A Characterization of Convex Problems in Decentralized Control*

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***Abstract*—We consider the problem of constructing optimal decentralized controllers. We formulate this problem as one of minimizing the closed-loop norm of a feedback system subject to constraints on the controller structure. We define the notion of quadratic invariance of a constraint set with respect to a system, and show that if the constraint set has this property, then the constrained minimum-norm problem may be solved via convex programming. We also show that quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback. These results are developed in a very general framework, and are shown to hold in both continuous and discrete time, for both stable and unstable systems, and for any norm. This notion unifies many previous results identifying specific tractable decentralized control problems, and delineates the largest known class of convex problems in decentralized control. As an example, we show that optimal stabilizing controllers may be efficiently computed in the case where distributed controllers can communicate faster than their dynamics propagate. We also show that symmetric synthesis is included in this classification, and provide a test for sparsity constraints to be quadratically invariant, and thus amenable to convex synthesis.**

***Index Terms*—Convex optimization, decentralized control, delayed control, extended linear spaces, networked control.**

I. INTRODUCTION

MUCH OF conventional controls analysis assumes that the controllers to be designed all have access to the same measurements. With the advent of complex systems, decentralized control has become increasingly important, where one has multiple controllers each with access to different information. Examples of such systems include flocks of aerial vehicles, autonomous automobiles on the freeway, the power distribution grid, spacecraft moving in formation, and paper machining.

In a standard controls framework, the decentralization of the system manifests itself as sparsity or delay constraints on the controller to be designed. Therefore, a canonical problem one would like to solve in decentralized control is to minimize a norm of the closed-loop map subject to a subspace constraint as follows:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S. \end{aligned}$$

For a general linear time-invariant plant P and subspace S there is no known tractable algorithm for computing the optimal K. It has been known since 1968 [26] that even the simplest versions of this problem can be extremely difficult. In fact, certain cases have been shown to be intractable [3], [11]. However, there are also several special cases of this problem for which efficient algorithms have been found [2], [6], [8], [12], [22], [23]. This paper unifies these cases and identifies a simple condition, which we call ***quadratic invariance***, under which the above problem may be recast as a convex optimization problem. The notion of quadratic invariance allows us to better understand this dichotomy between tractable and intractable optimal decentralized control problems. It further delineates the largest known class of decentralized problems for which optimal controllers may be efficiently synthesized.

Quadratic invariance is a simple algebraic condition relating the plant and the constraint set. The main results of this paper hold for continuous-time or discrete-time systems, for stable or unstable plants, and for the minimization of any norm.

In Section II, we define quadratic invariance, and present some of its characteristics. In Section III, we show that quadratic invariance is necessary and sufficient for the constraint set to be invariant under a linear fractional transformation (LFT), namely, the map from K to $K(I - GK)^{-1}$. This allows for convex synthesis of optimal controllers when the plant is stable. In Section IV, we show that for possibly unstable plants, as long as a controller exists which is both stable and stabilizing, this invariance implies that the information constraint is equivalent to affine constraints on the Youla parameter. Thus synthesizing optimal stabilizing controllers subject to quadratically invariant constraints is a convex optimization problem.

In Section V, we apply these results to specific constraint classes. We first consider a distributed control problem, and find that optimizing the closed-loop norm may be formulated as a convex optimization problem when the controllers can communicate faster than the dynamics propagate. We further show that this result still holds in the presence of computational delay.

We show that optimal synthesis of a symmetric controller for a symmetric plant is also quadratically invariant and thus amenable to convex synthesis. This is important because this

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problem, while formerly known to be solvable, defied other efforts to classify tractable problems. We develop an explicit test for the quadratic invariance of sparsity constraints, and thus show that optimal synthesis subject to such constraints which pass the test may be cast as a convex optimization problem. As a consequence of the test, we show that block diagonal constraints are never quadratically invariant unless the plant is block diagonal as well.

These results all hold for the minimization of an arbitrary norm. In Section VI we show that if the norm of interest is the $\mathcal{H}_2$-norm, then the constrained convex optimization problem derived in Section IV may be further reduced to an unconstrained convex optimization problem, and then readily solved. We then provide a numerical example.

A. Prior Work

Decentralized control has been studied from many perspectives over the past half century, and there have been many striking results which illustrate the complexity of this problem. Important early work includes that of Radner [13], who developed sufficient conditions under which minimal quadratic cost for a linear system is achieved by a linear controller. An important example was presented in 1968 by Witsenhausen [26] where it was shown that for quadratic stochastic optimal control of a linear system, subject to a decentralized information constraint called *nonclassical information*, a nonlinear controller can achieve greater performance than any linear controller. An additional consequence of the work of [10], [26] is to show that under such a nonclassical information pattern the cost function is no longer convex in the controller variables, a fact which today has increasing importance.

With the difficulty of the general problem elucidated and the myth of ubiquitous linear optimality refuted, efforts followed to classify when linear controllers were indeed optimal, to discern when finding the optimal linear controller could be cast as a convex optimization problem, and to understand the complexity of decentralized control problems. In a later paper [27], Witsenhausen summarized several important results on decentralized control at that time, and gave sufficient conditions under which the problem could be reformulated so that the standard linear quadratic Gaussian (LQG) theory could be applied. Under these conditions, an optimal decentralized controller for a linear system could be chosen to be linear. Ho and Chu [8], in the framework of *team theory*, defined a more general class of information structures, called *partially nested*, for which they showed the optimal LQG controller to be linear. Roughly speaking, a plant-controller system is called partially nested if whenever the information of controller A is affected by the decision of a controller B, then A has access to all of the information that B has.

The study of the computational complexity of decentralized control problems has shown certain problems to be intractable. Blondel and Tsitsiklis [3] showed that the problem of finding a stabilizing decentralized static output feedback is NP-complete. This is also the case for a discrete variant of Witsenhausen's counterexample [11].

For particular information structures, the controller optimization problem may have a tractable solution, and in particular, it was shown by Voulgaris [22] that the so-called *one-step delay information sharing pattern* problem has this property. In [6], the LEQG problem is solved in for this information pattern, and in [22] the $\mathcal{H}_2, \mathcal{H}_\infty$, and L_1 control synthesis problems are solved. A class of structured spatio-temporal systems has also been analyzed in [2], and shown to be reducible to a convex program. Several information structures are identified in [12] for which the problem of minimizing multiple objectives is reduced to a finite-dimensional convex optimization problem.

In this paper , we define a property called *quadratic invariance*, show that it is necessary and sufficient for the constraint set to be preserved under feedback, and that this allows optimal stabilizing decentralized controllers to be synthesized via convex programming. The tractable structures of [2], [6], [8], [12], [22], [23], and [27] can all be shown to satisfy this property.

B. Preliminaries

Given topological vector spaces $\mathcal{X}, \mathcal{Y}$, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the set of all maps $T : \mathcal{X} \rightarrow \mathcal{Y}$ such that T is linear and continuous. Note that if $\mathcal{X}, \mathcal{Y}$ are normed spaces, as in Lemma 12, then all such T are bounded, but that T may be unbounded in general. We abbreviate $\mathcal{L}(\mathcal{X}, \mathcal{X})$ with $\mathcal{L}(\mathcal{X})$.

Suppose $P \in \mathcal{L}(\mathcal{W} \times \mathcal{U}, \mathcal{Z} \times \mathcal{Y})$. Partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

so that $P_{11} : \mathcal{W} \rightarrow \mathcal{Z}, P_{12} : \mathcal{U} \rightarrow \mathcal{Z}, P_{21} : \mathcal{W} \rightarrow \mathcal{Y}$ and $P_{22} : \mathcal{U} \rightarrow \mathcal{Y}$. Suppose $K \in \mathcal{L}(\mathcal{Y}, \mathcal{U})$. If $I - P_{22}K$ is invertible, define $f(P, K) \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ by

$$f(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

The map $f(P, K)$ is called the (lower) ***linear fractional transformation*** (LFT) of P and K; we will also refer to this as the ***closed-loop map***. In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$.

Given a linear vector space $\mathcal{X}$, let $\mathcal{X}^*$ denote the dual-space of $\mathcal{X}$, let $\langle x, x^* \rangle$ denote the dual pairing of any $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$, and define

$$S^\perp = \{x^* \in \mathcal{X}^* \mid \langle x, x^* \rangle = 0, \text{ for all } x \in S\}.$$

Kronecker Products: Given $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{s \times q}$ let $A \otimes B \in \mathbb{C}^{ms \times nq}$ denote the *Kronecker product* of A and B.

Given $A \in \mathbb{C}^{m \times n}$, we may write A in term of its columns as

$$A = [a_1 \quad \cdots \quad a_n]$$

and then associate a vector $\text{vec}(A) \in \mathbb{C}^{mn}$ defined by

$$\text{vec}(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Lemma 1: Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{s \times q}, X \in \mathbb{C}^{n \times s}$. Then

$$\text{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\text{vec}(X)$$

Proof: See, for example, [9]. ■

Transfer Functions: We use the following standard notation. Denote the imaginary axis by

$$j\mathbb{R} = \{z \in \mathbb{C} \,|\, \Re(z) = 0\}$$

and the closed right half of the complex plane by

$$\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \Re(z) \geq 0\}.$$

We define transfer functions for continuous-time systems therefore determined on $j\mathbb{R}$, but we could also define transfer functions for discrete-time systems determined on the unit circle. A rational function $G : j\mathbb{R} \to \mathbb{C}$ is called ***real-rational*** if the coefficients of its numerator and denominator polynomials are real. Similarly, a matrix-valued function $G : j\mathbb{R} \to \mathbb{C}^{m\times n}$ is called real-rational if G_{ij} is real-rational for all i, j. It is called ***proper*** if

$$\lim_{\omega\to\infty} G(j\omega) \quad \text{exists and is finite}$$

and it is called ***strictly proper*** if

$$\lim_{\omega\to\infty} G(j\omega) = 0.$$

Denote by $\mathcal{R}_p^{m\times n}$ the set of matrix-valued real-rational proper transfer matrices

$$\mathcal{R}_p^{m\times n} = \{G : j\mathbb{R} \to \mathbb{C}^{m\times n} \,|\, G \text{ proper, real-rational}\}$$

and let $\mathcal{R}_{sp}^{m\times n}$ be

$$\mathcal{R}_{sp}^{m\times n} = \{G \in \mathcal{R}_p^{m\times n} \,|\, G \text{ strictly proper}\}.$$

Also, let $\mathcal{RH}_\infty$ be the set of real-rational proper stable transfer matrices

$$\mathcal{RH}_\infty^{m\times n} = \{G \in \mathcal{R}_p^{m\times n} \,|\, G \text{ has no poles in } \mathbb{C}_+\}.$$

It can be shown that functions in $\mathcal{RH}_\infty$ are determined by their values on $j\mathbb{R}$, and thus we can regard $\mathcal{RH}_\infty$ as a subspace of $\mathcal{R}_p$. If $A \in \mathcal{R}_p^{n\times n}$ we say A is ***invertible*** if $\lim_{\omega\to\infty} A(j\omega)$ is an invertible matrix and $A(j\omega)$ is invertible for almost all $\omega \in \mathbb{R}$. Note that this is different from the definition of invertibility for the associated multiplication operator on L_2. If A is invertible we write $B = A^{-1}$ if $B(j\omega) = A(j\omega)^{-1}$ for almost all $\omega \in \mathbb{R}$. Note that, if $G \in \mathcal{R}_{sp}^{n_y\times n_u}$ then $I - GK$ is invertible for all $K \in \mathcal{R}_p^{n_u\times n_y}$, which will be consistent with the definition of invertibility for operators on extended spaces.

Topology: Let $\mathcal{X}$ be a vector space and $\{\|\cdot\|_\alpha | \alpha \in I\}$ be a family of seminorms on $\mathcal{X}$. The family is called ***sufficient*** if for all $x \in \mathcal{X}$ such that $x \neq 0$ there exists $\alpha \in I$ such that $\|x\|_\alpha \neq 0$. The topology generated by all open $\|\cdot\|_\alpha$-balls is called the topology generated by the family of seminorms. Convergence in this topology is equivalent to convergence in every seminorm, and continuity of a linear operator is equivalent to continuity in every seminorm. See, for example, [20] and [31].

Extended Spaces: We introduce some new notation for extended linear spaces. These spaces are utilized extensively in [5] and [25].

We define the truncation operator P_T for all $T \in \mathbb{R}_+$ on all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f_T = P_T f$ is given by

$$f_T(t) = \begin{cases} f(t), & \text{if } t \leq T \\ 0, & \text{if } t > T \end{cases}$$

and, hereafter, abbreviate $P_T f$ as f_T. We make use of the standard L_p Banach spaces equipped with the usual p-norm, and the extended spaces

$$L_{pe} = \{f : \mathbb{R}_+ \to \mathbb{R} \mid f_T \in L_p, \text{ for all } T \in \mathbb{R}_+\}, \quad \text{for all } p \geq 1.$$

We let the topology on L_{2e} be generated by the sufficient family of seminorms $\{\|\cdot\|_T \,|\, T \in \mathbb{R}_+\}$ where $\|f\|_T = \|P_T f\|_{L_2}$, and let the topology on $\mathcal{L}(L_{2e}^m, L_{2e}^n)$ be generated by the sufficient family of seminorms $\{\|\cdot\|_T \mid T \in \mathbb{R}_+\}$ where $\|A\|_T = \|P_T A\|_{L_2^m \to L_2^n}$.

We use similar notation for discrete time. As is standard, we extend the discrete-time Banach spaces ℓ_p to the extended space

$$\ell_e = \{f : \mathbb{Z}_+ \to \mathbb{R} \,|\, f_T \in \ell_\infty, \text{ for all } T \in \mathbb{Z}_+\}.$$

Note that in discrete time, all extended spaces contain the same elements, since the common requirement is that the sequence is finite at any finite index. This motivates the abbreviated notation of ℓ_e.

We let the topology on ℓ_e be generated by the sufficient family of seminorms $\{\|\cdot\|_T \,|\, T \in \mathbb{Z}_+\}$ where $\|f\|_T = \|P_T f\|_{\ell_2}$, and let the topology on $\mathcal{L}(\ell_e^m, \ell_e^n)$ be generated by the sufficient family of seminorms $\{\|\cdot\|_T \mid T \in \mathbb{Z}_+\}$ where $\|A\|_T = \|P_T A\|_{\ell_2^m \to \ell_2^n}$.

When the dimensions are implied by context, we omit the superscripts of $\mathcal{R}_p^{m\times n}, \mathcal{R}_{sp}^{m\times n}, \mathcal{RH}_\infty^{m\times n}, L_{pe}^{m\times n}, \ell_e^{m\times n}$. We will indicate the restriction of an operator A to $L_2[0,T]$ or $\ell_e[0,T]$ by $A\,|_T$, and the restriction and truncation of an operator as $A_T = P_T A\,|_T$. Thus, for every seminorm in this paper, one may write $\|A\|_T = \|A_T\|$. Given a set of operators S, we also denote $S_T = \{P_T A\,|_T; A \in S\}$.

C. Problem Formulation

Suppose $S \subseteq \mathcal{R}_p^{n_u\times n_y}$ is a subspace. Given $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, we would like to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S. \end{aligned} \tag{1}$$

Here, $\|\cdot\|$ is any norm on $\mathcal{R}_p^{n_z\times n_w}$, chosen to encapsulate the control performance objectives, and S is a subspace of admissible controllers which encapsulates the decentralized nature of the system. The norm on $\mathcal{R}_p^{n_z\times n_w}$ may be either a deterministic measure of performance, such as the induced norm, or a stochastic measure of performance, such as the $\mathcal{H}_2$ norm. Many decentralized control problems may be formulated in this form, and some examples are shown below. We call the subspace S the ***information constraint***.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved by a simple change of variables, as discussed in Section IV-B. For specific norms, the problem may also be solved using a state–space approach. Note that the cost function $\|f(P,K)\|$ is in general a nonconvex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

II. Quadratic Invariance

We now turn to the main focus of this paper, which is the characterization of constraint sets S that lead to tractable solutions for problem (1).

Definition 2: Suppose $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$. The set S is called ***quadratically invariant*** under G if

$$KGK \in S, \qquad \text{for all } K \in S$$

Note that, given G, we can define a quadratic map $\Psi : \mathcal{L}(\mathcal{Y},\mathcal{U}) \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is $\Psi(S) \subseteq S$.

Definition 3: Given a constraint set $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$, we define a complementary set $S^\star \subseteq \mathcal{L}(\mathcal{U},\mathcal{Y})$ by

$$S^\star = \{G \in \mathcal{L}(\mathcal{U},\mathcal{Y}) \,|\, S \text{ is quadratically invariant under } G\}.$$

Theorem 4: If S is a subspace, $S^\star$ is quadratically invariant under K for all $K \in S$.

Proof: Suppose $K_1, K_2 \in S$ and $G \in S^\star$. First note that

$$K_1GK_2 + K_2GK_1 = (K_1+K_2)G(K_1+K_2) - K_1GK_1 - K_2GK_2$$

and since all terms on the right hand side of this equation are in S, we have $K_1GK_2 + K_2GK_1 \in S$. Then, we have

$$\begin{aligned} 2K_2GK_1GK_2 &= (K_2 + K_1GK_2 + K_2GK_1)G(K_2 + K_1GK_2 + K_2GK_1) \\ &\quad - (K_1GK_2 + K_2GK_1)G(K_1GK_2 + K_2GK_1) - K_2GK_2 \\ &\quad + (K_1 - K_2GK_2)G(K_1 - K_2GK_2) - K_1GK_1 \end{aligned}$$

and since all terms on the right-hand side of this equation are in S, we have $K_2GK_1GK_2 \in S$ for all $K_1, K_2 \in S$ and for all $G \in S^\star$. This implies $GK_1G \in S^\star$ for all $K_1 \in S$ and for all $G \in S^\star$, and the desired result follows. ■

This tells us that the complementary set is quadratically invariant under any element of the constraint set, which will be very useful in proving the main result of Section IV.

We give another general lemma on quadratic invariance which will be useful throughout the remainder of this paper.

Lemma 5: Suppose $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$ is a subspace. If S is quadratically invariant under G, then

$$K(GK)^n \in S, \qquad \text{for all } K \in S,\ n \in \mathbb{Z}_+.$$

Proof: We prove this by induction. By assumption, given $K \in S$, we have that $KGK \in S$. For the induction step, assume that $K(GK)^n \in S$ for some $n \in \mathbb{Z}_+$. Then

$$2K(GK)^{n+1} = (K + K(GK)^n)G(K + K(GK)^n) - KGK - K(GK)^{2n+1}$$

and since all terms on the right-hand side of this equation are in S, we have $K(GK)^{n+1} \in S$. ■

III. Invariance Under Feedback

In this section, we focus on time-invariant causal operators and show that quadratic invariance is necessary and sufficient for the constraint set to be invariant under a linear fractional transformation, namely, the map from K to $K(I-GK)^{-1}$.

We define the map $h : \mathcal{L}(\mathcal{U},\mathcal{Y}) \times \mathcal{L}(\mathcal{Y},\mathcal{U}) \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ by

$$h(G,K) = -K(I-GK)^{-1}$$

for all G, K such that $I - GK$ is invertible. We will also make use of the notation $h_G(K) = h(G,K)$, which is then defined for all $K \in \mathcal{L}(\mathcal{Y},\mathcal{U})$ such that $I - GK$ is invertible. Given $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, we note that h_G is an involution on this set, as a straightforward calculation shows that $h_G(h_G(K)) = K$.

From Lemma 5, we see that if we could express $K(I - GK)^{-1}$ as $\sum_{n=0}^{\infty} K(GK)^n$, then this mapping would lie in S for all $K \in S$, provided that S was a closed subspace. Noting that K can be pulled outside of the sum due to continuity, we thus seek conditions under which $\sum_{n=0}^{\infty}(GK)^n$ converges. We would like to be able to utilize this expansion not just for small K, as in the small gain theorem, but for arbitrarily large K as well. We consider the plant and controller as operators on extended spaces both because that will allow us to achieve this, and also so that unstable operators may be considered.

In Section III-A, we develop conditions under which this Neumann series is guaranteed to converge in the topologies defined in Section I-B. These topologies were first utilized in [17]. We then prove in Section III-B that under very broad assumptions quadratic invariance in necessary and sufficient for the constraint set to be preserved under feedback. These conditions include, but are not limited to, the case we are often interested in where $G \in \mathcal{R}_{sp}$ and $S \subseteq \mathcal{R}_p$.

A. Convergence of Neumann Series

We first analyze convergence of the *Neumann series*

$$(I-W)^{-1} = \sum_{n=0}^{\infty} W^n$$

where W is a general causal linear operator on extended spaces. Note that while most of the results in this paper have analogs in both continuous-time and discrete-time, the proofs in these cases are different. We first analyze the continuous-time case, and begin by providing a preliminary lemma which states that if a sequence of impulse responses converge in a particular sense, then their associated operators do as well.

Lemma 6: Suppose $W_n \in \mathcal{L}(L_{2e}^m)$ is causal and time-invariant for all $n \in \mathbb{Z}_+, w^{(n)} \in L_{\infty e}$ is the impulse response of $W_n, a \in L_{\infty e}$ and $(w^{(n)})_T$ converges uniformly to a_T as $n \to \infty$ for all $T \in \mathbb{R}_+$. Then W_n converges to $A \in \mathcal{L}(L_{2e}^m)$, where A is given by $Au = a * u$.

Proof: Given $u \in L_{2e}^m$ and $T \in \mathbb{R}_+$

$$(a*u)_T = (a_T * u_T)_T$$

since $a(t)=0$ and $u(t)=0$ for $t<0$. Hence, $(a*u)_T \in L_2$, since $a_T \in L_1$ and $u_T \in L_2$, by [21, Th. 65]. Therefore, we can define $A \in \mathcal{L}(L_{2e}^m)$ by $Au = a*u$.

For any $n \in \mathbb{Z}_+$ and any $T \in \mathbb{R}_+$

$$\begin{aligned}\|A-W_n\|_T^2 &= \sup_{u\in L_2,\|u\|_2=1} \|P_T Au - P_T W_n u\|_2^2 \\ &\leq \sup_{u\in L_2,\|u\|_2=1} \left\|\left(a_T - \left(w^{(n)}\right)_T\right) * u\right\|_2^2 \\ &\leq \sup_{u\in L_2,\|u\|_2=1} \sum_{i=1}^m \sum_{j=1}^m \left\|\left(a_T - \left(w^{(n)}\right)_T\right)_{ij}\right\|_1^2 \|u_j\|_2^2\end{aligned}$$

and hence

$$\|A-W_n\|_T^2 \leq \sum_{i=1}^m \sum_{j=1}^m \left\|\left(a_T - \left(w^{(n)}\right)_T\right)_{ij}\right\|_1^2.$$

Since $(w_{ij}^{(n)})_T$ converges uniformly to a_T, for any $\epsilon > 0$ we can choose N such that for all $n \geq N$ and for all $i,j = 1,\ldots,m, |(a_{ij})_T(t) - (w_{ij}^{(n)})_T(t)| < \epsilon/(mT)$ for all $t \in [0,T]$ and, thus, $\|A-W_n\|_T < \epsilon$. So W_n converges to A in $\mathcal{L}(L_{2e}^m)$. ■

We can now prove convergence of the Neumann series under the given conditions by showing the convergence of impulse responses. The method for showing this is similar to that used for spatio-temporal systems in [2, App.].

Theorem 7: Suppose $W \in \mathcal{L}(L_{2e}^m)$ is causal and time-invariant with impulse response matrix w such that $w \in L_{\infty e}$. Then $\sum_{n=0}^\infty W^n$ converges to an element $B \in \mathcal{L}(L_{2e}^m)$ such that $B = (I-W)^{-1}$.

Proof: Let $q(T) = \sup_{t\in[0,T]} \|w(t)\| < \infty$ for all $T \in \mathbb{R}_+$, and let $w^{(n)}$ be the impulse response matrix of W^n. First we claim that $\|w^{(n)}(T)\| \leq (T^{n-1}/(n-1)!)q(T)^n$ for all integers $n \geq 1$. This is true immediately for $n=1$. For the inductive step

$$\begin{aligned}\left\|w^{(n+1)}(T)\right\| &= \left\|\int_{t=0}^T w(T-t)w^{(n)}(t)\,dt\right\| \\ &\leq \int_{t=0}^T \|w(T-t)\|\cdot\|w^{(n)}(t)\|\,dt \\ &\leq q(T)\int_{t=0}^T \|w^{(n)}(t)\|\,dt \\ &\leq q(T)\int_{t=0}^T \frac{t^{n-1}}{(n-1)!}q(t)^n\,dt \\ &\leq \frac{T^n}{n!}q(T)^{n+1}.\end{aligned}$$

Then $|w_{ij}^{(n)}(t)| \leq (T^{n-1}/(n-1)!)q(T)^n$ for all $t \in [0,T]$, for all $n \geq 1$, and for all $i,j = 1,\ldots,m$. $\sum_{n=1}^\infty (T^{n-1}/(n-1)!)q(T)^n$ converges to $q(T)e^{Tq(T)}$, so by the Weierstrass M-test, $\sum_{n=1}^\infty (w_{ij}^{(n)})_T$ converges uniformly and absolutely for all $i,j=1,\ldots,m$. ■

Let $a = \sum_{n=1}^\infty w^{(n)}$. Then $a_{ij} \in L_{\infty e} \subseteq L_{1e}$ for all $i,j = 1,\ldots,m$, and we can define $A, B \in \mathcal{L}(L_{2e}^m)$ by $Au = a*u$ and $B = I + A$.

Then, by Lemma 6, $\sum_{k=1}^n W^k$ converges to A in $\mathcal{L}(L_{2e}^m)$, and thus $\sum_{k=0}^n W^k$ converges to B in $\mathcal{L}(L_{2e}^m)$.

Finally,

$$\begin{aligned}B(I-W) &= (I-W)B \\ &= \sum_{n=0}^\infty W^n - \sum_{n=1}^\infty W^n = I.\end{aligned}$$

■

A simple example of the utility of this result is as follows. Consider W represented by the transfer function $2/(s+1)$. Then, $I - W = (s-1)/(s+1)$ is not invertible in $\mathcal{L}(L_2)$. However, using the previous theorem, the inverse in $\mathcal{L}(L_{2e})$ is given by $\sum_{n=0}^\infty (2/(s+1))^n = (s+1)/(s-1)$.

We now move on to analyze the discrete-time case. Let $r(\,\cdot\,)$ denote spectral radius.

Theorem 8: Suppose $W \in \mathcal{L}(\ell_e^m)$ is causal and time-invariant with impulse response matrix w such that $w \in \ell_e$ and $r(w(0)) < 1$. Then $\sum_{n=0}^\infty W^n$ converges to an element $B \in \mathcal{L}(\ell_e^m)$ such that $B = (I-W)^{-1}$.

Proof: We may represent $P_T W|_T$ with the block lower triangular Toeplitz matrix

$$W_T = \begin{bmatrix} w(0) & & \\ w(1) & \ddots & \\ \vdots & & \\ w(T) & \cdots & w(0)\end{bmatrix}$$

Since $w \in \ell_e, W_T \in \mathbb{R}^{mT\times mT}$. Then, $r(W_T) = r(w(0)) < 1$, which implies that $\sum_{n=0}^\infty (W_T)^n$ converges in $\mathbb{R}^{mT\times mT}$. Thus we can define $B \in \mathcal{L}(\ell_e^m)$ by $(Bu)_T = (\sum_{n=0}^\infty (W_T)^n)u_T$ for any $u \in \ell_e^m$ and any $T \in \mathbb{Z}_+$. It is then immediate that $\|B - \sum_{n=0}^\infty W^n\|_T \to 0$ as $n \to \infty$ for all T, and thus $\sum_{n=0}^\infty W^n$ converges to B in $\mathcal{L}(\ell_e^m)$.

Finally,

$$\begin{aligned}B(I-W) &= (I-W)B \\ &= \sum_{n=0}^\infty W^n - \sum_{n=1}^\infty W^n = I.\end{aligned}$$

■

Note that while the conditions of Theorem 8 are necessary for convergence as well as sufficient, the conditions of Theorem 7 are not.

In particular, the aforementioned results imply the following corollaries, which show convergence of the Neumann series for strictly proper systems, possibly with delay.

Corollary 9: Suppose $W \in \mathcal{L}(L_{2e}^m)$ is given by $W_{ij} = D_{\tau_{ij}} G_{ij}$ where $\tau_{ij} \geq 0$ and $G_{ij} \in \mathcal{R}_{sp}$. Then, $\sum_{n=0}^\infty W^n$ converges to an element $B \in \mathcal{L}(L_{2e}^m)$ such that $B = (I-W)^{-1}$.

Corollary 10: Suppose $W \in \mathcal{L}(\ell_e^m)$ is given by $W_{ij} \in \mathcal{R}_{sp}$. Then, $\sum_{n=0}^\infty W^n$ converges to an element $B \in \mathcal{L}(\ell_e^m)$ such that $B = (I-W)^{-1}$.

B. LFT Invariance

This subsection contains the main technical results of this paper. In particular, we show that for a broad class of systems, quadratic invariance is necessary and sufficient for the information constraint S to be invariant under an important LFT.

We first state two lemmas which will help with the converse of our main result.

Lemma 11: Suppose $S \subseteq \mathcal{L}(L_{2e}^m, L_{2e}^n)$ or $S \subseteq \mathcal{L}(\ell_e^m, \ell_e^n)$, and $C \notin S$. Then there exists T such that $C_T \notin S_T$.

Proof: Suppose not. Then for every positive $T, C_T \in S_T$. Thus for every T, there exists $K \in S$ such that $P_T C|_T = P_T K|_T$, or $\|C - K\|_T = 0$. Since $\|A\|_T = 0$ only if $\|A\|_\tau = 0$ for all $\tau \leq T$, it follows that there exists $K \in S$ such that $\|C - K\|_T = 0$ for all T. However, then $C - K = 0$, and so $C \in S$ and we have a contradiction. ∎

Lemma 12: Suppose $\mathcal{U}, \mathcal{Y}$ are Banach spaces, $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y}), S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a closed subspace, and S is not quadratically invariant under G. Then there exists $K \in S$ such that $I - GK$ is invertible and $h_G(K) \notin S$.

Proof: There exists $K_0 \in S$ such that $K_0 G K_0 \notin S$. We will construct $K \in S$ such that $h_G(K) \notin S$. Without loss of generality we may assume $\|K_0\| = 1$. Choose $\Gamma \in S^\perp$ with $\|\Gamma\| = 1$ such that

$$\beta = \langle K_0 G K_0, \Gamma \rangle \in \mathbb{R} \quad \text{and} \quad \beta > 0$$

and choose $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \frac{\beta}{\|G\|(\beta + \|G\|)}.$$

Let $K = \alpha K_0$. Then $\|GK\| < 1, K \in S$, and

$$\langle K(I - GK)^{-1}, \Gamma \rangle = \sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma \rangle.$$

Thus

$$\begin{aligned} |\langle K(I - GK)^{-1}, \Gamma \rangle| &= \left| \sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma \rangle \right| \\ &= \left| \alpha^2 \beta + \sum_{i=2}^{\infty} \langle K(GK)^i, \Gamma \rangle \right| \\ &\geq \alpha^2 \beta - \alpha \sum_{i=2}^{\infty} \|G\|^i \alpha^i \\ &= \alpha^2 \left(\frac{\beta - \alpha \|G\|(\beta + \|G\|)}{1 - \alpha \|G\|} \right) \\ &> 0. \end{aligned}$$

Hence, $K(I - GK)^{-1} \notin S$ as required. ∎

We define a broad class of sets of controllers for which the closed-loop map will always be well-defined. Note that this includes the case which is often of interest where $G \in \mathcal{R}_{sp}$ and $S \subseteq \mathcal{R}_p$.

Definition 13: We say that $S \subseteq \mathcal{L}(L_{2e}^{n_u}, L_{2e}^{n_y})$ is ***inert*** with respect to G if for all $K \in S, (gk)_{ij} \in L_{\infty e}$ for all $i, j = 1, \ldots, m$ where (gk) is the impulse response matrix of GK. We overload our notation and also define $S \subseteq \mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ to be ***inert*** if for all $K \in S, (gk)_{ij} \in \ell_e$ for all $i, j = 1, \ldots, m$ and $r((gk)(0)) < 1$ where (gk) is the discrete impulse response matrix of GK.

Main Result 1: The following theorem is the main result of this section. It states that quadratic invariance of the constraint set is necessary and sufficient for the set to be invariant under the LFT defined by h_G.

Theorem 14: Suppose $G \in \mathcal{L}(L_{2e}^{n_u}, L_{2e}^{n_y})$ or $G \in \mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$, and S is an inert closed subspace. Then

$$S \text{ is quadratically invariant under } G \iff h_G(S) = S.$$

Proof: ($\Rightarrow$) Suppose $K \in S$. We first show that $h_G(K) \in S$.

$$\begin{aligned} K(I - GK)^{-1} &= K \sum_{n=0}^{\infty} (GK)^n \\ &= \sum_{n=0}^{\infty} K(GK)^n \end{aligned}$$

where the first equality follows from Theorems 7 and 8 and the second follows from the continuity of K.

By Lemma 5, we have $K(GK)^n \in S$ for all $n \in \mathbb{Z}_+$, and hence $K(I - GK)^{-1} \in S$ since S is a closed subspace.

So $K \in S \Rightarrow h_G(K) \in S$. Thus $h_G(S) \subseteq S$, and since h_G is involutive it follows that $h_G(S) = S$, which was the desired result.

($\Leftarrow$) We now turn to the converse of this result. Suppose that S is not quadratically invariant under G. Then there exists $K \in S$ such that $KGK \notin S$, and thus by Lemma 11, there exists a finite T such that $P_T KGK|_T \notin S_T$. Since K and G are causal, we then have

$$\begin{aligned} &K_T G_T K_T \notin S_T \quad \text{where} \\ &K_T = P_T K P_T \in S_T \quad \text{and} \quad G_T = P_T G P_T \end{aligned}$$

and, thus, S_T is not quadratically invariant under G_T. Then, by Lemma 12, there exists $\tilde{K} \in S_T$ such that

$$\tilde{K}(I - G_T \tilde{K})^{-1} = \sum_{n=0}^{\infty} \tilde{K}(G_T \tilde{K})^n \notin S_T.$$

By definition of S_T, there exists $K_0 \in S$ such that $\tilde{K} = P_T K_0|_T$. Then, by causality of K_0 and G

$$P_T \left(\sum_{n=0}^{\infty} K_0 (G K_0)^n \right) \Bigg|_T \notin S_T$$

and, thus, $h_G(K_0) = -\sum_{n=0}^{\infty} K_0 (GK_0)^n \notin S$. ∎

IV. Optimal Stabilizing Controllers

In this section, we address the problem posed in Section I-C; finding optimal stabilizing controllers subject to an information constraint. These results apply both to continuous-time and discrete-time systems. Note that throughout this section, the constraint set S is always inert, since $G \in \mathcal{R}_{sp}$ and $S \subseteq \mathcal{R}_p$.

There have been several key results regarding controller parameterization and optimization which we will extend for decentralized control, relying heavily on our result from the previous section. The celebrated Youla parameterization [29] showed that given a coprime factorization of the plant, one may

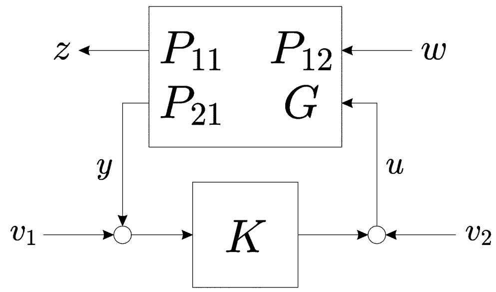

Fig. 1. Linear fractional interconnection of P and K.

parameterize all stabilizing controllers. The set of closed-loop maps achievable with stabilizing controllers is then affine in this parameter, an important result which converts the problem of finding the optimal stabilizing controller to a convex optimization problem, given the factorization. Zames proposed a two-step compensation scheme [30] for strongly stabilizable plants, that is, plants which can be stabilized with a stable compensator. In the first step, one finds any controller which is both stable and stabilizing, and in the second one optimizes over a parameterized family of systems. This idea has been extended to nonlinear control [1], and in this section we show that it may be extended to decentralized control when the constraint set is quadratically invariant, as first shown in [16].

Our approach starts with a single decentralized controller which is both stable and stabilizing, and uses it to parameterize all stabilizing decentralized controllers. The resulting parameterization expresses the closed-loop system as an affine function of a stable parameter, allowing the next step, optimization of closed-loop performance, to be achieved with convex programming. Techniques for finding an initial stabilizing controller for decentralized systems are discussed in detail in [19], and conditions for decentralized stabilizability were developed in [24].

A. Stabilization

We say that K ***stabilizes*** P if in Fig. 1 the nine transfer matrices from w, v_1, v_2 to z, u, y belong to $\mathcal{RH}_\infty$. We say that K ***stabilizes*** G if in the figure the four transfer matrices from v_1, v_2 to u, y belong to $\mathcal{RH}_\infty$. P is called ***stabilizable*** if there exists $K \in \mathcal{R}_p^{n_u \times n_y}$ such that K stabilizes P, and it is called is called ***strongly stabilizable*** if there exists $K \in \mathcal{RH}_\infty^{n_u \times n_y}$ such that K stabilizes P. We denote by $C_{\text{stab}} \subseteq \mathcal{R}_p^{n_u \times n_y}$ the set of controllers $K \in \mathcal{R}_p^{n_u \times n_y}$ which stabilize P. The following standard result relates stabilization of P with stabilization of G.

Theorem 15: Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and suppose P is stabilizable. Then, K stabilizes P if and only if K stabilizes G.

Proof: See, for example, [7, Ch. 4]. ■

B. Parameterization of Stabilizing Controllers

In this section, we review one well-known approach to solution of the feedback optimization problem (1) when the constraint that K lie in S is not present. In this case, one may use the following standard change of variables.

For a given system P, all controllers that stabilize the system may be parameterized using the well-known Youla parameterization [29], stated below.

Theorem 16: Suppose we have a doubly coprime factorization of G over $\mathcal{RH}_\infty$, that is, $M_l, N_l, X_l, Y_l, M_r, N_r, X_r, Y_r \in \mathcal{RH}_\infty$ such that $G = N_r M_r^{-1} = M_l^{-1} N_l$ and

$$\begin{bmatrix} X_l & -Y_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_r \\ N_r & X_r \end{bmatrix} = I.$$

Then the set C_{stab} of all controllers in $\mathcal{R}_p$ which stabilize G is

$$C_{\text{stab}} = \{(Y_r - M_r Q)(X_r - N_r Q)^{-1} \mid X_r - N_r Q \text{ is invertible}, Q \in \mathcal{RH}_\infty\}.$$

Furthermore, the set of all closed-loop maps achievable with stabilizing controllers is

$$\begin{aligned} &\{f(P,K) \mid K \in \mathcal{R}_p, K \text{ stabilizes } P\} \\ &= \{T_1 - T_2 Q T_3 \mid X_r - N_r Q \text{ is invertible}, Q \in \mathcal{RH}_\infty\} \end{aligned} \quad (2)$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by

$$\begin{aligned} T_1 &= P_{11} + P_{12} Y_r M_l P_{21} \\ T_2 &= P_{12} M_r \\ T_3 &= M_l P_{21} \end{aligned}$$

Proof: See, for example, [7, Ch. 4]. ■

This parameterization is particularly simple to construct in the case where we have a nominal stabilizing controller $K_{\text{nom}} \in \mathcal{RH}_\infty$; that is, a controller that is both stable and stabilizing.

Theorem 17: Suppose G is strictly proper, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty$. Then, all stabilizing controllers are given by

$$C_{\text{stab}} = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \,\middle|\, Q \in \mathcal{RH}_\infty \right\}$$

and all closed-loop maps are given by (2) where

$$\begin{aligned} T_1 &= P_{11} + P_{12} K_{\text{nom}} (I - G K_{\text{nom}})^{-1} P_{21} \\ T_2 &= -P_{12} (I - K_{\text{nom}} G)^{-1} \\ T_3 &= (I - G K_{\text{nom}})^{-1} P_{21}. \end{aligned} \quad (3)$$

Proof: A doubly coprime factorization for G over $\mathcal{RH}_\infty$ is given by

$$\begin{aligned} M_l &= (I - G K_{\text{nom}})^{-1} & M_r &= -(I - K_{\text{nom}} G)^{-1} \\ N_l &= G (I - K_{\text{nom}} G)^{-1} & N_r &= -G (I - K_{\text{nom}} G)^{-1} \\ X_l &= -I \quad Y_l = -K_{\text{nom}} & X_r &= I \quad Y_r = K_{\text{nom}}. \end{aligned}$$

Then

$$\begin{aligned} &(Y_r - M_r Q)(X_r - N_r Q)^{-1} \\ &= K_{\text{nom}} + Q (I + G (I - K_{\text{nom}} G)^{-1} Q)^{-1} \\ &= K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \end{aligned}$$

so all stabilizing controllers are given by

$$C_{\text{stab}} = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \,\middle|\, Q \in \mathcal{RH}_\infty \right\}.$$

The invertibility condition is met since G, and thus N_r, is strictly proper. ■

This theorem tells us that if the plant is strongly stabilizable, that is, it can be stabilized by a stable controller, then given such a controller, we can parameterize the set of all stabilizing controllers. See [30] for a discussion of this, and [1] for an extension to nonlinear control. The parameterization above is very useful, since in the absence of the constraint $K \in S$, problem (1) can be reformulated as

$$\begin{aligned} &\text{minimize} && \|T_1 - T_2QT_3\| \\ &\text{subject to} && Q \in \mathcal{RH}_\infty. \end{aligned} \tag{4}$$

The closed-loop map is now affine in Q, and its norm is therefore a convex function of Q. This problem is readily solvable by, for example, the techniques in [4]. After solving this problem to find Q, one may then construct the optimal K for problem (1) via $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$.

C. Parameterization of Admissible Controllers

We now wish to extend the above result to parameterize all stabilizing controllers $K \in \mathcal{R}_p$ that also satisfy the information constraint $K \in S$. Applying the above change of variables to problem (1), we arrive at the following optimization problem:

$$\begin{aligned} &\text{minimize} && \|T_1 - T_2QT_3\| \\ &\text{subject to} && Q \in \mathcal{RH}_\infty \\ &&& K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S. \end{aligned} \tag{5}$$

However, the set of all Q which satisfy this last constraint is not convex in general, and hence this problem is not easily solved. We thus develop conditions under which this set is in fact convex, so that the optimization problem (5) may be solved via convex programming. First, we state a preliminary lemma.

Lemma 18: Suppose $G \in \mathcal{R}_{sp}, S \subseteq \mathcal{R}_p$ is a closed subspace, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. Then S is quadratically invariant under $h(K_{\text{nom}}, G)$ if and only if

$$S = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \,\middle|\, Q \in S \right\}.$$

Proof: ($\Rightarrow$) Suppose S is quadratically invariant under $h(K_{\text{nom}}, G)$, and further suppose there exists $Q \in S$ such that

$$K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big).$$

Since S is quadratically invariant under $h(K_{\text{nom}}, G)$ and S is an inert subspace, Theorem 14 implies that $h\big(h(K_{\text{nom}}, G), Q\big) \in S$, and since $K_{\text{nom}} \in S$ as well, $K \in S$. ■

Now, suppose $K \in S$. Let

$$Q = h\big(h(K_{\text{nom}}, G), K_{\text{nom}} - K\big).$$

We know $K_{\text{nom}} - K \in S$, and since S is quadratically invariant under $h(K_{\text{nom}}, G)$, then by Theorem 14, we also have $Q \in S$. ($\Leftarrow$) Now suppose S is not quadratically invariant under $h(K_{\text{nom}}, G)$. Then, by Theorem 14, there exists $Q \in S$ such that $h\big(h(K_{\text{nom}}, G), Q\big) \notin S$, and thus $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \notin S$. ■

This lemma shows that if we can find a stable $K_{\text{nom}} \in S$ which is stabilizing, and if the condition that S is quadratically invariant under $h(K_{\text{nom}}, G)$ holds, then the set of all stabilizing admissible controllers can be easily parameterized with the same change of variables from Theorem 17. We now simplify this condition.

Main Result 2: The following theorem is the main result of this section. It states that if the constraint set is quadratically invariant under the plant, then the information constraints on K are equivalent to *affine constraints* on the Youla parameter Q. Specifically, the constraint $K \in S$ is equivalent to the constraint $Q \in S$.

Theorem 19: Suppose $G \in \mathcal{R}_{sp}, S \subseteq \mathcal{R}_p$ is a closed subspace, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. If S is quadratically invariant under G then

$$S = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \,\middle|\, Q \in S \right\}.$$

Proof: If S is quadratically invariant under G, then $G \in S^\star$. Further, by Theorem 4, $S^\star$ is quadratically invariant under K_{nom}, and then by Theorem 14, we have $h(K_{\text{nom}}, S^\star) = S^\star$. We then have $h(K_{\text{nom}}, G) \in S^\star$ and, therefore, S is quadratically invariant under $h(K_{\text{nom}}, G)$. By Lemma 18, this yields the desired result. ■

Remark 20: When P is stable, we can choose $K_{\text{nom}} = 0$ and the result reduces to that analyzed in [15].

Remark 21: When $S = \mathcal{R}_p^{n_u \times n_y}$, which corresponds to centralized control, then the quadratic invariance condition is met and the result reduces to Theorem 17.

D. Equivalent Convex Problem

When the constraint set is quadratically invariant under the plant, we now have the following equivalent problem. Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subseteq \mathcal{R}_p^{n_u \times n_y}$ is a closed subspace. Then K is optimal for problem (1) if and only if $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$ and Q is optimal for

$$\begin{aligned} &\text{minimize} && \|T_1 - T_2QT_3\| \\ &\text{subject to} && Q \in \mathcal{R}H_\infty \\ &&& Q \in S \end{aligned} \tag{6}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by (3). This problem may be solved via convex programming.

V. Specific Constraint Classes

In this section, we apply these results to specific constraint classes. Armed with our findings on quadratic invariance, many useful results easily follow. In Section V-A, we consider a distributed control problem, and show that if the controllers can communicate faster than the dynamics propagate, then norm-optimal controllers may be found via convex programming. In Section V-B, we show that symmetric synthesis is quadratically

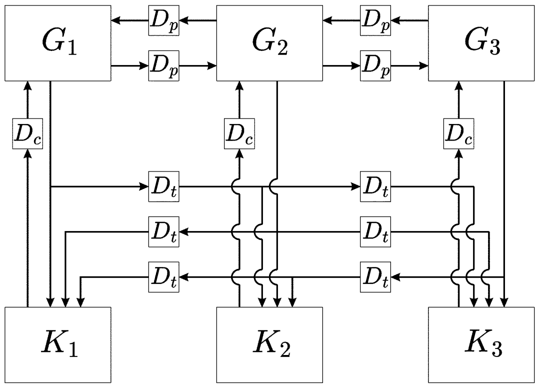

Fig. 2. Distributed control problem.

invariant, and thus convex. In Section V-C, we consider sparsity constraints. We develop a computational test for quadratic invariance of sparsity constraints, and easily show that norm minimization subject to such constraints which pass the test is a convex optimization problem. We also see an interesting negative result, that perfectly decentralized control is never quadratically invariant.

A. Distributed Control With Delays

We now consider the distributed control problem shown in Fig. 2. Suppose there are n subsystems with transmission delay $t \geq 0$, propagation delay $p \geq 0$ and computational delay $c \geq 0$. When expressed in linear-fractional form, we define the allowable set of controllers as follows. Let D be the delay operator on L_{2e} or ℓ_e. Then, $K \in S$ if and only if

$$K = \begin{bmatrix} D_c H_{11} & D_{t+c} H_{12} & \cdots & D_{(n-1)t+c} H_{1n} \\ D_{t+c} H_{21} & D_c H_{22} & \cdots & D_{(n-2)t+c} H_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)t+c} H_{n1} & \cdots & & D_c H_{nn} \end{bmatrix}$$

for some $H_{ij} \in \mathcal{R}_p$ of appropriate spatial dimensions. The corresponding system G is given by

$$G = \begin{bmatrix} A_{11} & D_p A_{12} & \cdots & D_{(n-1)p} A_{1n} \\ D_p A_{21} & A_{22} & \cdots & D_{(n-2)p} A_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)p} A_{n1} & \cdots & & A_{nn} \end{bmatrix}$$

for some $A_{ij} \in \mathcal{R}_{sp}$.

We define $\text{Delay}(\cdot)$ to give the delay associated with a time-invariant causal operator

$$\text{Delay}(W) = \inf\{\tau \geq 0 \,|\, w(\tau) \neq 0\} \quad \text{where } w \text{ is the impulse response of } W.$$

Theorem 22: Suppose that G and S are defined as above, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. Then, if

$$t \leq p + \frac{c}{(n-1)}$$

we have

$$S = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\middle|\; Q \in S \right\}.$$

Proof: Given $K \in S$

$$KGK \in S \iff \text{Delay}\big((KGK)_{kl}\big) \geq c + t|k-l| \quad \text{for all } k, l = 1, \ldots, n.$$

We now seek conditions which cause this to hold

$$(KGK)_{kl} = \sum_i \sum_j K_{ki} G_{ij} K_{jl}$$

and so, assuming w.l.o.g. that $k \leq l$

$$\begin{aligned} &\text{Delay}((KGK)_{kl}) \\ &\quad \geq \min_{i,j} \{\text{Delay}(K_{ki}) + \text{Delay}(G_{ij}) + \text{Delay}(K_{jl})\} \\ &\quad \geq \min_{i,j} \{2c + t(|k-i| + |j-l|) + p|i-j|\} \\ &\quad = \min_{k \leq i,j \leq l} \{2c + t(|k-i| + |j-l|) + p|i-j|\} \\ &\quad = \min_{k \leq i,j \leq l} \{2c + t(|k-l| - |i-j|) + p|i-j|\} \\ &\quad = 2c + t|k-l| + \min_{k \leq i,j \leq l} \{(p-t)|i-j|\} \\ &\quad = 2c + \min\{t,p\}|k-l|. \end{aligned}$$

So, the condition for quadratic invariance is

$$2c + \min\{t,p\}|k-l| \geq c + t|k-l| \quad \text{for all } k, l.$$

This is equivalent to $c - (t - \min\{t,p\})(n-1) \geq 0$, which is equivalent to $t \leq p + c/(n-1)$. So when this inequality holds, S is quadratically invariant under G, and the desired result follows from Theorem 19. ■

Thus, we see that finding the minimum-norm controller may be reduced to the convex optimization problem (6) when the controllers can transmit information faster than the dynamics propagate; that is, when $t \leq p$. We also see that the presence of computational delay causes this condition to be surprisingly relaxed. This result has been generalized considerably [14].

B. Symmetric Constraints

The following shows that when the plant is symmetric, the methods introduced in this paper could be used to find the optimal symmetric stabilizing controller. Symmetric synthesis is a well-studied problem, and there are many techniques which exploit its structure. Therefore, the methods in this paper are possibly not the most efficient. However, it is important to note the quadratic invariance of this structure because it defied earlier attempts to classify solvable problems. This arises because a symmetric matrix multiplied by another, i.e., KG, is not guaranteed to yield a symmetric matrix, but a symmetric matrix left and right multiplied by the same symmetric matrix, i.e., KGK, will indeed.

Theorem 23: Suppose

$$\mathbb{H}^n = \{A \in \mathbb{C}^{n \times n} \,|\, A = A^*\}$$

and

$$S = \{K \in \mathcal{R}_p \,|\; K(j\omega) \in \mathbb{H}^n \text{ for almost all } \omega \in \mathbb{R}\}.$$

Further suppose $K_{\rm nom} \in C_{\rm stab} \cap \mathcal{RH}_\infty \cap S$ and $G \in \mathcal{R}_{sp}$ with $G(j\omega) \in \mathbb{H}^n$ for almost all $\omega \in \mathbb{R}$. Then

$$S = \left\{ K_{\rm nom} - h\big(h(K_{\rm nom}, G), Q\big) \;\middle|\; Q \in S \right\}.$$

Proof: Follows immediately from Theorem 19. ■

C. Sparsity Constraints

Many problems in decentralized control can be expressed in the form of problem (1), where S is the set of controllers that satisfy a specified sparsity constraint. In this section, we provide a computational test for quadratic invariance when the subspace S is defined by block sparsity constraints. First, we introduce some notation.

Suppose $A^{\rm bin} \in \{0,1\}^{m\times n}$ is a binary matrix. We define the subspace

$$\text{Sparse}(A^{\rm bin}) = \{B \in \mathcal{R}_p \,|\, B_{ij}(j\omega) = 0 \text{ for all } i, j \text{ such that } A^{\rm bin}_{ij} = 0 \text{ for almost all } \omega \in \mathbb{R}\}.$$

Also, if $B \in \mathcal{R}_{sp}$, let $A^{\rm bin} =$ Pattern(B) be the binary matrix given by

$$A^{\rm bin}_{ij} = \begin{cases} 0, & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1, & \text{otherwise} \end{cases}$$

Note that in this section, we assume that matrices of transfer functions are indexed by blocks, so that above, the dimensions of $A^{\rm bin}$ may be much smaller than those of B. Then, $K^{\rm bin}_{kl}$ determines whether controller k may use measurements from subsystem l, K_{kl} is the map from the outputs of subsystem l to the inputs of subsystem k, and G_{ij} represents the map from the inputs to subsystem j to the outputs of subsystem i.

We seek an explicit test for quadratic invariance of a constraint set defined by such a binary matrix. We first prove two preliminary lemmas.

Lemma 24: Suppose $S =$ Sparse$(K^{\rm bin})$, and let $G^{\rm bin} =$ Pattern(G). If S is quadratically invariant under G, then

$$K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \text{ such that } K^{\rm bin}_{kl} = 0, G^{\rm bin}_{ij} = 1, K \in S.$$

Proof: Suppose there exists (i,j,k,l) and K such that

$$K^{\rm bin}_{kl} = 0, G^{\rm bin}_{ij} = 1, \; K \in S$$

but

$$K_{ki} \neq 0 \text{ and } K_{jl} \neq 0.$$

Then, we must have

$$K^{\rm bin}_{ki} = 1, \; K^{\rm bin}_{jl} = 1, \; i \neq l, \; j \neq k.$$

Consider $K \in S$ such that

$$K_{ab} = 0 \text{ if } (a,b) \notin \{(k,i),(j,l)\}.$$

Then

$$(KGK)_{kl} = \sum_r \sum_s K_{kr} G_{rs} K_{sl} = K_{ki} G_{ij} K_{jl}.$$

Since $G_{ij} \neq 0$, we can easily choose K_{ki} and K_{jl} such that $(KGK)_{kl} \neq 0$. So $KGK \notin S$ and S is not quadratically invariant. ■

Lemma 25: Suppose $S =$ Sparse$(K^{\rm bin})$, and let $G^{\rm bin} =$ Pattern(G). If

$$K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \text{ such that } K^{\rm bin}_{kl} = 0, G^{\rm bin}_{ij} = 1, K \in S.$$

Then

$$K^{\rm bin}_{ki} K^{\rm bin}_{jl} = 0 \text{ for all } (i,j,k,l) \text{ such that } K^{\rm bin}_{kl} = 0, G^{\rm bin}_{ij} = 1.$$

Proof: We show this by contradiction. Suppose there exists (i,j,k,l) such that

$$K^{\rm bin}_{kl} = 0, \quad G^{\rm bin}_{ij} = 1, \quad K^{\rm bin}_{ki} K^{\rm bin}_{jl} \neq 0.$$

Then

$$K^{\rm bin}_{ki} = K^{\rm bin}_{jl} = 1$$

and, hence, it must follow that there exists $K \in S$ such that $K_{ki} \neq 0$ and $K_{jl} \neq 0$. ■

The following is the main result of this section. It provides a computational test for quadratic invariance when S is defined by sparsity constraints. It also equates quadratic invariance with a stronger condition.

Theorem 26: Suppose $S =$ Sparse$(K^{\rm bin})$, and let $G^{\rm bin} =$ Pattern(G). Then, the following are equivalent:

i) S is quadratically invariant under G;
ii) $KGJ \in S$ for all $K, J \in S$;
iii) $K^{\rm bin}_{ki} G^{\rm bin}_{ij} K^{\rm bin}_{jl}(1 - K^{\rm bin}_{kl}) = 0$ for all $i, l = 1, \ldots, n_y$ and $j, k = 1, \ldots, n_u$.

Proof: We will show that i) ⇒ iii) ⇒ ii) ⇒ i). Suppose S is quadratically invariant under G. Then, by Lemma 24

$$K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \text{ such that } K^{\rm bin}_{kl} = 0, \; G^{\rm bin}_{ij} = 1, \; K \in S$$

and by Lemma 25

$$K^{\rm bin}_{ki} K^{\rm bin}_{jl} = 0 \text{ for all } (i,j,k,l) \text{ such that } K^{\rm bin}_{kl} = 0, \; G^{\rm bin}_{ij} = 1$$

which can be restated

$$K^{\rm bin}_{ki} G^{\rm bin}_{ij} K^{\rm bin}_{jl} \left(1 - K^{\rm bin}_{kl}\right) = 0$$

and which implies that

$$K_{ki} = 0 \text{ or } J_{jl} = 0 \text{ for all } (i,j,k,l), K, J \text{ such that } K^{\rm bin}_{kl} = 0, \; G^{\rm bin}_{ij} = 1, \; K, J \in S$$

which clearly implies

$$(KGJ)_{kl} = \sum_i \sum_j K_{ki} G_{ij} J_{jl} = 0 \text{ for all } (k,l), K, J \text{ such that } K^{\rm bin}_{kl} = 0, \; K, J \in S$$

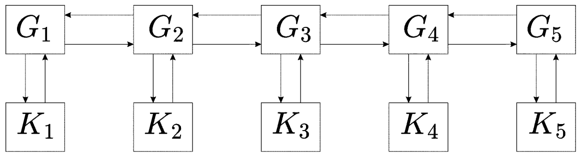

Fig. 3. Perfectly decentralized control.

and thus

$$KGJ \in S \quad \text{for all } K, J \in S$$

which is a stronger condition than quadratic invariance and hence implies i). ■

This result shows us several things about sparsity constraints. In this case, quadratic invariance is equivalent to another condition which is stronger in general. When G is symmetric, for example, the subspace consisting of symmetric K is quadratically invariant but does not satisfy condition ii). Condition iii), which gives us the computational test we desired, shows that quadratic invariance can be checked in time $O(n^4)$, where $n = \max\{n_u, n_y\}$. It also shows that, if S is defined by sparsity constraints, then S is quadratically invariant under G if and only if it is quadratically invariant under all systems with the same sparsity pattern.

Perfectly Decentralized Control: We now show an interesting negative result. Let $n_u = n_y$, so that each subsystem has its own controller as in Fig. 3.

Corollary 27: Suppose there exists i, j, with $i \neq j$, such that $G_{ij} \neq 0$. Suppose K^{bin} is diagonal and $S = \text{Sparse}(K^{\text{bin}})$. Then, S is not quadratically invariant under G.

Proof: Let $G^{\text{bin}} = \text{Pattern}(G)$. Then

$$K_{ii}^{\text{bin}} G_{ij}^{\text{bin}} K_{jj}^{\text{bin}} (1 - K_{ij}^{\text{bin}}) = 1.$$

The result then follows from Theorem 26. ■

It is important to note that the plant and controller do not have to be square to apply this result because of the block notation used in this section. This corollary tells us that if each subsystem has its own controller which may only use sensor information from its own subsystem, and any subsystem affects any other, then the system *is not* quadratically invariant. In other words, perfectly decentralized control is never quadratically invariant except for the trivial case where no subsystem affects any other.

Sparse Synthesis: The following theorem shows that for sparsity constraints, the test in Section V-C can be used to identify tractable decentralized control problems.

Theorem 28: Suppose $G \in \mathcal{R}_{sp}$ and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. Further, suppose $G^{\text{bin}} = \text{Pattern}(G)$ and $S = \text{Sparse}(K^{\text{bin}})$ for some $K^{\text{bin}} \in \{0,1\}^{n_u \times n_y}$. If

$$K_{ki}^{\text{bin}} G_{ij}^{\text{bin}} K_{jl}^{\text{bin}} \left(1 - K_{kl}^{\text{bin}}\right) = 0, \qquad \text{for all } i, l = 1, \ldots, n_y \text{ and } j, k = 1, \ldots, n_u$$

then

$$S = \left\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \,\middle|\, Q \in S \right\}.$$

Proof: Follows immediately from Theorems 19 and 26. ■

VI. Computation of Optimal Controllers

We show in this section that if we wish to minimize the $\mathcal{H}_2$-norm, then we can convert (6) to an unconstrained problem which may be readily solved. We focus on sparsity constraints, as in [18], but the vectorization techniques in this section are easily applied to the other constraint classes of Section V as well. A similar method was used for symmetric constraints in [28].

For ease of presentation, we now make a slight change of notation from Section V-C. We no longer assume that the plant and controller are divided into blocks, so that K_{kl}^{bin} now determines whether the kl index of the controller may be nonzero, rather than determining whether controller k may use information from subsystem l, and G_{ij} similarly represents the ij index of the plant. K^{bin} therefore has the same dimension as the controller itself. n_u and n_y represent the total number of inputs and outputs, respectively.

Let

$$a = \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} K_{ij}^{\text{bin}}$$

such that a represents the number of admissible controls, that is, the number of indexes for which K is not constrained to be zero.

The following theorem gives the equivalent unconstrained problem.

Theorem 29: Suppose x is an optimal solution to

$$\begin{aligned} &\text{minimize} && \|b + Ax\|_2 \\ &\text{subject to} && x \in \mathcal{RH}_\infty \end{aligned} \tag{7}$$

where $D \in \mathbb{R}^{n_u n_y \times a}$ is a matrix whose columns form an orthonormal basis for $\text{vec}(S)$, and

$$b = \text{vec}(T_1) \quad A = -(T_3^{\text{T}} \otimes T_2)D.$$

Then, $Q = \text{vec}^{-1}(Dx)$ is optimal for (6) and the optimal values are equivalent.

Proof: We know that

$$Q \in \mathcal{RH}_\infty^{n_u \times n_y} \cap S \iff \text{vec}(Q) = Dx \quad \text{for some } x \in \mathcal{RH}_\infty^{a \times 1}.$$

Since

$$\begin{aligned} &\|T_1 - T_2 Q T_3\|_2 \\ &= \|\text{vec}(T_1 - T_2 Q T_3)\|_2 \quad \text{by definition of the } \mathcal{H}_2\text{-- norm} \\ &= \|\text{vec}(T_1) - \left(T_3^{\text{T}} \otimes T_2\right) \text{vec}(Q)\|_2 \quad \text{by Lemma 1} \\ &= \|\text{vec}(T_1) - \left(T_3^{\text{T}} \otimes T_2\right) Dx\|_2 \\ &= \|b + Ax\|_2 \end{aligned}$$

we have the desired result. ■

Therefore, we can find the optimal x for problem (7) using many available tools for unconstrained $\mathcal{H}_2$-synthesis, with

$$P_{11} = b \quad P_{12} = A \quad P_{21} = 1 \quad P_{22} = 0^{1 \times a}$$

then find the optimal Q for problem (6) as $Q = \text{vec}^{-1}(Dx)$, and finally, find the optimal K for problem (1) as $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$.

A. Numerical Example

Consider an unstable lower triangular plant

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s-1} \end{bmatrix}$$

with P given by

$$P_{11} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} G \\ I \end{bmatrix} \quad P_{21} = [G \quad I]$$

and a sequence of sparsity constraints $K_1^{\rm bin}, \ldots, K_6^{\rm bin}$

$$K_1^{\rm bin} = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&1 \end{bmatrix} \qquad K_2^{\rm bin} = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&1 \end{bmatrix}$$

$$K_3^{\rm bin} = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&0\\1&1&0&0&1 \end{bmatrix} \qquad K_4^{\rm bin} = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&0\\1&1&1&0&1 \end{bmatrix}$$

$$K_5^{\rm bin} = \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\1&1&1&0&1 \end{bmatrix} \qquad K_6^{\rm bin} = \begin{bmatrix} 1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1 \end{bmatrix}$$

defining a sequence of information constraints

$$S_i = \text{Sparse}(K_i^{\rm bin})$$

such that each subsequent constraint is less restrictive, and such that each is quadratically invariant under G. We also use S_7 as the set of controllers with no sparsity constraints, i.e., the centralized case. A stable and stabilizing controller which lies in the subspace defined by any of these sparsity constraints is given by

$$K_{\rm nom} = \begin{bmatrix} 0&0&0&0&0\\0&\frac{-6}{s+3}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&\frac{-6}{s+3} \end{bmatrix}.$$

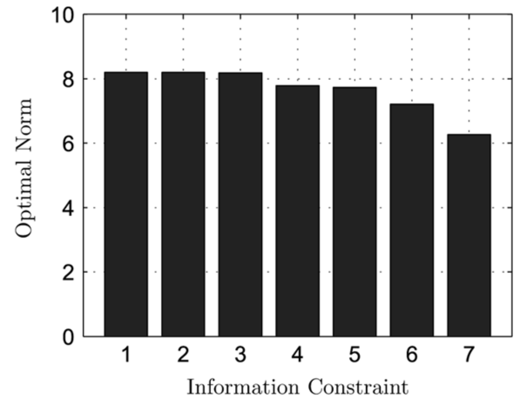

Fig. 4. Optimal norm with information constraints.

We can then find T_1, T_2, T_3 as in (3), and then find the stabilizing controller which minimizes the closed-loop norm subject to the sparsity constraints by solving problem (7), as outlined in Section VI. The graph in Fig. 4 shows the resulting minimum $\mathcal{H}_2$ norms for the six sparsity constraints as well as for a centralized controller.

VII. Conclusion

We defined the notion of quadratic invariance of a constraint set with respect to a plant. We showed in Theorem 14 that quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback. In Theorem 19, we then proved that quadratic invariance allows us to choose a controller parameterization such that the information constraint is equivalent to an affine constraint on the Youla parameter. Thus synthesizing optimal decentralized controllers becomes a convex optimization problem.

We then applied this to some specific constraint classes. We showed that for distributed systems with delays, optimal controllers may be synthesized in this manner if the communication delay is less than the propagation delay. We then noted that symmetric synthesis is included in this classification. We also provided a test for sparsity constraints to be quadratically invariant, and thus amenable to convex synthesis.

We thus characterized a broad and useful class of tractable decentralized control problems, and unified many previous results regarding specific structures.

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