Paper Title:
Nonclassical Filtering Problems in Quantum Electronics

From the Proceedings:
The 1975 IEEE Decision and Control Conference
pp. 804-806

Houston, Texas
December 1975
NONCLASSICAL FILTERING PROBLEMS IN QUANTUM ELECTRONICS

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Abstract

Filtering problems arising in quantum electronics are investigated. In particular the minimum variance linear filtering problem incorporating quantum mechanical measurements is solved for both scalar and vector signal processes. The simplification of the filter under certain assumptions is demonstrated. Examples from optical communication systems illustrate the results.

Summary

With the advent of lasers detection and estimation problems in quantum electronics became of primary importance [1,2]. More recently linear filtering of a random signal sequence utilizing quantum measurements has been considered [3,4,8,9]. The basic problem we consider is the linear filtering of a random sequence \( \{x_k\} \), which influences a quantum field, based on quantum mechanical measurements. The following optical communication problem gives a concrete example. At each time \( k \) a laser modulated in some fashion by \( \{x_k\} \) is received in a cavity and a device is used to perform a measurement on the captured field. Then the cavity is cleansed and repeats to process at time \( k+1 \). We wish to select optimally the measuring device at each time along with the postprocessing scheme of past and current measurement outcomes, in order to estimate \( \{x_k\} \). This is clearly a nonclassical filtering problem.

There are two cases, depending on whether \( \{x_k\} \) is a scalar or vector process, with marked difference in the complexity of the analysis. First we describe our results for the scalar case. For details we refer to [3,4]. The formulation is as follows. The quantum state of the captured field is described by a density operator \( \rho(x_k) \) and is by which does not depend explicitly on \( k \) due to the cleansing of the cavity.

The device which measures one scalar physical quantity at time \( j \) is represented by a selfadjoint operator \( V_j \) on \( \mathcal{K} \) (an observable) [5]. The outcome \( v_j \) of this measurement is a random variable. The statistics of the outcomes \( \{v_0, \ldots, v_n\} \) are described by the joint probability measure

\[
\mu_k(\mathbf{x}_0, \ldots, \mathbf{x}_k) = \pi^{k-j} \cdot \mathrm{Tr}[\rho(\mathbf{x}_j)M(B_1, \ldots, B_k)F(\mathbf{x}_0, \ldots, \mathbf{x}_k)]
\]

where \( M_i \) is the projection valued measure associated with \( V_i \) [5], \( F(x_0, \ldots, x_k) \) is the joint distribution function of \( \{x_0, \ldots, x_k\} \) and \( B \in \mathcal{B} \), the Borel \( \sigma \)-algebra of \( \mathbb{R} \). Given the outcomes \( v_0, \ldots, v_{k-1} \) choose the measurement \( V_k \) at time \( k \) and the constants \( c_i(k) \), so that to minimize \( E |x_k - \hat{x}_k|^2 \) where

\[
\hat{x}_k = \sum_{i=0}^{k} c_i(k)v_i
\]

The MMSE estimate of \( x_k \) without postprocessing (i.e. set \( c_i(k) = 0 \), \( i=0, \ldots, k-1 \), in (2)) is given [6] by the outcome \( \hat{x}_k \). The solution to this filtering problem is \([3,4]\):

**Theorem:** The optimum observable \( \hat{V}_k \) and optimal processing coefficients \( \hat{c}_i(k), i=0,1,\ldots,k-1 \) always exist and are given as solutions of the equations:

\[
\hat{V}_k = T_k \hat{V}_k + T_{k-1} (\hat{c}_0(k)\sigma_{k0} + \cdots + \hat{c}_{k-1}(k)\sigma_{k(k-1)} + \delta_{k0})
\]

where \( \delta_{ki} = 0 \) for \( k 
eq i \) and \( \delta_{ki} = 1 \) for \( k = i \) and

\[
\sum_{j=0}^{k-1} \hat{c}_j(k)\mathrm{Tr}[\rho(x_j)C_j]\sigma_{ji} = 0, \quad j=0, \ldots, k-1
\]

where \( \sigma_{ki} = \sigma_{ki} + \delta_{ki} = 0 \) for \( k 
eq i \) and \( \sigma_{ki} = 1 \) for \( k = i \).
This general filter is not very satisfactory from practical considerations. The number of measuring devices needed for implementing the filter is large and $V_k$ may depend in a significant structural way on $k$ and on new data. In addition if $x_k$ satisfies a recursion like

$$x_{k+1} = A_k x_k + w_k$$  

where $A_k$ is a sequence of scalars and $\{w_k\}$ is a sequence of independent, Gaussian random variables with zero mean and variance $Q_k$, a recursive filter is highly desirable. Additional assumptions on the problem produce highly simplified filter structures. A crucial result in this direction is the following "separation" theorem from [3, 4]:

**Theorem 2:** Suppose the signal process $\{x_k\}$ is pairwise Gaussian, the measurements $V_i x_k$ are optimally chosen (according to Theorem 1) and that the outcomes $\tau_i$ of the measurements represented by $\hat{A}_i$ have the property that $(\tau_i, x_i)$ are jointly Gaussian for each $i$. Then the quantum mechanical linear filtered estimate of $x_k$ (i.e., $\hat{x}_k$) is equal to the classical linear MMSE estimate of $x_k$ given the random variables $\{\tau_i, i=0, 1, \ldots, k\}$.

Consider now the problem where $x_k$ satisfies a recursion like (9) and is transmitted as the (real) amplitude of a laser (assumed monochromatic) and received, along with thermal noise, in a single mode cavity. Then the optimal filter becomes

$$K = \sum_{i=0}^{N-1} P_i (N - \frac{1}{4})^{-1}, P_i = 2 \left[ 4 \left( 1 - K_{i-1} \right) \right] Q_{i-1}$$

Next we describe our results when the signal process is an $\mathbb{R}^N$ vector process. This case is more delicate and difficult because of the fundamental "compatibility" constraint in quantum mechanics. That is, only "compatible" measurements can be made simultaneously [5]. However by adjoining auxiliary apparatus to the original system [1, 7] one can perform "compatible" measurements on the augmented system which correspond satisfactorily to "non compatible" measurements on the original system. The effect of this on our problem is that instead of considering measurements as represented by projection valued measures (and thus self adjoint operators) we have to consider representations via positive operator valued measures (p.o.m.). Such measurements are called extended measurements [1]. Considering now a sequence of measurements represented by the p.o.m.'s $M_k$, we have that the probability measure characterizing the joint statistics of the vector measurement outcomes $\{v_0, \ldots, v_k\}$ is given by (I) with the appropriate changes from scalar to vector processes. The corresponding filtering problem is to find a p.o.m. $M_k$ and $N \times N$ matrices $C_i(k)$, $i=0, 1, \ldots, k$ so as to minimize $E\left[ \| x(k) - \hat{x}(k) \|^2 \right]$.

$$\hat{x}_k = \sum_{i=0}^{k} C_i(k) v_i$$

The existence of solution to this optimization problem has been established in [8, 9] and various necessary or sufficient conditions have been found. Here we have also a "separation" of the filter under additional assumptions as described in the sequel. We restrict to p.o.m.'s with a base [1], that is p.o.m.'s that can be expressed as

$$x(B) = \int_0^B P(u) \mu (du)$$

for some positive-operator valued function $P$ and a measure $\mu$ on $\mathbb{R}^N$. Let $M_i, i=0, \ldots, k$ be the optimal measurements when we use post processing (with outcomes $v_i$) and let $\hat{z}_i, i=0, \ldots, k$ be the optimal measurements (with outcomes $z_i$) when we do not use postprocessing. Then we have [8, 9]:

**Theorem 3:** Suppose that the vector signal sequence $\{x_k\}$ is pairwise Gaussian, and that $(z_i, x_i)$ are jointly Gaussian for $i=0, \ldots, k$. Then the quantum mechanical linear filtered estimate of $x_k$ is equal to the classical linear MMSE estimate of $x_k$ given the random variables $\{z_i, i=0, 1, \ldots, k\}$.

As a multiparameter recursive filtering example consider the problem of estimating the two dimensional state $x_k$ which is transmitted as the in-phase $\phi_k$ and quadrature $\psi_k$ amplitudes of a laser (assumed monochromatic) and received along with thermal noise, in a single mode cavity. Suppose that $x_k$ satisfies the vector analog of (9). Then the filter becomes [8, 9]:

$$P(k) = \sum_{i=0}^{k} P(k-1) \left[ K(k) + Q(k-1) \right]^{-1}$$

where $K(k) = P(k) \left( \begin{array}{cc} n_0 & 1 \\ 2 & 2 \end{array} \right)^{-1}$, $P(k-1) = \left( \begin{array}{cc} n_0 & 1 \\ 2 & 2 \end{array} \right)^{-1}$, $P(k) = \left( \begin{array}{cc} n_0 & 1 \\ 2 & 2 \end{array} \right)^{-1}$ and $Q(k-1) = \left( \begin{array}{cc} n_0 & 1 \\ 2 & 2 \end{array} \right)^{-1}$.
References


