example 1

where ρ evolves according to the ODEs

$$\dot{\widehat{x}}(t) = (A + \gamma^{-2}P(t)Q)\widehat{x}(t) + Bu(t) + P(t)C'(y(t) - C\widehat{x}(t)),$$
$$\widehat{x}(0) = \widehat{x},$$

$$\dot{P}(t) = P(t)A' + AP(t) - P(t)(C'C - \gamma^{-2}Q)P(t) + I,$$
$$P(0) = P,$$

$$\dot{\phi}(t) = \tfrac{1}{2}(< \widehat{x}(t), Q\widehat{x}(t) > + < u(t), u(t) > -\gamma^2||y(t) - C\widehat{x}(t)||^2),$$
$$\phi(0) = \phi.$$

Notice that the ODE which describes the evolution of P is the finite horizon version of the estimation Riccati Equation (for K) of Theorem 1.3.2. The domain of $\mathcal{D}$ the information state value function $W(\rho)$ can be expressed as [YJ93]

$$\mathcal{D} = \{(\widehat{x}, P, \phi, t) \in \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R} \times [t_0, t_f] \ : \ Z(t)P < \gamma^2 I\ \}, \qquad (5.1.2)$$

where $Z(t)$ solves the regulator-type Riccati differential equation

$$-\dot{Z}(t) = A'Z + ZA - Z(BB' - \gamma^{-2}I)Z + Q, \ Z(t_f) = 0. \qquad (5.1.3)$$

Equation (5.1.3) is the finite horizon version of the control Riccati Equation of Theorem 1.3.2. In addition, the domain of the information state value function $\mathcal{D}$ is exactly the region on which the coupling condition is satisfied, viz Theorem 1.3.2. Thus the familiar coupling condition required for the solution of the H_∞ output feedback problem for linear systems is exactly a requirement that the information state value function be finite.

Now consider a particular linear system Σ_L where $A = -0.5$, $B = 1$, $C = 1$, and $Q = 2$. For this system the H_∞ optimal L_2 gain can be computed to be $\gamma^* = 1.789$. From Equation (5.1.2), we see that the domain $\mathcal{D}$ is affected only by the component P of the information state ρ. In Figure 2 the plots of $W(0, P)$ computed for $\gamma = 1.8$ are displayed at several different iterations. From these plots it is evident that the domain is a non-increasing function of the time horizon. If we let $\bar{Z}$ denote the steady state solution of Equation (5.1.3), then the lower bound on the size of the domain is given by $\gamma/\bar{Z}$.

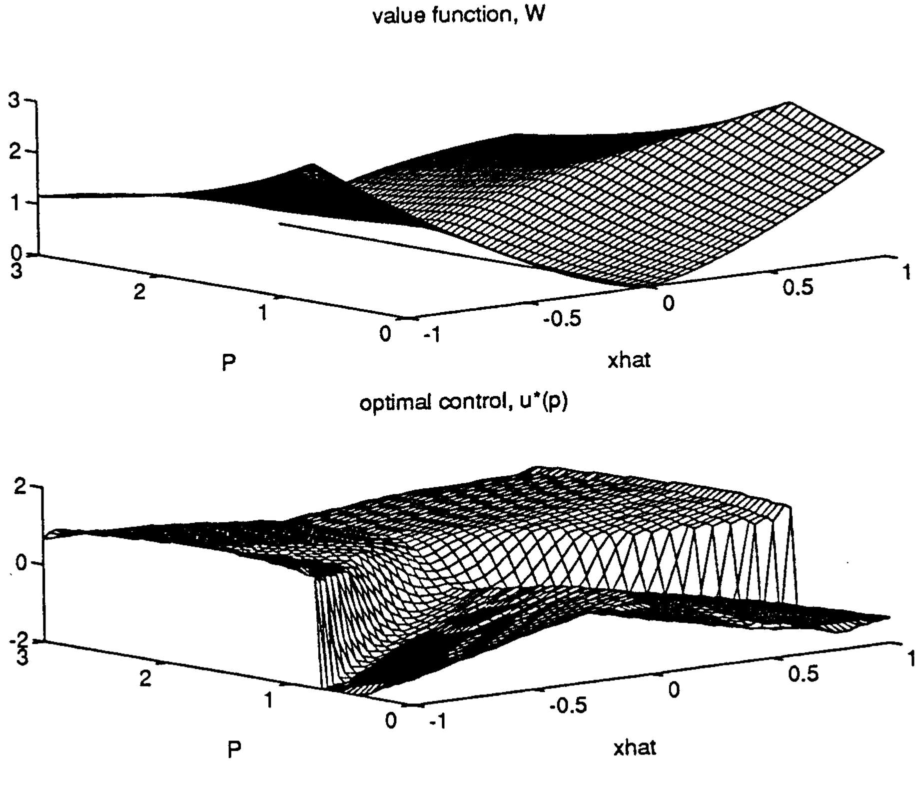

Figure 3: Value Function W and Optimal Control u* for Example 2

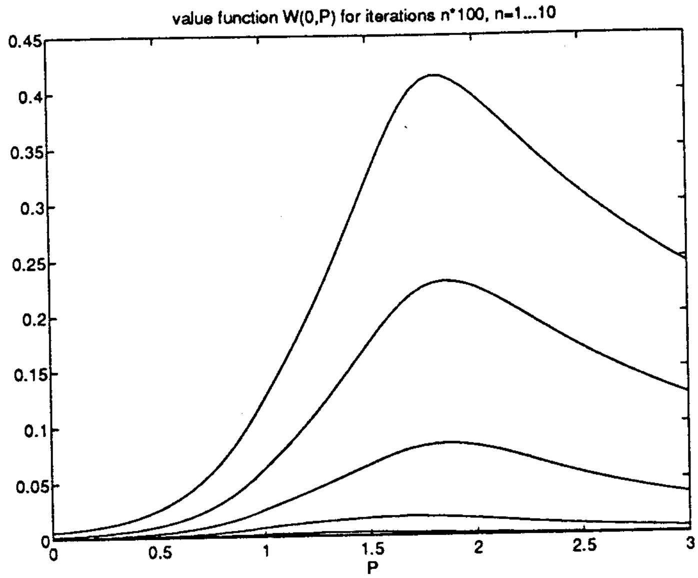

Figure 4: Domain of Value Function for Example 2

Example 2 Consider a bilinear system with the state space model

$$\Sigma_{SB} \begin{cases} \dot{x}(t) &= (-0.5 + 0.5u(t))x(t) + w(t), \\ y(t) &= x(t) + v(t), \\ z(t) &= \begin{bmatrix} 2x(t) \\ u(t) \end{bmatrix}. \end{cases} \tag{5.1.4}$$

The information state for Σ_{SB} can be identified with the finite dimensional vector $\rho = (\widehat{x}, P, \phi, t)$ where ρ evolves according to the ODEs

$$\begin{aligned} \dot{\widehat{x}}(t) &= (-0.5 + 0.5u(t) + 2\gamma^{-2}P(t))\widehat{x}(t) + P(t)(y(t) - \widehat{x}(t)), \\ \widehat{x}(0) &= \widehat{x}, \end{aligned}$$

$$\begin{aligned} \dot{P}(t) &= 2P(t)(-0.5 + 0.5u(t)) - (1 - 2\gamma^{-2})P(t)^2 + 1, \\ P(0) &= P, \end{aligned}$$

$$\begin{aligned} \dot{\phi}(t) &= \tfrac{1}{2}(2\widehat{x}(t)^2 + u(t)^2 - \gamma^2|y(t) - \widehat{x}(t)|^2), \\ \phi(0) &= \phi. \end{aligned}$$

Using a value of $\gamma = 2.0$ the information state value function and control have been computed. Figure 3 (top) shows the value function $W(\widehat{x}, P)$ and (bottom) shows the optimal control $u^*(\widehat{x}, P)$ at 100 iterations. Plots of $W(0, P)$ at several different iterations are depicted in Figure 4. For linear systems the value of P

at which $W(0,P)$ becomes nonzero indicates the size of the domain $\mathcal{D}$. The domain of $W(\widehat{x},P)$ decreases as the time horizon increases.

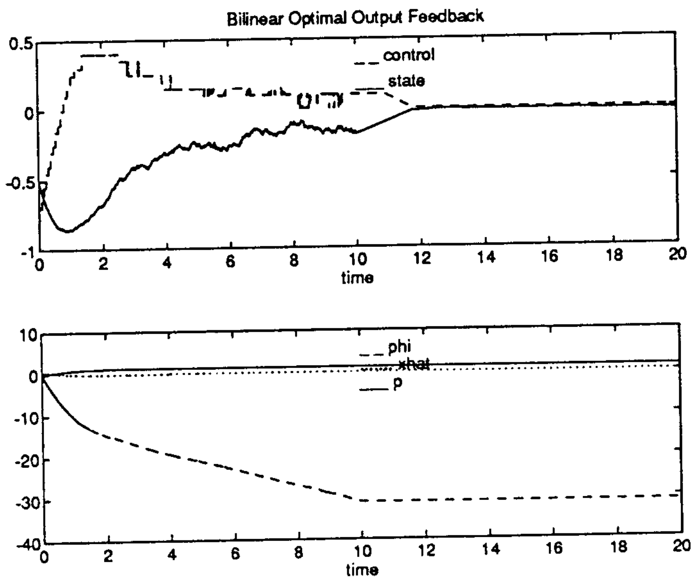

Figure 5: Stabilization of Unstable Bilinear System from Example 3

Example 3 Consider an open loop ($u \equiv 0$) unstable bilinear system with the state space model

$$\Sigma_{UB}\begin{cases} \dot{x}(t) &= (0.1+0.01u(t))x(t)+u(t)+w(t), \\ y(t) &= x(t)+v(t), \\ z(t) &= \begin{bmatrix} \sqrt{2}x(t) \\ u(t) \end{bmatrix}. \end{cases} \tag{5.1.5}$$

The information state for Σ_{US} can be identified with the finite dimensional vector $\rho = (\widehat{x}, P, \phi, t)$ where ρ evolves according to the ODEs

$$\begin{aligned} \dot{\widehat{x}}(t) &= (0.1+0.01u(t)+\gamma^{-2}P(t))\widehat{x}(t)+u(t)+P(t)(y(t)-\widehat{x}(t)), \\ \widehat{x}(0) &= \widehat{x}, \end{aligned}$$

$$\begin{aligned} \dot{P}(t) &= 2P(t)(0.1+0.01u(t))-(1-\gamma^{-2})P(t)^2+1, \\ P(0) &= P, \end{aligned}$$

$$\begin{aligned} \dot{\phi}(t) &= \tfrac{1}{2}(\widehat{x}(t)^2+u(t)^2-\gamma^2|y(t)-\widehat{x}(t)|^2), \\ \phi(0) &= \phi. \end{aligned}$$

We compute the information state value function and optimal control using the L_2 gain parameter $\gamma = 5.0$. Figure 5 demonstrates the performance of the information state control for this system in the presence of Gaussian state and measurement noise w and v respectively. In Figure 5 (top) are plotted the trajectories of state x and the control u. In Figure 5 (bottom) are plotted the trajectories of the information state $\rho = (\widehat{x}, P, \phi, t)$. At $t = 10$ seconds, the noise is pulled out from the system at which time the state approaches zero rapidly and moreover, the information state components $\widehat{x}(t), P(t)$ and ϕ remain bounded. Thus, the controller is stabilizing and robust to noise.

5.1.2 Quadratic Approximation of Information State

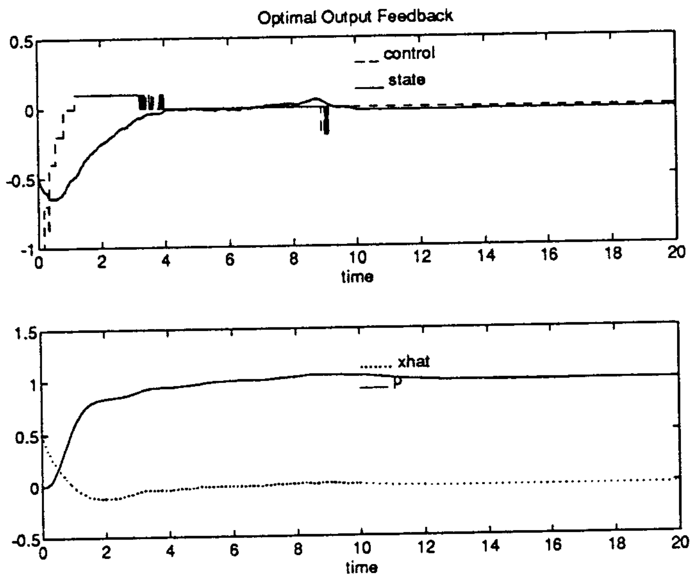

Figure 6: Stabilization of Nonlinear System for Example 4

Example 4 Consider a general nonlinear system which is open loop ($u \equiv 0$) unstable and has the state space model

$$\Sigma_{NL} \begin{cases} \dot{x}(t) &= x^2(t) + u(t) + w(t), \\ y(t) &= \sin(x(t)) + v(t), \\ z(t) &= \begin{bmatrix} \sqrt{2}x(t) \\ u(t) \end{bmatrix}. \end{cases} \tag{5.1.6}$$

Because this system does not admit a finite dimensional information state, we shall approximate its information state by the quadratic

$$p(x,t) = P_t - \frac{\gamma^2}{2} < x - \widehat{x}_t, P_t(x - \widehat{x}_t) >$$

where $\rho = (\widehat{x}, P, \phi)$ evolves according to the ODE's

$$\Sigma_\rho \begin{cases} \dot{\widehat{x}}_t &= \widehat{x}_t^2 + u(t) + \frac{2}{\gamma^2} P_t \widehat{x}_t - P_t \cos(\widehat{x}_t)(\sin(\widehat{x}_t) - y(t)), \\ \dot{P}_t &= 4 P_t \widehat{x}_t + 1 + P_t^2 (\frac{2}{\gamma^2} - \cos(\widehat{x}_t)^2), \\ \dot{\phi} &= \frac{1}{2}(\widehat{x}_t^2 + u(t)^2) - \frac{\gamma^2}{2}(h(\widehat{x}_t) - y(t))^2, \end{cases} \tag{5.1.7}$$

viz, Theorem 2.5.1. We implement the information state controller with the L_2 gain parameter $\gamma = 5.0$. Figure 6 demonstrates the performance of the information state control for this system in the presence of Gaussian state and measurement noise w and v respectively. In Figure 6 (top) are plotted the trajectories of state x and the control u. In Figure 6 (bottom) are plotted the trajectories of the information state $\rho = (\widehat{x}, P)$. At $t = 10$ seconds, the noise is pulled out from the system at which time the state approaches zero rapidly and moreover, the information state components $\widehat{x}(t)$ and $P(t)$ remain bounded. Thus, the controller is observed to be stabilizing and robust to noise.

5.2 Certainty Equivalence Control

In Chapter 3 it was shown that under certain conditions the Certainty Equivalence Controller (CEC), which is computationally simpler than the information state feedback controller, is optimal. Recall that under such conditions we say that the Certainty Equivalence Principal (CEP) holds. The conditions under which the CEP holds, however, are often difficult to verify. Moreover, it is known that such conditions do not hold in general [Jam93b]. Computational and practical considerations thus lead us to consider the implementation of suboptimal controllers. In this section we give empirical evidence that the implementation of the CEC, although suboptimal, can be locally stabilizing and robust to noise for many nonlinear systems.

Example 5 Consider the open loop ($u \equiv 0$) unstable nonlinear system Σ_{NL} from Example 4. We implement the CEC with the L_2 gain parameter $\gamma = 10.0$. Figure 7 demonstrates the performance of the CEC for this system in the presence of Gaussian state and measurement noise w and v respectively. In Figure 7 are plotted the trajectories of the state x, the output y, the minimum stress estimate $\bar{x}$, and the control u. At $t = 10$ seconds, the noise is pulled out from the system at which time the state approaches zero rapidly. Thus, the controller is observed to be stabilizing and robust to noise for this nonlinear system.

5.2.1 Quadratic Approximation of Information State

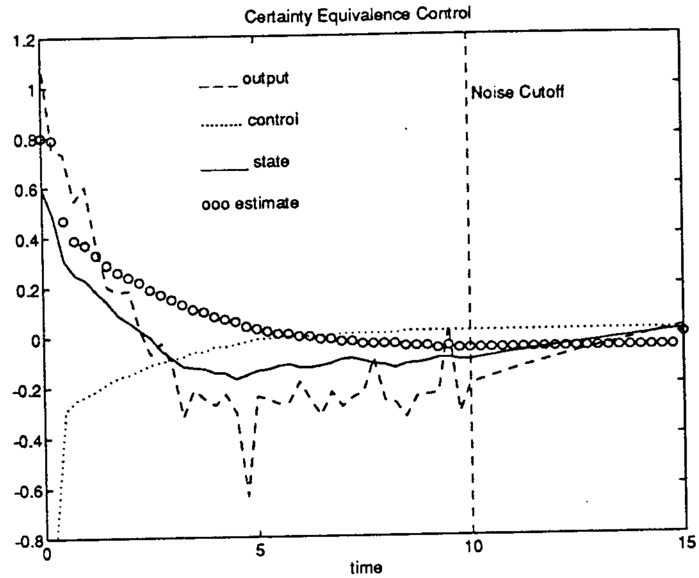

Figure 7: Stabilization of Nonlinear System for Example 5

Example 6 Consider the open loop ($u \equiv 0$) unstable nonlinear system Σ_{NL} from Example 4. We approximate the information state by a quadratic which can be identified with the finite dimensional vector $\rho = (\widehat{x}, P, \phi)$ where the components of ρ evolve according to Σ_ρ Equation (5.1.7). We implement the approximate CEC with the L_2 gain parameter $\gamma = 10.0$. Figure 8 demonstrates the performance of the CEC for this system in the presence of Gaussian state and measurement noise w and v respectively. In Figure 8 (top) are plotted the trajectories of the state x, the minimum stress estimate $\bar{x}$, and the control u. In Figure 8 (bottom) are plotted the trajectories of the information state $\rho = (\widehat{x}, P)$. At $t = 5$ seconds, the noise is pulled out from the system at which time the state approaches zero rapidly and moreover, the information state components $\widehat{x}(t)$ and $P(t)$ remain bounded. Thus, the controller is observed to be stabilizing and robust to noise.

5.2.2 Modified Extended Kalman Filter

Example 7 Consider again the open loop ($u \equiv 0$) unstable nonlinear system Σ_{NL} from Example 4. We again approximate the information state by a quadratic which can be identified with the finite dimensional vector $\rho = (\widehat{x}, P, \phi)$ where the components of ρ evolve according to Σ_ρ Equation (5.1.7). In addition, we approximate the full state value function by a quadratic with its minimum at the origin. The value function can then be parameterized by Π, viz., Theorem 3.4.1. To compute Π we integrate the following ODE until

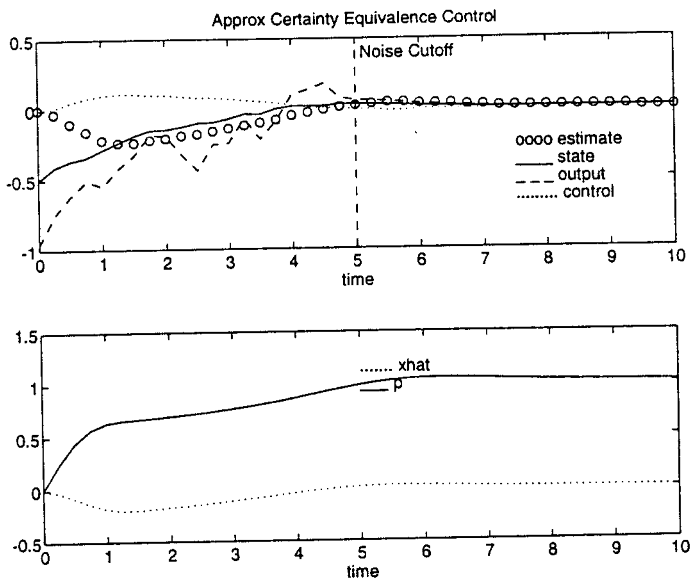

Figure 8: Stabilization of Nonlinear System for Example 6

a steady state value is achieved

$$\dot{\Pi}_t = (\frac{1}{\gamma^2} - 1)\Pi_t^2 + 1.$$

The state estimate is then given by $\bar{x}_t \approx \widehat{x}_t/(1 - \frac{1}{\gamma^2}P_t\Pi)$ and the control by $u_{CEQ} \approx -\Pi\bar{x}_t$, viz., Theorem 3.4.1. We implement the modified EKF with the L_2 gain parameter $\gamma = 5.0$. Figure 9 demonstrates the performance of the EKF for this system in the presence of Gaussian state and measurement noise w and v respectively. In Figure 9 (top) are plotted the trajectories of the state x, the output $\bar{y}$, and the control u. In Figure 9 (bottom) are plotted the trajectories of the information state $\rho = (\widehat{x}, P)$ and value function parameter Π. At $t = 2$ seconds, the noise is pulled out from the system at which time the state approaches zero rapidly and moreover, the information state components $\widehat{x}(t), P(t)$ and value function parameter Π remain bounded. Thus, the controller is stabilizing and robust to noise for this nonlinear system.

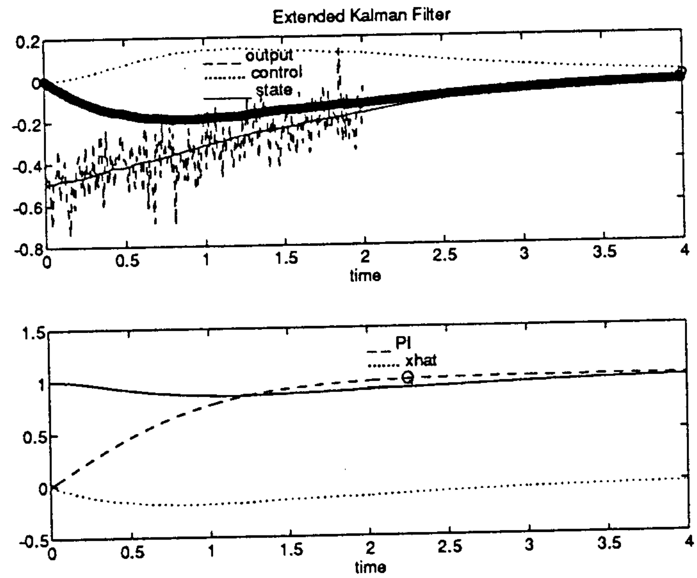

Figure 9: Stabilization of Nonlinear System for Example 7

Chapter 6

Discrete Time

For the discrete time robust H_∞ output feedback control problem the mathematical difficulties associated with differentiation do not exist. Although James and Baras [JB94b] have provided a complete theoretical solution to the discrete time robust output feedback control problem, difficult implementation issues still remain. As with the continuous time information state control, direct digital implementation is impossible except under the special circumstances in which the information state is finite dimensional. Finding conditions in which the information state is finite dimensional for discrete time systems is currently an open problem.

In this chapter we consider the discrete time version of the Certainty Equivalence Controller (CEC). In Section 6.1 the discrete time CEC is described and in Section 6.2 the implementation is discussed and convergence results are given. Finally in Section 6.3 examples are given which demonstrate that this controller is locally stabilizing and robust to noise for highly nonlinear systems. In addition an example is given to emphasize the difficulties associated with choosing the L_2 gain parameter γ.

6.1 Certainty Equivalence

Consider the discrete-time partially observed dynamic game problem for the system

$$\begin{cases} x_{k+1} &= b(x_k, u_k) + w_k, \\ y_{k+1} &= h(x_k) + v_k \end{cases} \tag{6.1.1}$$

on the finite time interval $k = 0, 1, 2, \dots M$, with cost

$$J(u) = \sup_{(w,v)\in\ell_2[0,M-1]} \sup_{x_0\in\mathbf{R}^n} \left\{ \sum_{l=0}^{M-1} L(x_l, u_l) + \Phi(x_M) - \gamma^2 \sum_{l=0}^{M-1} (|w_l|^2 + |v_l|^2) \right\}. \tag{6.1.2}$$

The terminal cost $\Phi(\cdot)$ is taken to be zero in our implementation since this more closely approximates the infinite-horizon situation. The running cost, $L(\cdot,\cdot)$, is a positive function of the state and control; for our implementation we have chosen $L(x,u) = |x|^2 + |u|^2$. For convergence of the algorithm, γ must be chosen *large enough.* Clearly γ must be chosen greater than the H_∞ optimal level for a suboptimal controller to exist.

The admissible controls $u \in \mathcal{O}_{0,M-1}$ are any U-valued sequence which is a non-anticipating function of the observation path. The partially observed dynamic game problem entails finding an admissible sequence $u \in \mathcal{O}_{0,M-1}$ such that

$$J(u^*) = \inf_{u\in\mathcal{O}_{0,M-1}} J(u).$$

The certainty equivalence controller [JBE94, Whi81] is given by the following two infinite dimensional dynamic programming equations. The sequence of information states $\{p_k\}$ is given recursively by the dynamic programming equation

$$\begin{cases} p_k(z) &= \sup_{\xi\in\mathbf{R}^n} \left\{ L(\xi, u_{k-1}) - \gamma^2 |z - b(\xi, u_{k-1})|^2 - \gamma^2 |h(\xi) - y_k|^2 + p_{k-1}(\xi) \right\}, \\ p_0(z) &= 0. \end{cases} \tag{6.1.3}$$

The sequence of upper values $\{\bar{f}_k\}$ of the fully observed dynamic game is given recursively by the dynamic programming equation

$$\begin{cases} \bar{f}_k(x) &= \inf_{u\in U} \sup_{w\in\mathbf{R}^n} \{ \bar{f}_{k+1}(b(x,u) + w) + L(x,u) - \frac{\gamma^2}{2} |w|^2 \}, \\ \bar{f}_M(x) &= \Phi(x). \end{cases} \tag{6.1.4}$$

If $\widetilde{u}_k^*(x)$ achieves the minimum in Equation (6.1.4), then $u_k^* = \widetilde{u}_k^*(x_k)$ is an optimal feedback policy for the completely observed game. The minimum stress estimate $\bar{x}_k$ is given by

$$\bar{x}_k \in \underset{x\in\mathbb{R}^n}{\text{argmax}} \{ p_k(x) + \bar{f}_k(x) \} \triangleq \widehat{x}_k$$

where $\widehat{x}_k$ set valued. The certainty equivalence controller is defined by

$$u_k^* = \widetilde{u}_k^*(\bar{x}_k)$$

and if the CEP holds then this controller is an optimal policy for the partially observed game.

6.2 Implementation

The sequence of upper values $\{\bar{f}_k\}$, and thus the optimal control policy for the fully observed game, is computable off line. The information state, playing a role similar to an observer, is dependent on the current output and control of the system. Thus, the information state is part of the controller dynamics which must be computed on line.

In our examples, given in Section 5.2, we have observed that convergence of the value function and optimal control to steady state is achieved after a relatively small number of iterations. Thus, in order to reduce the computational effort as well as the memory required for storage of the value function and optimal control, our implementation uses the steady state optimal control and value function for all simulations.

6.2.1 Convergence of Discretization

For the implementation of the controller on a digital computer the state space $X = \mathbf{R}^n$, the control space U, and the disturbance space $W = \mathbf{R}^n$ must be discretized. We consider the uniformly discretized grids of size Δ and denote the discretized spaces by X^Δ, U^Δ, and W^Δ respectively. One must then ask the question: Given a uniform discretization of the various spaces of grid size Δ, do the solutions $\widetilde{p}_k$ and $\widetilde{f}_k$ of the discretized Hamilton-Jacobi equations

$$\begin{aligned} \widetilde{p}_k(z_j) &= \sup_{\xi_i \in (\mathbf{R}^n)^\Delta} \left\{ L(\xi_i, u_{k-1}) - \gamma^2 |z_j - b(\xi_i, u_{k-1})|^2 - \gamma^2 |h(\xi_i) - y_k|^2 + \widetilde{p}_{k-1}(\xi_i) \right\}, \quad (6.2.1) \\ \widetilde{p}_0(z_j) &= 0, \end{aligned}$$

$$\begin{aligned} \widetilde{f}_k(x_i) &= \inf_{u_j \in U^\Delta} \sup_{w_\ell \in (\mathbf{R}^n)^\Delta} \{ \widetilde{f}_{k+1}(b(x_i, u_j) + w_\ell) + L(x_i, u_j) - \frac{\gamma^2}{2} |w_\ell|^2 \}, \quad (6.2.2) \\ \widetilde{f}_M(x_i) &= \Phi(x_i). \end{aligned}$$

converge to the solutions p_k and f_k of the true Hamilton-Jacobi Equations (6.1.3) and (6.1.4) respectively as $\Delta \to 0$? In order to answer this question we will first prove the following lemma.

Lemma 6.2.1 Given any function g which is bounded and jointly uniformly continuous in x and z, then $\sup_{\xi_j \in (\mathbf{R}^n)^\Delta} g(\xi_j, z)$ converges to $\sup_{x \in \mathbf{R}^n} g(x, z)$ uniformly as $\Delta \to 0$, i.e., for all $\epsilon > 0$ there exists $\Delta_0 > 0$ such that for any $\Delta \le \Delta_0$

$$\sup_{x \in \mathbf{R}^n} g(x, z) - \sup_{\xi_j \in (\mathbf{R}^n)^\Delta} g(\xi_j, z) < \epsilon.$$

Proof: Assume there exists $x^* \in \mathbb{R}^n$ which achieves the supremum of $g(x,z)$. Fix $\epsilon_0 > 0$ arbitrary. By uniform continuity of g, given any $\epsilon > 0$ there exists a $\Delta > 0$ such that $\|x_1 - x_2\| < \Delta$ implies $|g(x_1, z) - g(x_2, z)| < \epsilon$. Let Δ_0 be associated with ϵ_0. Now pick the $\xi^* \in (\mathbb{R}^n)_0^\Delta$ closest to x^*. Then ξ^* satisfies

$$\|x^* - \xi^*\| \leq \frac{\Delta}{2}$$

which implies $|g(x^*, z) - g(\xi^*, z)| < \epsilon$. Since we know

$$g(\xi^*, z) \leq \sup_{\xi_j \in (\mathbf{R}^n)^{\Delta_0}} g(\xi_j, z) \leq \sup_{x \in \mathbf{R}^n} g(x, z) = g(x^*, z)$$

we have our result under the assumption that there exists $x^* \in \mathbf{R}^n$ which achieves the supremum of $g(x,z)$.

When this is not the case g asymptotically approaches a finite value $\bar{g}$ as $\|x\| \to \infty$ and then both sides will be less than ϵ from $\bar{g}$ for all $x > \bar{x}$ for some $\bar{x}$, and thus less than ϵ from each other.

∎

Using this lemma, we can prove by induction a theorem which will imply that for systems which are bounded and uniformly continuous, i.e., L, b and h uniformly continuous, that the discretized information state (6.2.1) converges to the true information state (6.1.3) uniformly as the sampling interval Δ approaches zero.

Theorem 6.2.2 Assume g bounded and jointly uniformly continuous in $x, z \in \mathbf{R}^n$. Define q_k and $\tilde{q}_k$ as follows

$$\begin{aligned} q_k(z) &= \sup_{x \in \mathbf{R}^n} \{g(x,z) + q_{k-1}(x)\} \\ \tilde{q}_k(z) &= \sup_{\xi_i \in (\mathbf{R}^n)^\Delta} \{g(\xi_i, z) + \tilde{q}_{k-1}(\xi_i)\} \\ q_0(z) &= \tilde{q}_0(z) = 0. \end{aligned}$$

Then $\lim_{\Delta \to 0} \tilde{q}_k = q_k$ uniformly, i.e., for all $\epsilon > 0$ there exists a $\Delta_0 > 0$ such that for any $\Delta \leq \Delta_0$, $|q_k(z) - \tilde{q}_k(z)| < \epsilon$ for all $z \in \mathbb{R}^n$.

Proof: Now we will show by induction that (i) $\tilde{q}_k \leq q_k$, and (ii) $\lim_{\Delta \to 0} \tilde{q}_k = q_k$. These two induction proofs will be done in parallel although the first result is used to prove the second.

Show true for $k = 1$:

(i) $\tilde{q}_1(z) = \sup_{\xi_i \in (\mathbf{R}^n)^\Delta} \{g(\xi_i, z)\} \leq \sup_{x \in \mathbf{R}^n} \{g(x, z)\} = q_1(z)$

(ii) As $\Delta \to 0$ we know by Lemma 6.2.1 that $\widetilde{q}_1(z) = \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z)\} \to \sup_{x \in \mathbf{R}^n}\{g(x, z)\} = q_1(z)$ uniformly.

Assume true for $k-1$:

(i) Assume $\widetilde{q}_{k-1} \leq q_{k-1}$.

(ii) Assume $\lim \Delta \to 0$ $\widetilde{q}_{k-1} = q_{k-1}$ uniformly.

Prove true for k:

(i)

$$\begin{aligned}\widetilde{q}_k(z) &= \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z) + \widetilde{q}_{k-1}(\xi_i)\} \\ &\leq \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z) + q_{k-1}(\xi_i)\} \\ &\leq \sup_{x \in \mathbf{R}^n}\{g(x, z) + q_{k-1}(x)\} = q_k(z)\end{aligned}$$

where the first inequality follows from the induction assumption.

(ii) We know from induction proof part (i) that $\widetilde{q}_k \leq q_k$ for all k and Δ. Also, by induction assumption, we know that for all $\epsilon > 0$ there exists $\Delta_0 > 0$ such that for all sampling intervals $\Delta \leq \Delta_0$, $q_{k-1}(\xi_i) - \widetilde{q}_{k-1}(\xi_i) \leq \epsilon$. Thus

$$\begin{aligned}q_k(z) \geq \widetilde{q}_k(z) &= \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z) + \widetilde{q}_{k-1}(\xi_i)\} \\ &\geq \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z) + q_{k-1}(\xi_i) - \epsilon\}\end{aligned}$$

From Lemma 6.2.1 we know that there exists $\Delta_1 > 0$ such that

$$\sup_{x \in \mathbf{R}^n}\{g(x, z) + q_{k-1}(x)\} - \sup_{\xi_i \in (\mathbf{R}^n)^\Delta}\{g(\xi_i, z) + q_{k-1}(\xi_i)\} < \epsilon$$

for all $z \in \mathbb{R}^n$. Thus for any $\Delta \leq \min(\Delta_0, \Delta_1)$

$$q_k(z) \geq \widetilde{q}_k(z) \geq q_k(z) - 2\epsilon$$

and thus by induction we have shown uniform convergence.

■

Corollary 6.2.3 Consider the discrete time partially observed dynamic game problem given by Equation (6.1.1) with cost given by Equation (6.1.2). If the system and cost are bounded and uniformly continuous, i.e., L, b, h bounded and uniformly continuous, the discretized information state (6.2.1) converges to the true information state (6.1.3) uniformly as the sampling interval Δ approaches zero.

A similar result holds for the upper value (6.1.4) and discretized upper value (6.2.2). Theorem 6.2.4 implies that for systems which are bounded and uniformly continuous the discretized upper value converges to the true upper value uniformly as the sampling interval Δ approaches zero.

Theorem 6.2.4 Assume Φ bounded and uniformly continuous, and g and f bounded and jointly uniformly continuous in x, u and w. Define q_k and $\widetilde{q}_k$ as follows

$$\begin{aligned} q_k(x) &= \inf_{u\in U} \sup_{w\in \mathbf{R}^n} \{g(x,u,w) + q_{k+1}(f(x,u,w))\} \\ \widetilde{q}_k(x) &= \inf_{u_i\in U^\Delta} \sup_{w_j\in (\mathbf{R}^n)^\Delta} \{g(x,u_i,w_j) + \widetilde{q}_{k+1}(f(x,u_i,w_j))\} \\ q_M(x) &= \widetilde{q}_M(x) = \Phi(x). \end{aligned}$$

Then $\lim_{\Delta\to 0} \widetilde{q}_k = q_k$ uniformly, i.e., for all $\epsilon > 0$ there exists a $\Delta_0 > 0$ such that for any $\Delta \leq \Delta_0$, $|q_k(z) - \widetilde{q}_k(z)| < \epsilon$ for all $z \in \mathbb{R}^n$.

Proof: This result follows by induction.

- Show true for $k = M$:

$$q_M(x) = \widetilde{q}_M(x) = \Phi(x).$$

- Assume true for $k+1$:

  Assume $\lim \Delta \to 0$ $\widetilde{q}_{k+1} = q_{k+1}$ uniformly.

- Prove true for k:

  By induction assumption, for all $\epsilon > 0$ there exists $\Delta_0 > 0$ such that for all sampling intervals $\Delta \leq \Delta_0$, $-\epsilon \leq \widetilde{q}_{k+1}(\xi_i) - q_{k+1}(\xi_i) \leq \epsilon$ for all $x \in \mathbf{R}^n$. Thus

$$\begin{aligned} &\inf_{u_i\in U^\Delta} \sup_{w_j\in (\mathbf{R}^n)^\Delta} \{g(x,u_i,w_j) + q_{k+1}(f(x,u_i,w_j)) - \epsilon\} \leq \widetilde{q}_k(x) \\ &\leq \inf_{u_i\in U^\Delta} \sup_{w_j\in (\mathbf{R}^n)^\Delta} \{g(x,u_i,w_j) + q_{k+1}(f(x,u_i,w_j)) + \epsilon\}. \end{aligned}$$

  Lemma 6.2.1 implies there exists $\Delta_1 \leq \Delta_0$ such that for all sampling intervals $\Delta \leq \Delta_1$

$$\begin{aligned} &\inf_{u_i\in U^\Delta} \sup_{w\in \mathbf{R}^n} \{g(x,u_i,w) + q_{k+1}(f(x,u_i,w)) - 2\epsilon\} \leq \widetilde{q}_k(x) \\ &\leq \inf_{u_i\in U^\Delta} \sup_{w\in \mathbf{R}^n} \{g(x,u_i,w) + q_{k+1}(f(x,u_i,w)) + 2\epsilon\}. \end{aligned}$$

  A second application of Lemma 6.2.1 implies there exists $\Delta_2 \leq \Delta_1$ such that for all sampling intervals $\Delta \leq \Delta_2$

$$\begin{aligned} &\inf_{u\in U} \sup_{w\in \mathbf{R}^n} \{g(x,u,w) + q_{k+1}(f(x,u,w)) - 3\epsilon\} \leq \widetilde{q}_k(x) \\ &\leq \inf_{u\in U} \sup_{w\in \mathbf{R}^n} \{g(x,u,w) + q_{k+1}(f(x,u,w)) + 3\epsilon\}. \end{aligned}$$

■

Corollary 6.2.5 Consider the discrete time partially observed dynamic game problem given by Equation (6.1.1) with cost given by Equation (6.1.2). If the system and cost are bounded and uniformly continuous the discretized upper value (6.2.2) converges to the true upper value (6.1.4) uniformly as the sampling interval Δ approaches zero.

For implementation the state space X, the control space U, and the disturbance space W, are also truncated to compact sets centered at the origin, i.e., the cube $L^n \bigcap (\mathbb{R}^n)^\Delta$ where n is the dimension of the original space and L is the length of a side. Excursions from the respective truncated spaces are projected onto the boundary of the truncated space.

6.3 Examples

In each of the following examples the state space X, the control space U, and the disturbance space W are discretized uniformly with grid size $\Delta \approx 0.01$.

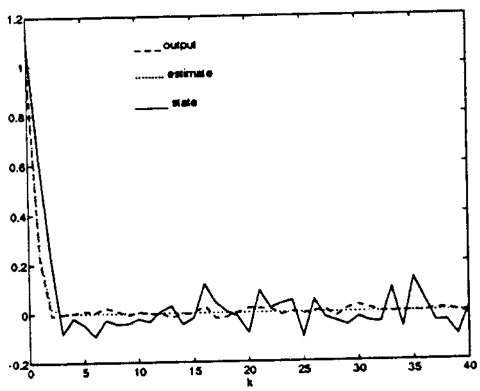

Figure 1: Performance of the CEC for Example 1 with Gaussian state and measurement noise

Example 1 Consider the following system

$$\Sigma_1 \begin{cases} x_{k+1} &= x_k^3 + x_k(x_k - 0.5)u_k + w_k, \\ y_{k+1} &= x^2 \sin(x_k) + v_k, \\ x_0 &= 1.1. \end{cases} \tag{6.3.1}$$

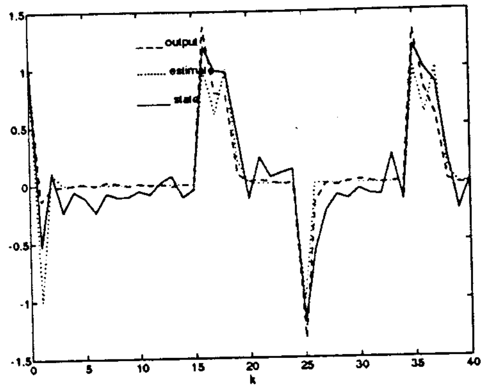

Figure 2: Performance of the CEC for Example 1 with Random jump disturbances

The nonlinear system Σ_1 of Equation (6.3.1) presents a challenging control problem since the popular technique of feedback linearization is not applicable even in the case that direct knowledge of the state is available. The problem with applying feedback linearization technique to this system is that the control required to linearize the system,

$$u_k = -\frac{x_k^2 + ax_k}{x_k - 0.5}$$

where $a \in \mathbb{R}^n$ is any positive constant, blows up at $x_k = 0.5$ and is therefore not a practical control to consider. Figure 1 demonstrates the performance of the certainty equivalence controller for this system where the state perturbations w_k and observation noise v_k have Gaussian distribution. The system is initialized so that in the case that there were no control the state trajectory would be unbounded. In the figure are plotted the state x_k, the output y_k, and the minimum stress estimate $\bar{x}_k$. It clearly illustrates that the CEC is locally stabilizing in the presence of Gaussian state and observation noise.

Figure 2 depicts the certainty equivalence control of system Σ_1 where the state and observation perturbations w_k and v_k are Gaussian with the addition of random jump disturbances. From Figure 2 it can be seen that (i) the CEC is locally stabilizing and (ii) the minimum stress estimate is able to closely track quick movements of the state.

Example 2 In this example we consider the system Σ_2, Equation (6.3.2) below, which has the same state equation as Σ_1 from Example 1. In Σ_2 we have increased the difficulty of the control problem by allowing

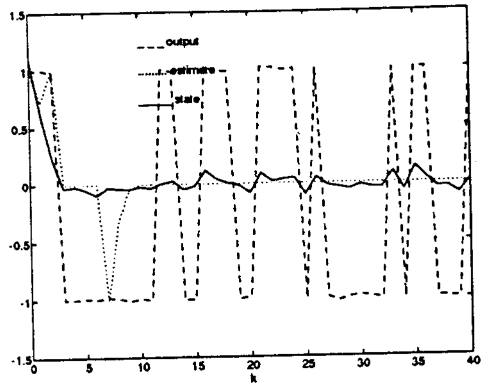

Figure 3: Performance of the CEC for Example 2 with Gaussian state and measurement noise

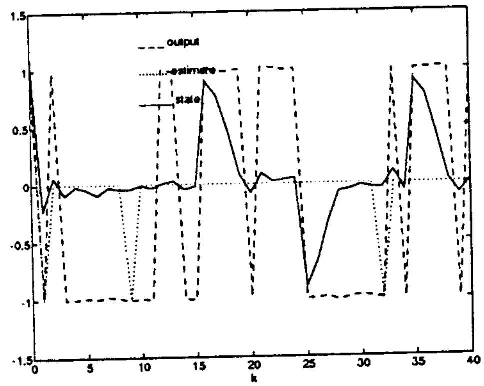

Figure 4: Performance of the CEC for Example 2 with Random jump disturbances

the controller access to only a noisy measurement of the sign of the state.

$$\Sigma_2 \begin{cases} x_{k+1} &= x_k^3 + x_k(x_k - 0.5)u_k + w_k, \\ y_{k+1} &= \text{sgn}(x_k) + v_k, \\ x_0 &= 1.1. \end{cases} \tag{6.3.2}$$

Figures 3 and 4 demonstrate the performance of the certainty equivalence controller for this system. In Figures 3 the control is performed in the presence of Gaussian state and measurement noises w_k and v_k respectively. In Figure 4 the control is complicated by the addition of random jump disturbances. In the figures are plotted the state x_k, the output y_k, and the minimum stress estimate $\bar{x}_k$. Again, the system is initialized so that in the case that there were no control the state trajectory would be unbounded. With such little information available to the controller the deterioration of the estimation performance is inevitable, yet Figures 3 and 4 provide clear evidence that the CEC is locally stabilizing in the presence of Gaussian state and observation noise and random jump disturbances.

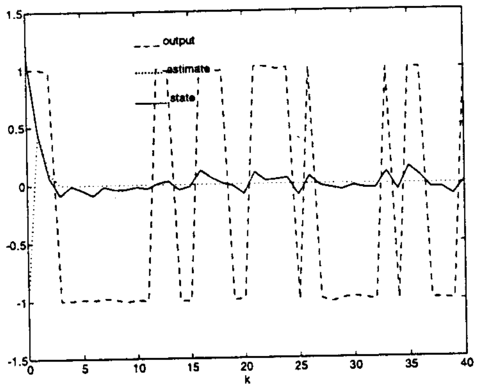

Figure 5: Performance of the CEC for Example 3 with Gamma large enough

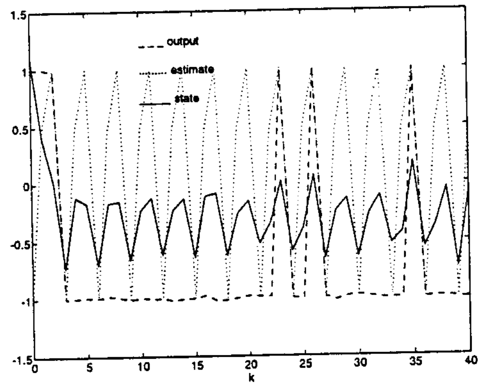

Figure 6: Performance of the CEC for Example 3 with Gamma too small

Example 3 In this example we demonstrate the importance of properly choosing the L_2 gain parameter γ. The parameter value of γ must be chosen larger than the H_∞ optimal level for a stabilizing controller to exist. No method currently exists for choosing a value of γ for which a stabilizing controller is guaranteed to exist. The choice is currently made by trial and error.

We consider the system

$$\Sigma_3 \begin{cases} x_{k+1} &= x_k^2 + u_k + w_k, \\ y_{k+1} &= \text{sgn}(x_k) + v_k, \\ x_0 &= 1.1. \end{cases} \tag{6.3.3}$$

Certainty equivalence controllers CEC_{γ_1} and CEC_{γ_2} are implemented for the system Σ_3 which employ two different values of the L_2 gain parameter γ. For each controller the system is initialized so that in the case that there were no control the state trajectory would be unbounded. In the figures are plotted the state x_k, the output y_k, and the minimum stress estimate $\bar{x}_k$.

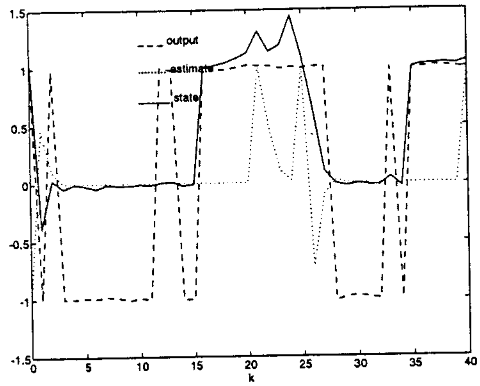

Figure 7: Performance of the CEC for Example 3 with Random jump disturbances

The controller CEC_{γ_1} is implemented with $\gamma_1 = \sqrt{10}$. For this value of L_2 gain parameter a stabilizing controller exists. Figure 5 illustrates the stabilizing certainty equivalence control of Σ_3 for $\gamma = \gamma_1$ in the presence of Gaussian state and measurement noises, w_k and v_k respectively.

The controller CEC_{γ_2} is implemented with $\gamma_2 = \sqrt{5}$. For this value of L_2 gain parameter a stabilizing controller does not exist. Figure 6 depicts the certainty equivalence control of Σ_3 for $\gamma = \gamma_2$. In this case the system is clearly not stabilized, in fact, the controller sets up oscillations of the state in a region which would be stable without control.

The system Σ_3 is a particularly interesting one from the point of view of output feedback control. The only information available to the controller is a noisy measurement of the sign of the state and in this case the sign of the state is not particularly useful information for the controller since assuming no state disturbance, i.e., $\{w_k\} \equiv 0$, the control u_k which drives the state to zero would always be negative. Figure 7 depicts the certainty equivalence control of Σ_3 for $\gamma = \gamma_1$ with the addition of random jump disturbances. The performance of the controler is a little sluggish, however, it is interesting that even with such little information the controller is able to prevent total instability.

Chapter 7

Conclusions and Future Directions

We have in this dissertation addressed the problem of determining H_∞ optimal output feedback controllers for a large class of partially observed nonlinear systems. This study has resulted in fundamental contributions to the field of nonlinear control in three broad areas: (i) theoretical development, (ii) digital implementation, and (iii) the construction of examples of both continuous and discrete time systems which validate the utility of our discretization and implementation approach. In the paragraphs below we summarize the major aspects of each these three areas. These are followed by a clear and concise enumeration of the major achievements resulting from this research and a list of a few of the ways in which this research may be continued.

Theory For the case of discrete time systems, James and Baras [JB94b] have presented an approach to the solution of the H_∞ output feedback robust control problem. We have in this dissertation formally extended their results to the continuous time case. In particular we have given necessary and sufficient conditions which establish the equivalence of a solution to the output feedback problem and a solution to a dissipation inequality which in turn yields an information state controller for continuous time systems. Further, we have provided a clear and concise proof which demonstrates the optimality of the Certainty Equivalence Controller (CEC) for systems with finite dimensional information state under the assumptions of (i) existence of a unique minimum stress estimate, and (ii) existence of a continuously differentiable information state and state feedback value function. A major advantage of our proof is that it clearly identifies the relationship which exists between the information state feedback and certainty equivalence controls.

Implementation With an eye toward digital implementation, our approach to implementing H_∞ output feedback controllers for continuous time systems has emphasized issues of paramount practical importance including (i) reasonable computational complexity of control, (ii) knowledge of observable quantities only, and (iii) implementability in finite time. Our approach depends fundamentally on the *information state* which allows the translation of the partially observed problem to that of an equivalent fully observed problem. The solution of the new fully observed problem is given in terms of the value function which is a function of the information state. Both the information state and the value function are in general complicated functions which are described by partial differential equations/inequalities. Our key contribution in the area of implementation lies in the discretization of the partial differential inequality which describes the value function (which in turn describes the optimal H_∞ control). In addition, we have provided several approximation techniques designed to decrease the on line computational burden for the continuous time information state controller and certainty equivalence controller. For discrete time systems, we have provided a state space discretization of the recursions which describe the full state information value function and the information state for the discrete time CEC.

Examples We have provided examples of continuous and discrete time systems in order to investigate the robustness properties of the H_∞ controllers which we have developed. Through direct simulation we have demonstrated that the information state controller, the certainty equivalence controller, and their approximations are all stabilizing and robust to noise for a wide variety of nonlinear systems. In addition, the discrete time CEC has been observed to be locally stabilizing and robust to noise for systems with even a large degree of nonlinearity.

Stated in the following list are the main contributions of this research:

- A formal extension to continuous of the discrete time results of James and Baras [JB94b] which provide necessary and sufficient conditions for obtaining a control which solves the robust H_∞ output feedback problem.
- A proof of the optimality of the Certainty Equivalence Controller (CEC) for systems with finite dimensional information state under the assumptions of (i) existence of a unique minimum stress estimate, and (ii) existence of a continuously differentiable information state and state feedback value function. This proof of the optimality of the CEC also clarifies the relation between the CEC and the information state feedback control. It shows that under the assumptions of the Certainty Equivalence Principle the function W_{CE}, from which the minimum stress estimate is obtained, is a solution of the dynamic programming equation for the Finite Time Robust H_∞ Output Feedback Problem.

- An extension of the numerical methods of [JY93] to apply to the discretization of the partial differential inequality which describes the value function (and in turn describes the optimal H_∞ control) for Robust H_∞ Output Feedback Problem in continuous time.
- A finite dimensional quadratic approximation of the information state controller. This approximation permits the implementation of the information state controller for general nonlinear systems and allows a faster implementation of the CEC due to the reduction of the on line computational complexity. In addition, by approximating the full state feedback value function by a quadratic, an approximation of the CEC is obtained which resembles an extended Kalman filter.
- Examples for continuous time systems which demonstrate that the information state controller, the CEC, and the approximations of these controllers are stabilizing and robust to noise for interesting nonlinear systems.
- State space discretizations of the recursions which describe the full state information value function and the information state for the discrete time CEC. Proofs of the convergence of the discretized recursions to the original recursions as the discretization parameter approaches zero are provided.
- Examples for discrete time systems which demonstrate that the discrete time CEC is locally stabilizing and robust to noise for highly nonlinear systems

There are numerous ways in which the research presented in this dissertation may be continued. We indicate a partial list of those here:

- Provide a rigorous extensions of the discrete time results of [JB94b] to the case of continuous time systems.
- Presently the value of γ is typically chosen in a trial and error fashion. We would like to develop mathematically well founded methods for choosing the L^2 gain parameter γ in the suboptimal case. In this vain, it is desirable to determine conditions under which the solvability of the fully observed *state* feedback problem for a given γ assures the solvability of the partially observed problem for a related γ. Specifically, given that we know that a solution to the full state feedback problem exists for a particular $\gamma = \gamma_{\textbf{state feedback}}$ is it possible to show that under certain conditions there exists a $\gamma_{\textbf{output feedback}} \geq \gamma_{\textbf{state feedback}}$ such that the Robust H_∞ Output Feedback Problem is solvable?
- Although we have developed our theory and implementations for the case of the H_∞ norm it is possible to derive similar results for other kinds of norms or even arbitrary costs using exactly the same framework. This quality gives the research presented here great extensibility since it may prove

useful in areas of optimal control theory which do not yet even exist. Thus, a rather broad area of future development lies in the exploration of alternative norms and/or cost functionals for optimal control in senses other than H_∞.

- As with the continuous time information state control, implementation of the discrete time information state controller would require preliminary approximations in the case that the information state is not finite dimensional before the discretization of the recursions can be applied. For continuous time systems we described two possible preliminary approximations: (i) approximation of the information state by a finite dimensional quantity, or (ii) application of the CEC. For discrete time systems, finding conditions in which the information state is finite dimensional remains an open problem. In addition, for general nonlinear systems for which the finite dimensional information state conditions are not satisfied, finding an approximation of the information state by a finite dimensional quantity is also an open problem. Providing an approximation of the discrete time information state by a finite dimensional quantity would permit the implementation of the discrete time information state controller for more general nonlinear systems and allow a faster implementation for the discrete time CEC.
- Test the digital controllers on more complex and/or realistic systems, e.g., high dimensional and systems modeling real chemical processes.

Bibliography

[BB91] T. Başar and P. Bernhard, *H_∞ Optimal Control and Related Minimax Design Problems*, Birkhauser, Boston, 1991.

[BH89a] J.A. Ball and J.W. Helton, H_∞ control for nonlinear plants: Connections with differential games, In *Proc. 28th IEEE Conference on Decision and Control, Tampa, FL*, pages 956–962, December 1989.

[BH89b] D. Bernstein and P. Haddad, LQG control with an H_∞ performance bound: A riccati equation approach, *IEEE Transactions on Automatic Control*, 34:293–305, 1989.

[BHW91] J.A. Ball, J.W. Helton, and M.L. Walker, H_∞ control for nonlinear systems with output feedback, *Preprint*, 1991.

[BvS85] A. Bensoussan and J.H. van Schuppen, Optimal control of partially observable stochastic systems with an exponential-of-integral performance index, *SIAM J. Control and Optimization*, 23:599–613, 1985.

[Byr92] C.I. Byrnes, On the Ricatti differential equation for nonlinear Bolza and Lagrange problems, Preprint, Department of Systems Science and Mathematics, Washington University, 1992.

[CEL84] M.G. Crandal, L.C. Evans, and P.L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Transactions of the American Mathematical Society*, 282:487–502, April 1984.

[CJ92] M. Campi and M.R. James, Discrete-time nonlinear risk-sensitive control, *To appear: International Journal of Robust & Nonlinear Control*, 1992.

[CL83] M.G. Crandal and P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Transactions of the American Mathematical Society*, 277:1–42, May 1983.

[DBB93] G. Didinsky, T. Başar, and P. Bernhard, Structural properties of minimax policies for a class of differential games arising in nonlinear H_∞-control and filtering, In *IEEE Conference on Decision and Control*, 1993.

[DGKF89] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, State space solutions to standard H_2 and H_∞ control problems, *IEEE Transactions on Automatic Control*, 34:831–846, August 1989.

[Ell93] R.J. Elliott, *Viscosity Solutions and Optimal Control*, Pitman Research Notes in Mathematics, 165. Wiley, New York, 1993.

[Eva92] L.C. Evans, *Partial Differential Equations*, Manuscript, UC Berkeley, 1992.

[FM91] W.H. Fleming and W.M. McEneaney, Risk sensitive optimal control and differential games, In *Proc. Conf. on Adaptive and Stochastic Control*. Springer Verlag, 1991.

[Fra87] B.A. Francis, *A Course in H_∞ Theory*, Springer-Verlag, Berlin-New York, 1987.

[Fri71] A. Friedman, *Differential Games*, Wiley, New York, 1971.

[FS93] W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1993.

[GD88] K. Glover and J.C. Doyle, State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity, *Systems & Control Letters*, 11:167–172, 1988.

[GL93] M. Gevers and G. Li, *Parametrizations in Control, Estimation, and Filtering Problems*, Springer-Verlag, New York, 1993.

[HM76] D. Hill and P. Moylan, The stability of nonlinear dissipative systems, *IEEE Transactions on Automatic Control*, 21:377–382, October 1976.

[HM77] D. Hill and P. Moylan, Stability results for nonlinear feedback systems, *Automatica*, 13:708–711, 1977.

[HM80] D. Hill and P. Moylan, Connections between finite gain and asymptotic stability, *IEEE Transactions on Automatic Control*, 25:931–936, 1980.

[IA92] A. Isidori and A. Astolfi, Disturbance attenuation and H_∞ control via measurement feedback in nonlinear systems, *IEEE Transactions on Automatic Control*, 37:1283–1293, 1992.

[Jac73] D.H. Jacobson, Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games, *IEEE Transactions on Automatic Control*, 18:124–131, 1973.

[Jam92] M.R. James, Asymptotic analysis of nonlinear stochastic risk-sensitive control and differential games, *Math. Control, Signals, Systems*, pages 401–417, 1992.

[Jam93a] M.R. James, A nonlinear partially observed differential game with a finite-dimensional information state, *Preprint*, 1993.

[Jam93b] M.R. James, On the certainty equivalence principle and optimal control of partially observed dynamic games, *To appear: IEEE Transactions on Automatic Control*, 1993.

[Jam93c] M.R. James, A partial differential inequality for dissipative nonlinear systems, *Systems & Controls Letters*, 21:315–320, 1993.

[JB94a] M.R. James and J.S. Baras, Partially observed differential games and nonlinear H_∞ control, *Preprint*, 1994.

[JB94b] M.R. James and J.S. Baras, Robust output feedback control for discrete-time nonlinear systems, *To appear: IEEE Transactions on Automatic Control*, 1994.

[JBE93a] M.R. James, J.S. Baras, and R.J. Elliott, Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems, In *IEEE Conference on Decision and Control*, 1993.

[JBE93b] M.R. James, J.S. Baras, and R.J. Elliott, Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems, Technical Report 93-24, Institute for Systems Research, University of Maryland, College Park, MD, 1993.

[JBE94] M.R. James, J.S. Baras, and R.J. Elliott, Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems, *To appear: IEEE Transactions on Automatic Control*, 1994.

[Joh82] F. John, *Partial Differential Equations*, Springer-Verlag, New York, fourth edition, 1982.

[JY93] M.R. James and S. Yuliar, Numerical approximation of the H_∞ norm for nonlinear systems, *Submitted:Automatica*, 1993.

[KD92] H.J. Kushner and P.G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, New York, 1992.

[Kha92] H.K. Khalil, *Nonlinear Systems*, Macmillan Publishing Company, New York, 1992.

[LAKG92] D.J.N. Limebeer, B.D.O. Anderson, P.P. Khargonekar, and M. Green, A game theoretic approach to H_∞ control for time varying systems, *SIAM J. Control and Optimization*, pages 262–283, 1992.

[Luk69] D.L. Lukes, Optimal regulation of nonlinear dynamical systems, *SIAM Journal of Control*, 7:75–100, February 1969.

[MA73] P.J. Moylan and B.D.O. Anderson, Nonlinear regulator theory and an inverse optimal control problem, *IEEE Transactions on Automatic Control*, 18:460–464, October 1973.

[Roy68] H.L. Royden, *Real Analysis*, MacMillan Publishing Co., Inc., New York, second edition, 1968.

[RS89] I. Rhee and J.L. Speyer, A game theoretic controller and its relationship to H_∞ and linear-exponential-gaussian synthesis, In *Proc. 28th IEEE Conference on Decision and Control*, pages 909–915, December 1989.

[Tit87] A.L. Tits, Optimal control, Course notes, Institute for Systems Research, University of Maryland, 1987.

[TYJB94] C.A. Teolis, S.X. Yuliar, M.R. James, and J.S. Baras, Robust H_∞ output feedback control fo bilinear systems, *Submitted: IEEE Conference on Decision and Control*, 1994.

[vdS91] A.J. van der Schaft, On a state space approach to nonlinear H_∞ control, *Systems & Controls Letters*, 16:1–8, 1991.

[vdS92] A.J. van der Schaft, L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control, *IEEE Transactions on Automatic Control*, 37:770–784, June 1992.

[vdS93] A.J. van der Schaft, Nonlinear state space H_∞ control theory, In *To appear: Perspectives In Control*, 1993.

[Whi81] P. Whittle, Risk-sensitive linear/quadratic/gaussian control, *Adv. Appl. Prob.*, 13:764–777, 1981.

[Whi91] P. Whittle, A risk-sensitive maximum principle: The case of imperfect state observation, *IEEE Transactions on Automatic Control*, 36:793–801, 1991.

[Wil72a] J.C. Willems, Dissipative dynamical systems part I: general theory, *Arch. Rational Mechanics and Analysis*, 45:321–351, 1972.

[Wil72b] J.C. Willems, Dissipative dynamical systems part II: linear systems with quadratic supply rates, *Arch. Rational Mechanics and Analysis*, 45:352–393, 1972.

[YJ93] S. Yuliar and M.R. James, Robust output feedback control: Linear systems, *Preprint*, 1993.

[Zam81] G. Zames, Feedback and optimal sensitivity: Model reference transforms, multiplicative semi-norms and approximate inverses, *IEEE Transactions on Automatic Control*, 26:585–601, 1981.